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Developing Complementary Rough Inclusion Functions

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Summary. We continue the formal development of rough inclusion functions (RIFs), continuing the research on the formalization of rough sets [15] – a well-known tool of modelling of incomplete or partially unknown information. In this article we give the formal characterization of complementary RIFs, following a paper by Gomolińska [4]. We expand this framework introducing Jaccard index, Steinhaus generate metric, and Marczewski-Steinhaus metric space [1]. This is the continuation of [9]; additionally we implement also parts of [2], [3], and the details of this work can be found in [7].

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0. Introduction

In the paper, continuing our development of rough inclusion functions (RIFs), we deal with functions complementary to RIFs, and consider distance operators obtained from such functions.

Quite large part of the Mizar formalization of rough sets [5], [8] was done by means of the notion of a generalized approximation space understood as a pair $\langle U, \rho \rangle$, where ρ is an indiscernibility relation defined on the universe U. This reflects the standpoint of Skowron and Stepaniuk [16], based on tolerance relations instead of equivalence relations (claimed by Pawlak) and further generalized by Zhu [17], among many others. The framework build in a similar manner is contained in [10] and [11].

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In the alternative approach, used by Gomolińska [3], approximation spaces are treated as triples of the form $\mathcal{A} = (U, I, \kappa)$, where U is a non-empty set called the universe, $I : U \mapsto \wp U$ is an uncertainty mapping, and $\kappa : \wp U \times \wp U \mapsto [0, 1]$ is a rough inclusion function. The formalization of uncertainty mappings was discussed in [13], and the current submission goes further in this direction. Still however, we can merge our existing approaches via theory merging mechanism [6], having in mind that we should avoid duplications in the repository of Mizar texts as much as we can [12].

After filling some gaps in the Mizar Mathematical Library, proving preliminary facts needed later, in Sect. 2 we continue the development of functions complementary to RIFs. Given arbitrary preRIF f (where preRIF stands for a general mapping from the Cartesian square of the powerset of the universe into the unit interval, without any additional assumptions), we introduce the Mizar functor CMap f (see Def. 1), which is of much more general interest. Then we prove a list of properties of the complementary function on three well-known RIFs (see [9]): $\kappa^{£}$, κ_1, and κ_2.

Let us briefly recall these three mappings. The first one, standard rough inclusion function, $\kappa^{£}$ based on the ideas of Jan Łukasiewicz [14] is defined as follows:

$$\kappa^{£}(X, Y) = \begin{cases} \frac{|X \cap Y|}{|X|}, & \text{if } X \neq \emptyset \\ 1, & \text{otherwise} \end{cases}$$

Two others are

$$\kappa_1(X, Y) = \begin{cases} \frac{|Y|}{|X \cup Y|}, & \text{if } X \cup Y \neq \emptyset \\ 1, & \text{otherwise} \end{cases}$$

and

$$\kappa_2(X, Y) = \frac{|(U - X) \cup Y|}{|U|}.$$

Additionally, we introduce a new type for an object complementary to RIF, called just co-RIF.

Our testbed for chosen formal approach was Section 4, where full formalization of Proposition 4 from [3] was presented. This was also a step towards defining three metrics: δ_L, δ_1, and δ_2 (Def. 3, 4, and 5, respectively). It is worth noticing that even if we can deal with fixed rough approximation space, say R, we give this variable explicitly both in definitions of all three κ functions, and consequently in corresponding distances δ.

Section 5 contains the definition and very basic properties of Jaccard similarity coefficient J_s, widely used in data mining and information retrieval. We adopt the setting allowing both sets to be empty at the same time (then the value of Jaccard index is set to 1). Based on that, in Sect. 6 we define Jaccard distance (or Marczewski distance) for arbitrary subsets A, B of the universe as $1 - J_s(A, B)$.

We met some difficulties in proofs of the triangle inequality for such metrics, and in order to make it easier for us, we decided to implement some more ideas from the theory of distances. Namely, we introduced the symmetric difference metric (Def. 11). Then, using newly defined construction of Steinhaus generate metric (Def. 10), we can obtain from any distance a new one. The crucial fact was that the Jaccard distance is precisely Steinhaus generate metric from symmetric difference distance, hence all ordinary properties of a metric space can be easily obtained by means of this construction.

