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Miscellaneous Graph Preliminaries

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Summary. This article contains many auxiliary theorems which were missing in the Mizar Mathematical Library [2] to the best of the author's knowledge. Most of them regard graph theory as formalized in the GLIB series (cf. [8]) and most of them are preliminaries needed in [7] or other forthcoming articles.

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0. Introduction

A generalized approach to graph theory as it was done in [3, 5] in contrast to [9], [4] is rather uncommon. To avoid duplication of the same theorems in different formalization frameworks in the Mizar Mathematical Library [1], a generalized approach to formalization is preferable (cf. [8], [6]). However, due to the sheer amount of "obvious facts" such an approach brings with it, it is only natural some of them not immediately needed slip the initial formalization process. This article aims to fill some of the gaps that emerged. Thereby, in most cases, preliminaries needed in [7] are provided.

Many theorems in this article regard the change of incident edge sets and degrees of a vertex when going from one graph to a related one (e.g. when reversing edge directions or adding an edge).

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1. Preliminaries not Directly Related to Graphs

Let us consider sets X, Y, Z. Now we state the propositions:

(1) If $Z \subseteq X$, then $X \cup Y \setminus Z = X \cup Y$.

(2) $X \cap Z$ misses $Y \setminus Z$.

(3) Let us consider objects x, y. Then $\{x, y\} \setminus \{$the element of $\{x, y\}\} = \emptyset$ if and only if $x = y$.

Let us consider objects a, b, x, y. Now we state the propositions:

(4) Suppose $a \neq b$ and $x =$ the element of $\{a, b\}$ and $y =$ the element of $\{a, b\} \setminus \{$the element of $\{a, b\}\}$. Then

(i) $x = a$ and $y = b$, or

(ii) $x = b$ and $y = a$.

(5) $\{a, b\} = \{x, y\}$ if and only if $x = a$ and $y = b$ or $x = b$ and $y = a$.

(6) Let us consider a set X, and a non empty set Y. Then $X \subset Y$ if and only if X is a proper subset of Y.

Let X be a non empty set. One can check that id_X is non irreflexive and $X \times X$ is non irreflexive and non asymmetric and there exists a binary relation on X which is non irreflexive and non asymmetric and there exists a binary relation on X which is symmetric, irreflexive, and non total and there exists a binary relation on X which is symmetric, non irreflexive, and non empty.

Let X be a non trivial set. Observe that id_X is non connected and there exists a binary relation on X which is symmetric and non connected and $X \times X$ is non antisymmetric and there exists a binary relation on X which is non antisymmetric.

Now we state the propositions:

(7) Let us consider binary relations R, S, and a set X. Then $(R \cup S)^{\circ} X = R^{\circ} X \cup S^{\circ} X$.

(8) Let us consider binary relations R, S, and a set Y. Then $(R \cup S)^{-1}(Y) = R^{-1}(Y) \cup S^{-1}(Y)$.

(9) Let us consider a binary relation R, and sets X, Y. Then $(Y \upharpoonleft R) \upharpoonright X = (Y \upharpoonleft R) \cap (R \upharpoonright X)$.

(10) Let us consider a symmetric binary relation R, and an object x. Then $R^{\circ} x = \mathrm{Coim}(R, x)$.

(11) Let us consider a set X, and a binary relation R on X. Then R is irreflexive if and only if id_X misses R.

(12) Let us consider objects x, y. Then $(\{\langle x, y \rangle\}$ **qua** binary relation$)^{\smile} = \{\langle y, x \rangle\}$.

(13) Let us consider a set X, objects x, y, and a symmetric binary relation R on X. If $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$.

Let a, b be cardinal numbers. Note that $a \cap b$ is cardinal and $a \cup b$ is cardinal.

Let X be a $\subseteq$-linear set. One can check that ${}^{\subseteq}{}_{X}$ is connected and $\langle X, \subseteq\rangle$ is connected.

Now we state the propositions:

(14) Let us consider a non empty set X. Suppose for every set a such that $a \in X$ holds a is a cardinal number. Then there exists a cardinal number A such that

(i) $A \in X$, and

(ii) $A = \bigcap X$.

Proof: Define $\mathcal{P}$[ordinal number] $\equiv$ $\$_1 \in X$ and $\$_1$ is a cardinal number. There exists an ordinal number A such that $\mathcal{P}[A]$. Consider A being an ordinal number such that $\mathcal{P}[A]$ and for every ordinal number B such that $\mathcal{P}[B]$ holds $A \subseteq B$. □

(15) Let us consider a set X. Suppose for every set a such that $a \in X$ holds a is a cardinal number. Then $\bigcap X$ is a cardinal number. The theorem is a consequence of (14).

