FORMALIZED MATHEMATICS
Vol. 28, No. 1, Pages 121–128, 2020
DOI: 10.2478/forma-2020-0011

sciendo
https://www.sciendo.com/

On Fuzzy Negations Generated by Fuzzy Implications

Adam Grabowski
Institute of Informatics
University of Białystok
Poland

Summary. We continue in the Mizar system [2] the formalization of fuzzy implications according to the book of Baczyński and Jayaram "Fuzzy Implications" [1]. In this article we define fuzzy negations and show their connections with previously defined fuzzy implications [4] and [5] and triangular norms and conorms [6]. This can be seen as a step towards building a formal framework of fuzzy connectives [10]. We introduce formally Sugeno negation, boundary negations and show how these operators are pointwise ordered. This work is a continuation of the development of fuzzy sets [12], [3] in Mizar [7] started in [11] and partially described in [8]. This submission can be treated also as a part of a formal comparison of fuzzy and rough approaches to incomplete or uncertain information within the Mizar Mathematical Library [9].

MSC: 03B52 68V20 03B35

Keywords: fuzzy set; fuzzy negation; fuzzy implication

MML identifier: FUZIMPL3, version: 8.1.09 5.60.1374

0. Introduction

The main aim of this Mizar article was to implement a formal counterpart of (the part of) Chapter 1.4, pp. 13–20 of Baczyński and Jayaram book "Fuzzy Implications" [1]. This is the fourth submission in the series formalizing this textbook, following [4], [5], and [6].

After filling some gaps – proving lemmas about monotone functions absent in the Mizar Mathematical Library, in Section 2 we recall the notion of conjugate

ISSN 1426–2630(Print), 1898-9934(Online)

fuzzy implications, and formally implement a method of generating a new fuzzy implication from a given one. We prove that I_f inherits corresponding properties of f, such as (NP) – the left neutrality property, (EP) – the exchange principle, (IP) – the identity principle, and (OP) – the ordering property, providing also registrations of clusters which guarantee the automatic handling of adjectives (their adjunction to the respective radix type), thus making a formalization work a bit easier.

Section 3, which is a fundamental part of this paper, contains elementary definitions needed to cope with fuzzy negations, and Sect. 4 provides a method of generating fuzzy negation from a given fuzzy implication. There are also concrete examples given in Section 5: the classical (standard) fuzzy complement N_{C} introduced at the beginning, two boundary (in the sense of the natural ordering of the functions) negations N_{D1} and N_{D2} (Def. 17 and 18, respectively). Section 6 shows which negations are generated from nine well-known fuzzy implications, so it can be treated as the formal counterpart of Table 1.7, p. 18 [1].

Fuzzy implication I	Fuzzy negation N_I
I_{LK}	$\mathrm{N_C}$
I_{GD}	$\mathrm{N_{D1}}$
I_{RC}	$\mathrm{N_C}$
I_{KD}	$\mathrm{N_C}$
I_{GG}	$\mathrm{N_{D1}}$
I_{RS}	$\mathrm{N_{D1}}$
I_{YG}	$\mathrm{N_{D1}}$
I_{WB}	$\mathrm{N_{D2}}$
I_{FD}	$\mathrm{N_C}$

Section 7 is devoted to Sugeno negation (Def. 21), which can be used as a useful method of constructing examples of fuzzy negations (for example, substituting $\lambda = 0$ in the Sugeno negation, we obtain the standard fuzzy complementation). We conclude with some properties of conjugate fuzzy negations.

1. Preliminaries

Now we state the proposition:

(1) Let us consider real numbers x, r. If $0 \leqslant x \leqslant 1$ and $r > -1$, then $x \cdot r + 1 > 0$.

Let us consider a real number z. Now we state the propositions:

(2) If $z \in [0,1]$ and $z \neq 0$, then there exists an element w of $[0,1]$ such that $w < z$.

(3) If $z \in [0,1]$ and $z \neq 1$, then there exists an element w of $[0,1]$ such that $w > z$.

Note that there exists a unary operation on $[0,1]$ which is bijective and increasing and every unary operation on $[0,1]$ which is bijective and non-decreasing is also increasing and every unary operation on $[0,1]$ which is bijective and increasing is also non-decreasing. Let f be a bijective, increasing unary operation on $[0,1]$. One can check that f^{-1} is real-valued and function-like and $(f \restriction [0,1])^{-1}$ is real-valued. Now we state the propositions:

(4) Let us consider a one-to-one unary operation f on $[0,1]$, and an element d of $[0,1]$. If $d \in \operatorname{rng} f$, then $(f^{-1})(d) \in \operatorname{dom} f$.

(5) Let us consider a bijective, increasing unary operation f on $[0,1]$. Then f^{-1} is increasing.