In the last section, we show that the value of Marczewski metric on two subsets A, B of given rough approximation space R is equal to $\delta_1(A, B)$. As δ_1 satisfies the triangle inequality, so does Marczewski metric.

1. Preliminaries

Let us consider finite sets x_1, x_2. Now we state the propositions:

(1) $\overline{\overline{x_1 \dot{-} x_2}} = \overline{\overline{x_1 \setminus x_2}} + \overline{\overline{x_2 \setminus x_1}}$.

(2) $\frac{2 \cdot \overline{\overline{x_1 \dot{-} x_2}}}{\overline{\overline{x_1}} + \overline{\overline{x_2}} + \overline{\overline{x_1 \dot{-} x_2}}} = \frac{\overline{\overline{x_1 \dot{-} x_2}}}{\overline{\overline{x_1 \cup x_2}}}$.

Now we state the propositions:

(3) Let us consider sets A, B, C. Then $A \dot{-} C = (A \dot{-} B) \dot{-} (B \dot{-} C)$.

(4) Let us consider finite sets A, B. Suppose $A \cup B \neq \emptyset$. Then $1 - \frac{\overline{\overline{A \cap B}}}{\overline{\overline{A \cup B}}} = \frac{\overline{\overline{A \dot{-} B}}}{\overline{\overline{A \cup B}}}$.

(5) Let us consider a finite set R, and subsets X, Y of R. Then $\overline{\overline{X \cup Y}} = \overline{\overline{X \cap Y}}$ if and only if $X = Y$.

Observe that there exists a metric space which is finite and non empty.

2. Complementary Rough Inclusion Functions

From now on R denotes a finite approximation space and X, Y, Z denote subsets of R.

Let R be a finite approximation space and f be a preRIF of R. The functor CMap f yielding a preRIF of R is defined by

(Def. 1) for every subsets x, y of R, $it(x, y) = 1 - f(x, y)$.

Now we state the propositions:

(6) Let us consider a preRIF f of R. Then $\mathrm{CMap}\,\mathrm{CMap}\,f = f$.
PROOF: Set $g = \mathrm{CMap}\,f$. For every element x of $2^{\alpha} \times 2^{\alpha}$, $(\mathrm{CMap}\,g)(x) = f(x)$, where α is the carrier of R. □

(7) If $X \neq \emptyset$, then $(\mathrm{CMap}\,\kappa^{\pounds}(R))(X, Y) = \frac{\overline{\overline{X \setminus Y}}}{\overline{\overline{X}}}$.

(8) If $X = \emptyset$, then $(\mathrm{CMap}\,\kappa^{\pounds}(R))(X, Y) = 0$.

(9) If $X \neq \emptyset$, then $(\mathrm{CMap}\,\kappa^{\pounds}(R))(X, Y) = \kappa^{\pounds}(X, Y^{c})$.

(10) If $X \cup Y \neq \emptyset$, then $(\mathrm{CMap}\,\kappa_1(R))(X, Y) = \frac{\overline{\overline{X \setminus Y}}}{\overline{\overline{X \cup Y}}}$.

(11) If $X \cup Y = \emptyset$, then $(\mathrm{CMap}\,\kappa_1(R))(X, Y) = 0$.

(12) $(\mathrm{CMap}\,\kappa_2(R))(X, Y) = \frac{\overline{\overline{X \setminus Y}}}{\overline{\overline{\Omega_R}}}$.

(13) Suppose $X \neq \emptyset$. Then $\kappa^{\pounds}(X, Y) = \frac{(\mathrm{CMap}\,\kappa_1(R))(X, Y^{c})}{\kappa_1(Y^{c}, X)} = \frac{(\mathrm{CMap}\,\kappa_2(R))(X, Y^{c})}{\kappa_2(\Omega_R, X)}$.

3. Introducing co-RIFs

Let us consider R. Let f be a preRIF of R. We say that f is co-RIF-like if and only if

(Def. 2) $\mathrm{CMap}\,f$ is a RIF of R.

Let f be a RIF of R. Let us observe that $\mathrm{CMap}\,f$ is co-RIF-like and there exists a preRIF of R which is co-RIF-like.

A co-RIF of R is a co-RIF-like preRIF of R.