Let f be a cardinal yielding function and x be an object. Note that $f(x)$ is cardinal.

Let X be a functional set. Note that $\bigcap X$ is function-like and relation-like.

Now we state the propositions:

(16) Let us consider a set X. Then $4 \subseteq \overline{\overline{X}}$ if and only if there exist objects w, x, y, z such that $w, x, y, z \in X$ and $w \neq x$ and $w \neq y$ and $w \neq z$ and $x \neq y$ and $x \neq z$ and $y \neq z$.

Proof: If $4 \subseteq \overline{\overline{X}}$, then there exist objects w, x, y, z such that w, x, y, $z \in X$ and $w \neq x$ and $w \neq y$ and $w \neq z$ and $x \neq y$ and $x \neq z$ and $y \neq z$. □

(17) Let us consider a set X. Suppose $4 \subseteq \overline{\overline{X}}$. Let us consider objects w, x, y. Then there exists an object z such that

(i) $z \in X$, and

(ii) $w \neq z$, and

(iii) $x \neq z$, and

(iv) $y \neq z$.

The theorem is a consequence of (16).

(18) Let us consider a set X. Then S_X misses $2\mathrm{Set}\,X$.

(19) Let us consider sets X, Y. Suppose $\overline{\overline{X}} = \overline{\overline{Y}}$. Then $\overline{\overline{\text{2Set}\,X}} = \overline{\overline{\text{2Set}\,Y}}$.
PROOF: Consider g being a function such that g is one-to-one and $\operatorname{dom} g = X$ and $\operatorname{rng} g = Y$. Define $\mathcal{K}(\text{set}) =$ the element of $\$_1$. Define $\mathcal{L}(\text{set}) =$ the element of $\$_1 \setminus \{\mathcal{K}(\$_1)\}$. Define $\mathcal{F}(\text{object}) = \{g(\mathcal{K}(\$_1(\in 2^X))), g(\mathcal{L}(\$_1(\in 2^X)))\}$. Consider f being a function such that $\operatorname{dom} f = \text{2Set}\,X$ and for every object x such that $x \in \text{2Set}\,X$ holds $f(x) = \mathcal{F}(x)$. $\square$

(20) Let us consider a finite set X. Then $\overline{\overline{\text{2Set}\,X}} = \binom{\overline{\overline{X}}}{2}$. The theorem is a consequence of (19).

2. Into GLIB_000

Now we state the propositions:

(21) Let us consider a graph G, a vertex v of G, and objects e, w. If v is isolated, then e does not join v and w in G.

(22) Let us consider a graph G, a vertex v of G, and objects e, w. Suppose v is isolated. Then

(i) e does not join v to w in G, and

(ii) e does not join w to v in G.

The theorem is a consequence of (21).

(23) Let us consider a graph G, and a vertex v of G. Then v is isolated if and only if $v \notin \operatorname{rng}(\text{the source of } G) \cup \operatorname{rng}(\text{the target of } G)$. The theorem is a consequence of (22).

(24) Let us consider a graph G, a vertex v of G, and an object e. If v is endvertex, then e does not join v and v in G.

(25) Let us consider a graph G, and a vertex v of G. Then

(i) v.edgesIn() = (the target of $G)^{-1}(\{v\})$, and

(ii) v.edgesOut() = (the source of $G)^{-1}(\{v\})$.

Let us consider a trivial graph G and a vertex v of G. Now we state the propositions:

(26) (i) v.edgesIn() = the edges of G, and

(ii) v.edgesOut() = the edges of G, and

(iii) v.edgesInOut() = the edges of G.

(27) (i) v.inDegree() = G.size(), and

(ii) v.outDegree() = G.size(), and

(iii) $v.\text{degree}() = G.\text{size}() + G.\text{size}()$.

The theorem is a consequence of (26).

(28) Let us consider a graph G, and sets X, Y. Then $G.\text{edgesBetween}(X, Y) = G.\text{edgesDBetween}(X, Y) \cup G.\text{edgesDBetween}(Y, X)$.

(29) Let us consider a graph G, and a vertex v of G. Then $v.\text{edgesInOut}() = G.\text{edgesBetween}(\text{the vertices of } G, \{v\})$. The theorem is a consequence of (28).

Let us consider a graph G and sets X, Y. Now we state the propositions:

(30) $G.\text{edgesDBetween}(X, Y) = G.\text{edgesOutOf}(X) \cap G.\text{edgesInto}(Y)$.