Let f be a bijective, increasing unary operation on $[0,1]$. Let us note that f^{-1} is increasing. Let us consider a unary operation f on $[0,1]$. Now we state the propositions:

(6) f is non-decreasing if and only if for every elements a, b of $[0,1]$ such that $a \leqslant b$ holds $f(a) \leqslant f(b)$.

(7) f is non-increasing if and only if for every elements a, b of $[0,1]$ such that $a \leqslant b$ holds $f(a) \geqslant f(b)$.

(8) f is decreasing if and only if for every elements a, b of $[0,1]$ such that $a < b$ holds $f(a) > f(b)$.

(9) f is increasing if and only if for every elements a, b of $[0,1]$ such that $a < b$ holds $f(a) < f(b)$.

(10) Let us consider an increasing, bijective unary operation f on $[0,1]$. Then

(i) $f(0) = 0$, and

(ii) $f(1) = 1$.

Let f be a bijective, increasing unary operation on $[0,1]$. Observe that f^{-1} is bijective and increasing as a unary operation on $[0,1]$.

2. Conjugate Fuzzy Implications

The functor Φ yielding a set is defined by the term

(Def. 1) the set of all f where f is a bijective, increasing unary operation on $[0,1]$.

Let f be a binary operation on $[0,1]$ and φ be a bijective, increasing unary operation on $[0,1]$. The functor f_{φ} yielding a binary operation on $[0,1]$ is defined by

(Def. 2) for every elements x_1, x_2 of $[0,1]$, $it(x_1, x_2) = (\varphi^{-1})(f(\varphi(x_1), \varphi(x_2)))$.

Let f, g be binary operations on $[0,1]$. We say that f, g are conjugate if and only if

(Def. 3) there exists a bijective, increasing unary operation φ on $[0,1]$ such that $g = f_\varphi$.

Let I be a fuzzy implication and f be a bijective, non-decreasing unary operation on $[0,1]$. Let us note that I_f is antitone w.r.t. 1st coordinate, isotone w.r.t. 2nd coordinate, 00-dominant, 11-dominant, and 10-weak.

(11) Let us consider a fuzzy implication I, and a bijective, increasing unary operation f on $[0,1]$. Then I_f is a fuzzy implication.

Let us note that there exists a fuzzy implication which satisfies (NP), (OP), (EP), and (IP). Let us consider a fuzzy implication I and a bijective, increasing unary operation f on $[0,1]$. Now we state the propositions:

(12) If I satisfies (NP), then I_f satisfies (NP). The theorem is a consequence of (10).

(13) If I satisfies (EP), then I_f satisfies (EP).

(14) If I satisfies (IP), then I_f satisfies (IP). The theorem is a consequence of (10).

(15) If I satisfies (OP), then I_f satisfies (OP).
Proof: Set $g = I_f$. If $g(x,y) = 1$, then $x \leqslant y$. $f(x) \leqslant f(y)$. $(f^{-1})(I(f(x), f(y))) = 1$. □

Let I be fuzzy implication satisfying (NP) and f be a bijective, increasing unary operation on $[0,1]$. Let us observe that I_f satisfies (NP). Let I be fuzzy implication satisfying (EP). Observe that I_f satisfies (EP). Let I be fuzzy implication satisfying (IP). Let us note that I_f satisfies (IP). Let I be fuzzy implication satisfying (OP). Note that I_f satisfies (OP). Now we state the proposition:

(16) Let us consider a fuzzy implication I, and a bijective, increasing unary operation f on $[0,1]$. Then $I_f = f^{-1} \cdot I \cdot (f \times f)$.
Proof: Set $g = I_f$. For every object x such that $x \in \operatorname{dom} g$ holds $g(x) = (f^{-1} \cdot I \cdot (f \times f))(x)$. □

3. Fuzzy Negations

Let N be a unary operation on $[0,1]$. We say that N is satisfying (N1) if and only if

(Def. 4) $N(0) = 1$ and $N(1) = 0$.

We say that N is satisfying (N2) if and only if

(Def. 5) N is non-increasing.

The functor N_C yielding a unary operation on $[0,1]$ is defined by

(Def. 6) for every element x of $[0,1]$, $it(x) = 1 - x$.

Note that N_C is satisfying (N1) and satisfying (N2) and N_C is bijective and decreasing and there exists a unary operation on $[0,1]$ which is bijective and decreasing and there exists a unary operation on $[0,1]$ which is satisfying (N1) and satisfying (N2).

A fuzzy negation is a satisfying (N1), satisfying (N2) unary operation on $[0,1]$. Let f be a unary operation on $[0,1]$. We say that f is continuous if and only if

(Def. 7) there exists a function g from $\mathbb{I}$ into $\mathbb{I}$ such that $f = g$ and g is continuous.