4. Proposition 6 from [4]

From now on κ denotes a RIF of R. Now we state the propositions:

(14) $(\mathrm{CMap}\,\kappa)(X, Y) = 0$ if and only if $X \subseteq Y$.

(15) $(\mathrm{CMap}\,\kappa^{\pounds}(R))(X, Y) = 0$ if and only if $X \subseteq Y$.
PROOF: If $(\mathrm{CMap}\,\kappa^{\pounds}(R))(X, Y) = 0$, then $X \subseteq Y$. □

(16) If $Y \subseteq Z$, then $(\mathrm{CMap}\,\kappa)(X, Z) \leqslant (\mathrm{CMap}\,\kappa)(X, Y)$.

(17) If $Y \subseteq Z$, then $(\mathrm{CMap}\,\kappa^{\pounds}(R))(X, Z) \leqslant (\mathrm{CMap}\,\kappa^{\pounds}(R))(X, Y)$.

(18) $(\mathrm{CMap}\,\kappa_2(R))(X, Y) \leqslant (\mathrm{CMap}\,\kappa_1(R))(X, Y) \leqslant (\mathrm{CMap}\,\kappa^{\pounds}(R))(X, Y)$.

(19) Let us consider real numbers a, b, c. If $a \leqslant b$ and $0 \leqslant c < b$ and $0 < b$, then $\frac{a}{b} \geqslant \frac{a-c}{b-c}$.

(20) If $X \neq \emptyset$ and $Y = \emptyset$, then $(\mathrm{CMap}\,\kappa_1(R))(X, Y) = 1$. The theorem is a consequence of (10).

(21) If $X = \emptyset$ and $Y \neq \emptyset$, then $(\mathrm{CMap}\,\kappa_1(R))(X, Y) = 0$. The theorem is a consequence of (10).

(22) $(\mathrm{CMap}\,\kappa_1(R))(X,Y)+(\mathrm{CMap}\,\kappa_1(R))(Y,Z) \geqslant (\mathrm{CMap}\,\kappa_1(R))(X,Z)$. The theorem is a consequence of (14) and (20).

(23) $0 \leqslant (\mathrm{CMap}\,\kappa^{\pounds}(R))(X,Y) \leqslant 1$.

(24) $0 \leqslant (\mathrm{CMap}\,\kappa_1(R))(X,Y) + (\mathrm{CMap}\,\kappa_1(R))(Y,X) \leqslant 1$. The theorem is a consequence of (11) and (10).

(25) $0 \leqslant (\mathrm{CMap}\,\kappa_2(R))(X,Y) + (\mathrm{CMap}\,\kappa_2(R))(Y,X) \leqslant 1$. The theorem is a consequence of (12).

(26) Suppose $X = \emptyset$ and $Y \neq \emptyset$ or $X \neq \emptyset$ and $Y = \emptyset$.
Then $(\mathrm{CMap}\,\kappa^{\pounds}(R))(X,Y)+(\mathrm{CMap}\,\kappa^{\pounds}(R))(Y,X) = (\mathrm{CMap}\,\kappa_1(R))(X,Y)$ $+(\mathrm{CMap}\,\kappa_1(R))(Y,X) = 1$.

Let us consider R. The functors: $\delta_L(R)$, $\delta_1(R)$, and $\delta_2(R)$ yielding preRIFs of R are defined by conditions

(Def. 3) for every subsets x, y of R, $\delta_L(R)(x,y) = \frac{(\mathrm{CMap}\,\kappa^{\pounds}(R))(x,y)+(\mathrm{CMap}\,\kappa^{\pounds}(R))(y,x)}{2}$,

(Def. 4) for every subsets x, y of R, $\delta_1(R)(x,y) =$ $(\mathrm{CMap}\,\kappa_1(R))(x,y) + (\mathrm{CMap}\,\kappa_1(R))(y,x)$,

(Def. 5) for every subsets x, y of R, $\delta_2(R)(x,y) =$ $(\mathrm{CMap}\,\kappa_2(R))(x,y) + (\mathrm{CMap}\,\kappa_2(R))(y,x)$,

respectively. Now we state the propositions:

(27) $(\delta_L(R))(X,Y) = 0$ if and only if $X = Y$. The theorem is a consequence of (14).