(31) $G.\text{edgesDBetween}(X, Y) \subseteq G.\text{edgesBetween}(X, Y)$.

Let us consider a graph G and a vertex v of G. Now we state the propositions:

(32) If for every object e, e does not join v and v in G, then $\overline{\overline{v.\text{edgesInOut}()}} = v.\text{degree}()$.

PROOF: $v.\text{edgesIn}() \cap v.\text{edgesOut}() = \emptyset$. $\square$

(33) v is isolated if and only if $v.\text{edgesIn}() = \emptyset$ and $v.\text{edgesOut}() = \emptyset$.

(34) v is isolated if and only if $v.\text{inDegree}() = 0$ and $v.\text{outDegree}() = 0$. The theorem is a consequence of (33).

(35) v is isolated if and only if $v.\text{degree}() = 0$. The theorem is a consequence of (34).

Let us consider a graph G and a set X. Now we state the propositions:

(36) $G.\text{edgesInto}(X) = \bigcup\{v.\text{edgesIn}()$, where v is a vertex of $G : v \in X\}$.

(37) $G.\text{edgesOutOf}(X) = \bigcup\{v.\text{edgesOut}()$, where v is a vertex of $G : v \in X\}$.

(38) $G.\text{edgesInOut}(X) = \bigcup\{v.\text{edgesInOut}()$, where v is a vertex of $G : v \in X\}$.

Let us consider a graph G and sets X, Y. Now we state the propositions:

(39) $G.\text{edgesDBetween}(X, Y) = \bigcup\{v.\text{edgesOut}() \cap w.\text{edgesIn}()$, where v, w are vertices of $G : v \in X$ and $w \in Y\}$.

(40) $G.\text{edgesBetween}(X, Y) \subseteq \bigcup\{v.\text{edgesInOut}() \cap w.\text{edgesInOut}()$, where v, w are vertices of $G : v \in X$ and $w \in Y\}$.

(41) Suppose X misses Y. Then $G.\text{edgesBetween}(X, Y) = \bigcup\{v.\text{edgesInOut}() \cap w.\text{edgesInOut}()$, where v, w are vertices of $G : v \in X$ and $w \in Y\}$. The theorem is a consequence of (40).

(42) Let us consider a graph G_1, a set E, a subgraph G_2 of G_1 with edges E removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) $v_2.\text{edgesIn}() = v_1.\text{edgesIn}() \setminus E$, and

(ii) $v_2.\text{edgesOut}() = v_1.\text{edgesOut}() \setminus E$, and

(iii) v_2.edgesInOut() $= v_1$.edgesInOut() $\setminus E$.

(43) Let us consider graphs G_1, G_2, and a set V. Then G_2 is a subgraph of G_1 with vertices V removed if and only if G_2 is a subgraph of G_1 with vertices $V \cap$ (the vertices of G_1) removed.

(44) Let us consider a graph G_1, a set V, a subgraph G_2 of G_1 with vertices V removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $V \subset$ the vertices of G_1 and $v_1 = v_2$. Then

(i) v_2.edgesIn() $= v_1$.edgesIn() $\setminus$ (G_1.edgesOutOf(V)), and

(ii) v_2.edgesOut() $= v_1$.edgesOut() $\setminus$ (G_1.edgesInto(V)), and

(iii) v_2.edgesInOut() $= v_1$.edgesInOut() $\setminus$ (G_1.edgesInOut(V)).

Proof: v_1.edgesOut()$\cap G_1$.edgesOutOf(V) $= \emptyset$. v_1.edgesIn()$\cap G_1$.edgesInto (V) $= \emptyset$. $\square$

(45) Let us consider a non trivial graph G_1, a vertex v of G_1, a subgraph G_2 of G_1 with vertex v removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_2.edgesIn() $= v_1$.edgesIn() $\setminus$ (v.edgesOut()), and

(ii) v_2.edgesOut() $= v_1$.edgesOut() $\setminus$ (v.edgesIn()), and

(iii) v_2.edgesInOut() $= v_1$.edgesInOut() $\setminus$ (v.edgesInOut()).

The theorem is a consequence of (44).

3. Into GLIB_002

Now we state the proposition:

(46) Let us consider a graph G, a component C of G, a vertex v_1 of G, and a vertex v_2 of C. Suppose $v_1 = v_2$. Then

(i) v_1.edgesIn() $= v_2$.edgesIn(), and

(ii) v_1.inDegree() $= v_2$.inDegree(), and

(iii) v_1.edgesOut() $= v_2$.edgesOut(), and

(iv) v_1.outDegree() $= v_2$.outDegree(), and

(v) v_1.edgesInOut() $= v_2$.edgesInOut(), and

(vi) v_1.degree() $= v_2$.degree().