Let N be a unary operation on $[0,1]$. We say that N is involutive if and only if

(Def. 8) for every element x of $[0,1]$, $N(N(x)) = x$.

We say that N is satisfying (N3) if and only if

(Def. 9) N is decreasing.

We say that N is satisfying (N4) if and only if

(Def. 10) N is continuous.

We say that N is satisfying (N5) if and only if

(Def. 11) N is involutive.

We say that N is strict if and only if

(Def. 12) N is satisfying (N3) and satisfying (N4).

We say that N is strong if and only if

(Def. 13) N is satisfying (N5).

We say that N is non-vanishing if and only if

(Def. 14) for every element x of $[0,1]$, $N(x) = 0$ iff $x = 1$.

We say that N is non-filling if and only if

(Def. 15) for every element x of $[0,1]$, $N(x) = 1$ iff $x = 0$.

4. Generating Fuzzy Negations from Fuzzy Implications

Now we state the proposition:

(17) Let us consider a decreasing, bijective unary operation f on $[0,1]$. Then

(i) $f(0) = 1$, and

(ii) $f(1) = 0$.

Let I be a binary operation on $[0,1]$. The functor N_I yielding a unary operation on $[0,1]$ is defined by

(Def. 16) for every element x of $[0,1]$, $it(x) = I(x, 0)$.

Let I be binary operation on $[0,1]$ satisfying (I1), (I3), and (I5). Note that N_I is satisfying (N1) and satisfying (N2).

Now we state the proposition:

(18) Let us consider a fuzzy implication I. Then N_I is a fuzzy negation.

5. Boundary Fuzzy Negations

The functors: $N_{\mathrm{D}1}$ and $N_{\mathrm{D}2}$ yielding unary operations on $[0,1]$ are defined by conditions

(Def. 17) for every element x of $[0,1]$, if $x = 0$, then $N_{\mathrm{D}1}(x) = 1$ and if $x \neq 0$, then $N_{\mathrm{D}1}(x) = 0$,

(Def. 18) for every element x of $[0,1]$, if $x = 1$, then $N_{\mathrm{D}2}(x) = 0$ and if $x \neq 1$, then $N_{\mathrm{D}2}(x) = 1$,

respectively. Let f_1, f_2 be unary operations on $[0,1]$. We say that $f_1 \leqslant f_2$ if and only if

(Def. 19) for every element a of $[0,1]$, $f_1(a) \leqslant f_2(a)$.

Let us note that $N_{\mathrm{D}1}$ is satisfying (N1) and satisfying (N2) and $N_{\mathrm{D}2}$ is satisfying (N1) and satisfying (N2).

Now we state the proposition:

(19) Let us consider a fuzzy negation N. Then $N_{\mathrm{D}1} \leqslant N \leqslant N_{\mathrm{D}2}$.

6. Fuzzy Negations Generated by Nine Fuzzy Implications

Now we state the propositions:

(20) $N_{I_{\mathrm{LK}}} = N_C$.

Proof: Set $I = I_{\mathrm{LK}}$. Set $f = N_I$. Set $g = N_C$. For every element x of $[0,1]$, $f(x) = g(x)$. □

(21) $N_{I_{\mathrm{GD}}} = N_{\mathrm{D}1}$.

(22) $N_{I_{\mathrm{RC}}} = N_C$.

(23) $N_{I_{\mathrm{KD}}} = N_C$.

Proof: Set $I = I_{\mathrm{KD}}$. Set $f = N_I$. Set $g = N_C$. For every element x of $[0,1]$, $f(x) = g(x)$. □

(24) $N_{I_{\mathrm{GG}}} = N_{\mathrm{D}1}$.

(25) $N_{I_{\mathrm{RS}}} = N_{\mathrm{D}1}$.

(26) $N_{I_{\mathrm{YG}}} = N_{\mathrm{D}1}$.

(27) $N_{I_{\mathrm{WB}}} = N_{\mathrm{D}2}$.

(28) $N_{I_{\rm FD}} = N_C$.

Proof: Set $I = I_{\rm FD}$. Set $f = N_I$. Set $g = N_C$. For every element x of $[0,1]$, $f(x) = g(x)$. □

(29) Let us consider binary operation I on $[0,1]$ satisfying (EP) and (OP). Then N_I is a fuzzy negation.

(30) Let us consider binary operation I on $[0,1]$ satisfying (EP) and (OP), and an element x of $[0,1]$. Then $x \leqslant (N_I)((N_I)(x))$.

(31) Let us consider binary operation I on $[0,1]$ satisfying (EP) and (OP). Then $(N_I) \cdot (N_I) \cdot (N_I) = N_I$. The theorem is a consequence of (7) and (30).