(28) $(\delta_L(R))(X,Y) = (\delta_L(R))(Y,X)$.

(29) If $X \neq \emptyset$ and $Y = \emptyset$ or $X = \emptyset$ and $Y \neq \emptyset$, then $(\delta_L(R))(X,Y) = \frac{1}{2}$.

(30) Suppose $X \neq \emptyset$ and $Y \neq \emptyset$. Then $(\delta_L(R))(X,Y) = \frac{\frac{\overline{\overline{X \setminus Y}}}{\overline{\overline{X}}}+\frac{\overline{\overline{Y \setminus X}}}{\overline{\overline{Y}}}}{2}$. The theorem is a consequence of (7).

(31) $(\delta_1(R))(X,Y) = \frac{\overline{\overline{X \dot{-} Y}}}{\overline{\overline{X \cup Y}}}$. The theorem is a consequence of (10) and (14).

(32) $(\delta_2(R))(X,Y) = \frac{\overline{\overline{X \dot{-} Y}}}{\overline{\overline{\Omega_R}}}$. The theorem is a consequence of (12).

(33) $(\delta_1(R))(X,Y) + (\delta_1(R))(Y,Z) \geqslant (\delta_1(R))(X,Z)$. The theorem is a consequence of (22).

(34) $(\delta_1(R))(X,Y) = 0$ if and only if $X = Y$. The theorem is a consequence of (14).

(35) $(\delta_1(R))(X,Y) = (\delta_1(R))(Y,X)$.

(36) $(\delta_2(R))(X,Y) = 0$ if and only if $X = Y$. The theorem is a consequence of (14).

(37) $(\delta_2(R))(X,Y) = (\delta_2(R))(Y,X)$.

(38) $(\mathrm{CMap}\,\kappa_2(R))(X,Y)+(\mathrm{CMap}\,\kappa_2(R))(Y,Z) \geqslant (\mathrm{CMap}\,\kappa_2(R))(X,Z)$. The theorem is a consequence of (12).

(39) $(\delta_2(R))(X,Y)+(\delta_2(R))(Y,Z) \geqslant (\delta_2(R))(X,Z)$. The theorem is a consequence of (38).

5. Jaccard Index Measuring Similarity of Sets

Let R be a finite set and A, B be subsets of R. The functor JaccardIndex(A,B) yielding an element of $[0,1]$ is defined by the term

(Def. 6) $$\begin{cases} \frac{\overline{\overline{A\cap B}}}{\overline{\overline{A\cup B}}}, & \textbf{if } A\cup B \neq \emptyset, \\ 1, & \textbf{otherwise}. \end{cases}$$

Let us consider a finite set R and subsets A, B of R. Now we state the propositions:

(40) JaccardIndex$(A,B)=1$ if and only if $A=B$. The theorem is a consequence of (5).

(41) JaccardIndex$(A,B)=$ JaccardIndex(B,A).

6. Marczewski-Steinhaus Metric

Let X be a non empty set and f be a function from $X\times X$ into $\mathbb{R}$. Observe that f is non-negative yielding if and only if the condition (Def. 7) is satisfied.

(Def. 7) for every elements x, y of X, $f(x,y)\geqslant 0$.

One can verify that there exists a function from $X\times X$ into $\mathbb{R}$ which is discernible, symmetric, reflexive, and triangle and every function from $X\times X$ into $\mathbb{R}$ which is reflexive, symmetric, and triangle is also non-negative yielding.

Now we state the proposition:

(42) Let us consider a non empty set X, a non-negative yielding, discernible, triangle, reflexive function f from $X\times X$ into $\mathbb{R}$, and elements x, y of X. If $x\neq y$, then $f(x,y)>0$.

Let R be a finite set. The functor JaccardDist R yielding a function from $2^R\times 2^R$ into $\mathbb{R}$ is defined by

(Def. 8) for every subsets A, B of R, $it(A,B)=1-$ JaccardIndex(A,B).

Let R be a finite 1-sorted structure. The functor MarczewskiDistance R yielding a function from $2^{(\text{the carrier of } R)}\times 2^{(\text{the carrier of } R)}$ into $\mathbb{R}$ is defined by the term

(Def. 9) JaccardDist Ω_R.