4. Into GLIB_006

Now we state the propositions:

(47) Let us consider a graph G_2, a set V, a supergraph G_1 of G_2 extended by the vertices from V, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_1.edgesIn() = v_2.edgesIn(), and

(ii) v_1.inDegree() = v_2.inDegree(), and

(iii) v_1.edgesOut() = v_2.edgesOut(), and

(iv) v_1.outDegree() = v_2.outDegree(), and

(v) v_1.edgesInOut() = v_2.edgesInOut(), and

(vi) v_1.degree() = v_2.degree().

(48) Let us consider a graph G_2, objects v, w, e, a supergraph G_1 of G_2 extended by e between vertices v and w, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$ and $v_2 \neq v$ and $v_2 \neq w$. Then

(i) v_1.edgesIn() = v_2.edgesIn(), and

(ii) v_1.inDegree() = v_2.inDegree(), and

(iii) v_1.edgesOut() = v_2.edgesOut(), and

(iv) v_1.outDegree() = v_2.outDegree(), and

(v) v_1.edgesInOut() = v_2.edgesInOut(), and

(vi) v_1.degree() = v_2.degree().

(49) Let us consider a graph G_2, vertices v, w of G_2, an object e, a supergraph G_1 of G_2 extended by e between vertices v and w, and a vertex v_1 of G_1. Suppose $e \notin$ the edges of G_2 and $v_1 = v$ and $v \neq w$. Then

(i) v_1.edgesIn() = v.edgesIn(), and

(ii) v_1.inDegree() = v.inDegree(), and

(iii) v_1.edgesOut() = v.edgesOut() $\cup \{e\}$, and

(iv) v_1.outDegree() = v.outDegree() $+ 1$, and

(v) v_1.edgesInOut() = v.edgesInOut() $\cup \{e\}$, and

(vi) v_1.degree() = v.degree() $+ 1$.

(50) Let us consider a graph G_2, vertices v, w of G_2, an object e, a supergraph G_1 of G_2 extended by e between vertices v and w, and a vertex w_1 of G_1. Suppose $e \notin$ the edges of G_2 and $w_1 = w$ and $v \neq w$. Then

(i) w_1.edgesIn() = w.edgesIn() $\cup \{e\}$, and

(ii) w_1.inDegree() = w.inDegree() + 1, and

(iii) w_1.edgesOut() = w.edgesOut(), and

(iv) w_1.outDegree() = w.outDegree(), and

(v) w_1.edgesInOut() = w.edgesInOut() $\cup \{e\}$, and

(vi) w_1.degree() = w.degree() + 1.

(51) Let us consider a graph G_2, a vertex v of G_2, an object e, a supergraph G_1 of G_2 extended by e between vertices v and v, and a vertex v_1 of G_1. Suppose $e \notin$ the edges of G_2 and $v_1 = v$. Then

(i) v_1.edgesIn() = v.edgesIn() $\cup \{e\}$, and

(ii) v_1.inDegree() = v.inDegree() + 1, and

(iii) v_1.edgesOut() = v.edgesOut() $\cup \{e\}$, and

(iv) v_1.outDegree() = v.outDegree() + 1, and

(v) v_1.edgesInOut() = v.edgesInOut() $\cup \{e\}$, and

(vi) v_1.degree() = v.degree() + 2.

5. Into GLIB_007

Now we state the propositions:

(52) Let us consider a graph G_3, a set E, a graph G_4 given by reversing directions of the edges E of G_3, a supergraph G_1 of G_3, and a graph G_2 given by reversing directions of the edges E of G_1. Suppose $E \subseteq$ the edges of G_3. Then G_2 is a supergraph of G_4.

(53) Let us consider a graph G_2, and an object v. Then every supergraph of G_2 extended by v is a supergraph of G_2 extended by vertex v and edges between v and $\emptyset$ of G_2.

(54) Let us consider a graph G_1, a set E, a graph G_2 given by reversing directions of the edges E of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$ and $E \subseteq$ the edges of G_1. Then

(i) v_2.edgesIn() = v_1.edgesIn() $\setminus E \cup v_1$.edgesOut() $\cap E$, and

(ii) v_2.edgesOut() = v_1.edgesOut() $\setminus E \cup v_1$.edgesIn() $\cap E$.

(55) Let us consider a graph G_1, a graph G_2 given by reversing directions of the edges of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_2.edgesIn() = v_1.edgesOut(), and

(ii) v_2.inDegree() = v_1.outDegree(), and

(iii) v_2.edgesOut() $= v_1$.edgesIn(), and

(iv) v_2.outDegree() $= v_1$.inDegree().