7. Sugeno Negation

Let x, λ be real numbers. We say that λ is greater than x if and only if

(Def. 20) $\lambda > x$.

One can verify that there exists a real number which is greater than (-1).

Let λ be a real number. Assume $\lambda > -1$. The functor SugenoNegation λ yielding a unary operation on $[0,1]$ is defined by

(Def. 21) for every element x of $[0,1]$, $it(x) = \frac{1-x}{1+\lambda \cdot x}$.

Now we state the proposition:

(32) $N_C =$ SugenoNegation 0.

Let λ be a greater than (-1) real number. Note that SugenoNegation λ is satisfying (N1) and satisfying (N2).

8. Conjugate Fuzzy Negations

Let f be a unary operation on $[0,1]$ and φ be a bijective, increasing unary operation on $[0,1]$. The functor f_φ yielding a unary operation on $[0,1]$ is defined by

(Def. 22) for every element x of $[0,1]$, $it(x) = (\varphi^{-1})(f(\varphi(x)))$.

Now we state the proposition:

(33) Let us consider a fuzzy negation I, and a bijective, increasing unary operation f on $[0,1]$. Then $I_f = f^{-1} \cdot I \cdot f$.

Proof: Set $g = I_f$. For every object x such that $x \in \operatorname{dom} g$ holds $g(x) = (f^{-1} \cdot I \cdot f)(x)$. □

Let f, g be unary operations on $[0,1]$. We say that f, g are conjugate if and only if

(Def. 23) there exists a bijective, increasing unary operation φ on $[0,1]$ such that $g = f_{\varphi}$.

Let N be a fuzzy negation and φ be a bijective, increasing unary operation on $[0,1]$. One can check that N_{φ} is satisfying (N1) and satisfying (N2).

Now we state the proposition:

(34) Let us consider a fuzzy implication I, and a bijective, increasing unary operation φ on $[0,1]$. Then $(N_I)_{\varphi} = N_{I_{\varphi}}$. The theorem is a consequence of (10).

References

[1] Michał Baczyński and Balasubramaniam Jayaram. *Fuzzy Implications*. Springer Publishing Company, Incorporated, 2008. doi:10.1007/978-3-540-69082-5.

[2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[3] Didier Dubois and Henri Prade. *Fuzzy Sets and Systems: Theory and Applications*. Academic Press, New York, 1980.

[4] Adam Grabowski. Formal introduction to fuzzy implications. *Formalized Mathematics*, 25(**3**):241–248, 2017. doi:10.1515/forma-2017-0023.

[5] Adam Grabowski. Fundamental properties of fuzzy implications. *Formalized Mathematics*, 26(**4**):271–276, 2018. doi:10.2478/forma-2018-0023.

[6] Adam Grabowski. Basic formal properties of triangular norms and conorms. *Formalized Mathematics*, 25(**2**):93–100, 2017. doi:10.1515/forma-2017-0009.

[7] Adam Grabowski. On the computer certification of fuzzy numbers. In M. Ganzha, L. Maciaszek, and M. Paprzycki, editors, *2013 Federated Conference on Computer Science and Information Systems (FedCSIS)*, Federated Conference on Computer Science and Information Systems, pages 51–54, 2013.

[8] Adam Grabowski and Takashi Mitsuishi. *Extending Formal Fuzzy Sets with Triangular Norms and Conorms*, volume 642: *Advances in Intelligent Systems and Computing*, pages 176–187. Springer International Publishing, Cham, 2018. doi:10.1007/978-3-319-66824-6_16.

[9] Adam Grabowski and Takashi Mitsuishi. Initial comparison of formal approaches to fuzzy and rough sets. In Leszek Rutkowski, Marcin Korytkowski, Rafal Scherer, Ryszard Tadeusiewicz, Lotfi A. Zadeh, and Jacek M. Zurada, editors, *Artificial Intelligence and Soft Computing – 14th International Conference, ICAISC 2015, Zakopane, Poland, June 14-18, 2015, Proceedings, Part I*, volume 9119 of *Lecture Notes in Computer Science*, pages 160–171. Springer, 2015. doi:10.1007/978-3-319-19324-3_15.

[10] Petr Hájek. *Metamathematics of Fuzzy Logic*. Dordrecht: Kluwer, 1998.

[11] Takashi Mitsuishi, Noboru Endou, and Yasunari Shidama. The concept of fuzzy set and membership function and basic properties of fuzzy set operation. *Formalized Mathematics*, 9(**2**):351–356, 2001.

[12] Lotfi Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965. doi:10.1016/S0019-9958(65)90241-X.

Accepted February 26, 2020