7. Steinhaus Generate Metric

Let X be a non empty set, p be an element of X, and f be a function from $X \times X$ into $\mathbb{R}$. The functor SteinhausGen(f, p) yielding a function from $X \times X$ into $\mathbb{R}$ is defined by

(Def. 10) for every elements x, y of X, $it(x, y) = \frac{2 \cdot f(x,y)}{f(x,p)+f(y,p)+f(x,y)}$.

Let f be a non-negative yielding function from $X \times X$ into $\mathbb{R}$. Observe that SteinhausGen(f, p) is non-negative yielding.

Let f be a non-negative yielding, reflexive function from $X \times X$ into $\mathbb{R}$. One can verify that SteinhausGen(f, p) is reflexive.

Let f be a non-negative yielding, discernible function from $X \times X$ into $\mathbb{R}$. Let us observe that SteinhausGen(f, p) is discernible.

Let f be a non-negative yielding, symmetric function from $X \times X$ into $\mathbb{R}$. Let us note that SteinhausGen(f, p) is symmetric.

Let f be a discernible, symmetric, triangle, reflexive function from $X \times X$ into $\mathbb{R}$. Let us observe that SteinhausGen(f, p) is triangle.

8. Marczewski-Steinhaus Metric is Generated by Symmetric Difference Metric

Let X be a finite set. The functor SymmetricDiffDist X yielding a function from $2^X \times 2^X$ into $\mathbb{R}$ is defined by

(Def. 11) for every subsets x, y of X, $it(x, y) = \overline{\overline{x \dot{-} y}}$.

One can check that SymmetricDiffDist X is reflexive, discernible, symmetric, and triangle.

The functor SymDifMetrSpace X yielding a metric structure is defined by the term

(Def. 12) $\langle 2^X$, SymmetricDiffDist $X \rangle$.

One can verify that SymDifMetrSpace X is non empty and SymDifMetrSpace X is reflexive, discernible, symmetric, and triangle.

Now we state the propositions:

(43) Let us consider a finite set R, and subsets A, B of R. Then (JaccardDist $R)(A, B) = \frac{\overline{\overline{A \dot{-} B}}}{\overline{\overline{A \cup B}}}$. The theorem is a consequence of (4).

(44) Let us consider a finite set X. Then JaccardDist X = SteinhausGen(SymmetricDiffDist $X, \emptyset_X)$. The theorem is a consequence of (43) and (2).

9. Steinhaus Metric Spaces

Let M be a finite, non empty metric space. One can check that the distance of M is symmetric, reflexive, discernible, and triangle.

Let M be a finite, non empty metric structure and p be an element of M. The functor SteinhausMetrSpace(M, p) yielding a metric structure is defined by the term

(Def. 13) $\langle$the carrier of M, SteinhausGen((the distance of M), p)$\rangle$.

Let M be a metric structure. We say that M is with nonnegative distance if and only if

(Def. 14) the distance of M is non-negative yielding.

Let A be a finite, non empty set. Note that the discrete metric of A is finite, non empty, and non-negative yielding and there exists a metric space which is finite, non empty, and with nonnegative distance.

Let M be a finite, non empty, with nonnegative distance metric structure and p be an element of M. Let us observe that SteinhausMetrSpace(M, p) is with nonnegative distance.

Let M be a finite, non empty, with nonnegative distance, discernible metric structure. Observe that SteinhausMetrSpace(M, p) is discernible.

Let M be a finite, non empty, with nonnegative distance, reflexive metric structure. Let us note that SteinhausMetrSpace(M, p) is reflexive.

Let M be a finite, non empty, with nonnegative distance, symmetric metric structure. Note that SteinhausMetrSpace(M, p) is symmetric.

Let M be a finite, non empty, discernible, symmetric, reflexive, triangle metric structure. Let us observe that SteinhausMetrSpace(M, p) is triangle.

Let R be a finite 1-sorted structure. Observe that MarczewskiDistance R is reflexive, discernible, and symmetric.

Now we state the proposition:

(45) Let us consider a finite approximation space R, and subsets A, B of R. Then (MarczewskiDistance R)$(A, B) = (\delta_1(R))(A, B)$. The theorem is a consequence of (43) and (31).

Let R be a finite 1-sorted structure. Note that MarczewskiDistance R is triangle.

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