(56) Let us consider a graph G_1, a set E, a graph G_2 given by reversing directions of the edges E of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_2.edgesInOut() $= v_1$.edgesInOut(), and

(ii) v_2.degree() $= v_1$.degree().

The theorem is a consequence of (54) and (2).

(57) Let us consider a graph G_2, an object v, a set V, a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, and a vertex w of G_1. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and $v = w$. Then

(i) w.allNeighbors() $= V$, and

(ii) w.degree() $= \overline{\overline{V}}$.

The theorem is a consequence of (29), (32), and (35).

(58) Let us consider a graph G_2, an object v, a set V, a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$ and $v_2 \notin V$. Then

(i) v_1.edgesIn() $= v_2$.edgesIn(), and

(ii) v_1.inDegree() $= v_2$.inDegree(), and

(iii) v_1.edgesOut() $= v_2$.edgesOut(), and

(iv) v_1.outDegree() $= v_2$.outDegree(), and

(v) v_1.edgesInOut() $= v_2$.edgesInOut(), and

(vi) v_1.degree() $= v_2$.degree().

(59) Let us consider a graph G_2, an object v, a subset V of the vertices of G_2, a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v \notin$ the vertices of G_2 and $v_1 = v_2$ and $v_2 \in V$. Then

(i) v_1.allNeighbors() $= v_2$.allNeighbors() $\cup \{v\}$, and

(ii) v_1.degree() $= v_2$.degree() $+ 1$.

(60) Let us consider a graph G_2, an object v, a set V, a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_1.degree() $\subseteq v_2$.degree() $+ 1$, and

(ii) v_1.inDegree() $\subseteq v_2$.inDegree() $+ 1$, and

(iii) v_1.outDegree() $\subseteq v_2$.outDegree() $+ 1$.

The theorem is a consequence of (58).

6. Into GLIB_008

Now we state the propositions:

(61) Let us consider a graph G. Then G is edgeless if and only if for every vertices v, w of G, v and w are not adjacent.

(62) Let us consider a loopless graph G. Then G is edgeless if and only if for every vertices v, w of G such that $v \neq w$ holds v and w are not adjacent. The theorem is a consequence of (61).

7. Into GLIB_009

Now we state the propositions:

(63) Let us consider a graph G. Then G.loops() $=$ dom((the source of G) $\cap$ (the target of G)).

(64) Let us consider graphs G_1, G_2, and a set E. Then G_2 is a graph given by reversing directions of the edges E of G_1 if and only if G_2 is a graph given by reversing directions of the edges $E \setminus (G_1$.loops()) of G_1.

(65) Let us consider a graph G_1, a subgraph G_2 of G_1 with loops removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_2.inNeighbors() $= v_1$.inNeighbors() $\setminus \{v_1\}$, and

(ii) v_2.outNeighbors() $= v_1$.outNeighbors() $\setminus \{v_1\}$, and

(iii) v_2.allNeighbors() $= v_1$.allNeighbors() $\setminus \{v_1\}$.

(66) Let us consider a graph G_1, a subgraph G_2 of G_1 with parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_2.allNeighbors() $= v_1$.allNeighbors().

(67) Let us consider a graph G_1, a subgraph G_2 of G_1 with directed-parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_2.inNeighbors() $= v_1$.inNeighbors(), and

(ii) v_2.outNeighbors() $= v_1$.outNeighbors(), and

(iii) v_2.allNeighbors() $= v_1$.allNeighbors().

(68) Let us consider a graph G_1, a simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then v_2.allNeighbors() = v_1.allNeighbors() $\setminus \{v_1\}$. The theorem is a consequence of (65) and (66).

(69) Let us consider a graph G_1, a directed-simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_2.inNeighbors() = v_1.inNeighbors() $\setminus \{v_1\}$, and

(ii) v_2.outNeighbors() = v_1.outNeighbors() $\setminus \{v_1\}$, and

(iii) v_2.allNeighbors() = v_1.allNeighbors() $\setminus \{v_1\}$.

The theorem is a consequence of (65) and (67).

Let G be a non loopless graph. One can verify that every subgraph of G with parallel edges removed is non loopless and every subgraph of G with directed-parallel edges removed is non loopless.

Let G be a non edgeless graph. Note that every subgraph of G with parallel edges removed is non edgeless and every subgraph of G with directed-parallel edges removed is non edgeless.

Now we state the propositions:

(70) Let us consider a graph G, and a representative selection of the parallel edges E of G. Then $\overline{\overline{E}} = \overline{\overline{\text{Classes EdgeParEqRel}(G)}}$.

PROOF: Define $\mathcal{F}(\text{object}) = [\$_1]_{\text{EdgeParEqRel}(G)}$. Consider f being a function such that $\operatorname{dom} f = E$ and for every object x such that $x \in E$ holds $f(x) = \mathcal{F}(x)$. □

(71) Let us consider a graph G, and a representative selection of the directed-parallel edges E of G. Then $\overline{\overline{E}} = \overline{\overline{\text{Classes DEdgeParEqRel}(G)}}$.

PROOF: Define $\mathcal{F}(\text{object}) = [\$_1]_{\text{DEdgeParEqRel}(G)}$. Consider f being a function such that $\operatorname{dom} f = E$ and for every object x such that $x \in E$ holds $f(x) = \mathcal{F}(x)$. □

(72) Let us consider a graph G, a set X, a subset E of the edges of G, and a graph H given by reversing directions of the edges X of G. Then E is a representative selection of the parallel edges of G if and only if E is a representative selection of the parallel edges of H.

(73) Let us consider a graph G, a non empty subset V of the vertices of G, a subgraph H of G induced by V, and a representative selection of the parallel edges E of G. Then $E \cap G$.edgesBetween(V) is a representative selection of the parallel edges of H.

(74) Let us consider a graph G, a non empty subset V of the vertices of G, a subgraph H of G induced by V, and a representative selection of the directed-parallel edges E of G. Then $E \cap G$.edgesBetween(V) is a representative selection of the directed-parallel edges of H.

Let us consider a graph G, a set V, a supergraph H of G extended by the vertices from V, and a subset E of the edges of G. Now we state the propositions:

(75) E is a representative selection of the parallel edges of G if and only if E is a representative selection of the parallel edges of H.

(76) E is a representative selection of the directed-parallel edges of G if and only if E is a representative selection of the directed-parallel edges of H.

Note that there exists a graph which is non non-multi and non-directed-multi.

Let G_F be a graph-yielding function. We say that G_F is plain if and only if

(Def. 1) for every object x such that $x \in \operatorname{dom} G_F$ there exists a graph G such that $G_F(x) = G$ and G is plain.

Let G be a plain graph. Note that $\langle G \rangle$ is plain and $\mathbb{N} \longmapsto G$ is plain.

Let G_F be a non empty, graph-yielding function. One can check that G_F is plain if and only if the condition (Def. 2) is satisfied.

(Def. 2) for every element x of $\operatorname{dom} G_F$, $G_F(x)$ is plain.

Let G_{Sq} be a graph sequence. Note that G_{Sq} is plain if and only if the condition (Def. 3) is satisfied.

(Def. 3) for every natural number n, $G_{Sq}(n)$ is plain.

Observe that every graph-yielding function which is empty is also plain and there exists a graph sequence which is plain and there exists a graph-yielding finite sequence which is non empty and plain.

Let G_F be a plain, non empty, graph-yielding function and x be an element of $\operatorname{dom} G_F$. Let us observe that $G_F(x)$ is plain. Let G_{Sq} be a plain graph sequence and x be a natural number. Let us observe that $G_{Sq}(x)$ is plain. Let p be a plain, graph-yielding finite sequence and n be a natural number. One can check that $p \restriction n$ is plain and $p_{\downharpoonright n}$ is plain. Let m be a natural number.

Observe that $\operatorname{smid}(p, m, n)$ is plain and $\langle p(m), \ldots, p(n) \rangle$ is plain. Let p, q be plain, graph-yielding finite sequences. One can check that $p \frown q$ is plain and $p \frown\!\frown q$ is plain. Let G_1, G_2 be plain graphs. Let us observe that $\langle G_1, G_2 \rangle$ is plain. Let G_3 be a plain graph. One can verify that $\langle G_1, G_2, G_3 \rangle$ is plain.

8. Into GLIB_010

Let us consider graphs G_1, G_2. Now we state the propositions:

(77) If $G_1 \approx G_2$, then there exists a partial graph mapping F from G_1 to G_2 such that $F = \mathrm{id}_{G_1}$ and F is directed-isomorphism.

(78) If $G_1 \approx G_2$, then G_2 is G_1-directed-isomorphic. The theorem is a consequence of (77).

(79) Let us consider a graph G_1, a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \mathrm{id}_{G_1}$, and

(ii) F is isomorphism.

(80) Let us consider a graph G_1, and a set E. Then every graph given by reversing directions of the edges E of G_1 is G_1-isomorphic. The theorem is a consequence of (79).

(81) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is directed-continuous and isomorphism. Then

(i) G_1 is non-directed-multi iff G_2 is non-directed-multi, and

(ii) G_1 is directed-simple iff G_2 is directed-simple.

Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a vertex v of G_1. Now we state the propositions:

(82) If $v \in \mathrm{dom}(F_{\mathbb{V}})$, then $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesInOut}()) \subseteq (F_{\mathbb{V}})_{/v}.\mathrm{edgesInOut}()$.

(83) Suppose F is directed and $v \in \mathrm{dom}(F_{\mathbb{V}})$. Then

(i) $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesIn}()) \subseteq (F_{\mathbb{V}})_{/v}.\mathrm{edgesIn}()$, and

(ii) $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesOut}()) \subseteq (F_{\mathbb{V}})_{/v}.\mathrm{edgesOut}()$.

(84) Suppose F is onto and semi-continuous and $v \in \mathrm{dom}(F_{\mathbb{V}})$. Then $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesInOut}()) = (F_{\mathbb{V}})_{/v}.\mathrm{edgesInOut}()$. The theorem is a consequence of (82).

(85) Suppose F is onto and semi-directed-continuous and $v \in \mathrm{dom}(F_{\mathbb{V}})$. Then

(i) $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesIn}()) = (F_{\mathbb{V}})_{/v}.\mathrm{edgesIn}()$, and

(ii) $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesOut}()) = (F_{\mathbb{V}})_{/v}.\mathrm{edgesOut}()$.

The theorem is a consequence of (83).

(86) If F is isomorphism, then $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesInOut}()) = (F_{\mathbb{V}})_{/v}.\mathrm{edgesInOut}()$. The theorem is a consequence of (84).

(87) Suppose F is directed-isomorphism. Then

(i) $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesIn}()) = (F_{\mathbb{V}})_{/v}.\mathrm{edgesIn}()$, and

(ii) $(F_{\mathbb{E}})^{\circ}(v.\mathrm{edgesOut}()) = (F_{\mathbb{V}})_{/v}.\mathrm{edgesOut}()$.

The theorem is a consequence of (85).

Let G_1 be a graph and G_2 be an edgeless graph. Note that every partial graph mapping from G_1 to G_2 is directed.

Let us consider graphs G_1, G_2 and a partial graph mapping F_0 from G_1 to G_2. Now we state the propositions:

(88) Suppose $F_{0\mathbb{E}}$ is one-to-one. Then there exists a subset E of the edges of G_2 such that for every graph G_3 given by reversing directions of the edges E of G_2.

There exists a partial graph mapping F from G_1 to G_3 such that $F = F_0$ and F is directed and if F_0 is not empty, then F is not empty and if F_0 is total, then F is total and if F_0 is one-to-one, then F is one-to-one and if F_0 is onto, then F is onto and if F_0 is semi-continuous, then F is semi-continuous and if F_0 is continuous, then F is continuous. The theorem is a consequence of (79).

(89) Suppose $F_{0\mathbb{E}}$ is one-to-one. Then there exists a subset E of the edges of G_2 such that for every graph G_3 given by reversing directions of the edges E of G_2.

There exists a partial graph mapping F from G_1 to G_3 such that $F = F_0$ and F is directed and if F_0 is weak subgraph embedding, then F is weak subgraph embedding and if F_0 is strong subgraph embedding, then F is strong subgraph embedding and if F_0 is isomorphism, then F is isomorphism. The theorem is a consequence of (88).

Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a vertex v of G_1. Now we state the propositions:

(90) Suppose F is directed and weak subgraph embedding. Then

(i) v.inDegree() $\subseteq (F_{\mathbb{V}})_{/v}$.inDegree(), and

(ii) v.outDegree() $\subseteq (F_{\mathbb{V}})_{/v}$.outDegree().

The theorem is a consequence of (83).

(91) If F is weak subgraph embedding, then v.degree() $\subseteq (F_{\mathbb{V}})_{/v}$.degree(). The theorem is a consequence of (89) and (56).

(92) Suppose F is onto and semi-directed-continuous and $v \in \mathrm{dom}(F_{\mathbb{V}})$. Then

(i) $(F_{\mathbb{V}})_{/v}$.inDegree() $\subseteq v$.inDegree(), and

(ii) $(F_{\mathbb{V}})_{/v}$.outDegree() $\subseteq v$.outDegree().

The theorem is a consequence of (85).

(93) If F is onto and semi-directed-continuous and $v \in \mathrm{dom}(F_{\mathbb{V}})$, then $(F_{\mathbb{V}})_{/v}$.degree() $\subseteq v$.degree(). The theorem is a consequence of (92).

(94) If F is directed-isomorphism, then v.inDegree() $= (F_{\mathbb{V}})_{/v}$.inDegree() and v.outDegree() $= (F_{\mathbb{V}})_{/v}$.outDegree(). The theorem is a consequence of (92) and (90).

(95) If F is isomorphism, then v.degree() $= (F_{\mathbb{V}})_{/v}$.degree(). The theorem is a consequence of (89), (94), and (56).

9. INTO CHORD

Now we state the proposition:

(96) Let us consider a graph G, and vertices u, v, w of G. Suppose v is endvertex and $u \neq w$. Then

(i) u and v are not adjacent, or

(ii) v and w are not adjacent.

PROOF: Consider e being an object such that v.edgesInOut() $= \{e\}$ and e does not join v and v in G. Consider v' being a vertex of G such that e joins v and v' in G. Consider e_8 being an object such that e_8 joins v and u in G. There exists no object e' such that e' joins v and w in G. □

Let us consider a graph G and a vertex v of G. Now we state the propositions:

(97) Suppose $3 \subseteq G$.order() and v is endvertex. Then there exist vertices u, w of G such that

(i) $u \neq v$, and

(ii) $w \neq v$, and

(iii) $u \neq w$, and

(iv) u and v are adjacent, and

(v) v and w are not adjacent.

The theorem is a consequence of (96).

(98) Suppose $4 \subseteq G$.order() and v is endvertex. Then there exist vertices x, y, z of G such that

(i) $v \neq x$, and

(ii) $v \neq y$, and

(iii) $v \neq z$, and

(iv) $x \neq y$, and

(v) $x \neq z$, and

(vi) $y \neq z$, and

(vii) v and x are adjacent, and

(viii) v and y are not adjacent, and

(ix) v and z are not adjacent.

The theorem is a consequence of (97), (17), and (96).

Let G_F be a graph-yielding function. We say that G_F is chordal if and only if

(Def. 4) for every object x such that $x \in \operatorname{dom} G_F$ there exists a graph G such that $G_F(x) = G$ and G is chordal.

Let G be a chordal graph. Let us note that $\langle G \rangle$ is chordal and $\mathbb{N} \longmapsto G$ is chordal.

Let G_F be a non empty, graph-yielding function. Note that G_F is chordal if and only if the condition (Def. 5) is satisfied.

(Def. 5) for every element x of $\operatorname{dom} G_F$, $G_F(x)$ is chordal.

Let G_{Sq} be a graph sequence. Let us note that G_{Sq} is chordal if and only if the condition (Def. 6) is satisfied.

(Def. 6) for every natural number n, $G_{Sq}(n)$ is chordal.

Let us observe that every graph-yielding function which is empty is also chordal and there exists a graph sequence which is chordal and there exists a graph-yielding finite sequence which is non empty and chordal.

Let G_F be a chordal, non empty, graph-yielding function and x be an element of $\operatorname{dom} G_F$. One can verify that $G_F(x)$ is chordal. Let G_{Sq} be a chordal graph sequence and x be a natural number. One can verify that $G_{Sq}(x)$ is chordal.

Let p be a chordal, graph-yielding finite sequence and n be a natural number. Note that $p \restriction n$ is chordal and $p_{\lfloor n}$ is chordal. Let m be a natural number. Let us observe that $\operatorname{smid}(p, m, n)$ is chordal and $\langle p(m), \ldots, p(n) \rangle$ is chordal.

Let p, q be chordal, graph-yielding finite sequences. Note that $p \frown q$ is chordal and $p \frown\!\frown q$ is chordal.

Let G_1, G_2 be chordal graphs. One can verify that $\langle G_1, G_2 \rangle$ is chordal. Let G_3 be a chordal graph. One can check that $\langle G_1, G_2, G_3 \rangle$ is chordal.

10. Into GLIB_011

Now we state the propositions:

(99) Let us consider non-directed-multi graphs G_1, G_2, a directed partial vertex mapping f from G_1 to G_2, and a vertex v of G_1. Suppose f is directed-isomorphism. Then

(i) v.inDegree() $= f_{/v}$.inDegree(), and

(ii) v.outDegree() $= f_{/v}$.outDegree().

The theorem is a consequence of (94).

(100) Let us consider non-multi graphs G_1, G_2, a partial vertex mapping f from G_1 to G_2, and a vertex v of G_1. If f is isomorphism, then v.degree() $= f_{/v}$.degree(). The theorem is a consequence of (95).

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