FORMALIZED MATHEMATICS
Vol. 27, No. 3, Pages 261–301, 2019
DOI: 10.2478/forma-2019-0024

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About Graph Mappings

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Summary. In this articles adjacency-preserving mappings from a graph to another are formalized in the Mizar system [7], [2]. The generality of the approach seems to be largely unpreceeded in the literature to the best of the author's knowledge. However, the most important property defined in the article is that of two graphs being isomorphic, which has been extensively studied. Another graph decorator is introduced as well.

MSC: 05C60 68T99 03B35

Keywords: graph homomorphism; graph isomorphism

MML identifier: `GLIB_010`, version: `8.1.09 5.59.1363`

0. Introduction

Writing this article has been rather challenging. "Much of graph theory is concerned with the study of simple graphs" [3, p. 3], so most graph theory books are only concerned with graph homomorphisms between simple graphs, if they are concerned with anything more general than isomorphisms at all. [3] writes about general graphs; isomorphisms are done in the first chapter while homomorphisms are only looked at in the context of vertex colorings in chapter 14. The book "Graphs and homomorphisms" [8] only handles (di)graphs without multiple parallel edges. The book "Graph coloring problems" [10] notes homomorphisms between loopless graphs, but doesn't elaborate. [6] only handles homomorphisms between simple graphs. [14] shortly describes homomorphisms between undirected graphs. [9] handles homomorphisms between

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ISSN 1426–2630(Print), 1898-9934(Online)

digraphs without parallel edges. [16] writes about general graphs but, like most graph books, only about isomorphisms. The best source so far has been [11], where graph homomorphisms are introduced for digraphs possibly containing loops and multiple parallel edges (just like graphs are formalized in [15]) but the focus is almost immediately shifted to homomorphisms between simple graphs. So a quick overview of the formalized notation seems in order.

A *graph* G consists of a non empty set $V(G)$ called vertices of G, a set $E(G)$ called edges of G and two functions $s(G), t(G) : E(G) \to V(G)$, the source and target of G. For $e \in E(G), v, w \in V(G)$ we write *e joins v to w* if $s(G)(e) = v$ and $t(G)(e) = w$, and we write *e joins v and w* if e joins v to w or e joins w to v. Let G_1, G_2 be graphs. A *partial graph mapping* from G_1 to G_2 is an ordered pair $F = \langle F_{\mathbb{V}}, F_{\mathbb{E}} \rangle$ with the following properties:

- $F_{\mathbb{V}}$ is a partial function from $V(G_1)$ to $V(G_2)$.
- $F_{\mathbb{E}}$ is a partial function from $E(G_1)$ to $E(G_2)$.
- For any $e \in \operatorname{dom} F_{\mathbb{E}}$ holds $s(G)(e), t(G)(e) \in \operatorname{dom} F_{\mathbb{V}}$.
- For any $e \in \operatorname{dom} F_{\mathbb{E}}$ and $v, w \in \operatorname{dom} F_{\mathbb{V}}$ such that e joins v and w holds $F_{\mathbb{E}}(e)$ joins $F_{\mathbb{V}}(v)$ and $F_{\mathbb{V}}(w)$.

Note that $\langle f, \emptyset \rangle$ is a valid partial graph mapping for any partial function $f : V(G_1) \to V(G_2)$, especially for $f = \emptyset$. Now define the following attributes:

- F is *empty* if $\operatorname{dom} F_{\mathbb{V}} = \emptyset$.
- F is *total* (or a homomorphism) if $\operatorname{dom} F_{\mathbb{V}} = V(G_1)$ and $\operatorname{dom} F_{\mathbb{E}} = E(G_1)$.
- F is *onto* (or surjective) if $\operatorname{rng} F_{\mathbb{V}} = V(G_2)$ and $\operatorname{rng} F_{\mathbb{E}} = E(G_2)$.
- F is *one-to-one* (or injective) if $F_{\mathbb{V}}$ and $F_{\mathbb{E}}$ are.
- F is *semi-continuous* if for any $e \in \operatorname{dom} F_{\mathbb{E}}$ and $v, w \in \operatorname{dom} F_{\mathbb{V}}$ such that $F_{\mathbb{E}}(e)$ joins $F_{\mathbb{V}}(v)$ and $F_{\mathbb{V}}(w)$ holds e joins v and w.
- F is *continuous* if for any $\tilde{e} \in E(G_2)$ and $v, w \in \operatorname{dom} F_{\mathbb{V}}$ such that $\tilde{e}$ joins $F_{\mathbb{V}}(v)$ and $F_{\mathbb{V}}(w)$ exists an $e \in \operatorname{dom} F_{\mathbb{E}}$ such that $F_{\mathbb{E}}(e) = \tilde{e}$ and e joins v and w.
- F is a *weak subgraph-embedding* if it is total and one-to-one.
- F is a *strong subgraph-embedding* if it is total, one-to-one and continuous.
- F is an *isomorphism* if it is total, one-to-one and onto.

Because modes in Mizar must always be inhabitated, partial graph mappings are the chosen foundation rather than homomorphisms, which may not exist between two graphs. The attributes *total*, *onto* and *one-to-one* were named like their function analogons from [4] and [5]. The *continuous* attribute was inspired by the continuous vertex mappings of [11] and is in fact sometimes different from *semi-continuous*. *Semi-continuous* seemed like the natural generalization of *continuous* for graph mappings instead of vertex mappings, but that turned out to be false. Still, a semi-continuous graph mapping already carries a lot of properties from G_1 to G_2, so the definition was kept. Corresponding attributes for directed graph mappings are given in this article as well.

If F is a weak subgraph-embedding, then G_1 is isomorphic to a subgraph of G_2. If F is a strong subgraph-embedding, then G_1 is isomorphic to an induced subgraph of G_2. The short term *embedding* was desperately avoided to be available for embeddings of graphs into the plane and other surfaces. If F is one-to-one, it is also semi-continuous. If F is semi-continuous and onto, it is also continuous.

Originally, only an article about graph isomorphisms was planned, but it was changed to provide a solid foundation of general graph mappings. Now this article also includes the restriction of F to subgraphs of G_1 or G_2, the domain and range of F defined as the plain subgraphs of G_1 and G_2 induced by $\operatorname{dom} F_{\mathbb{V}}$, $\operatorname{dom} F_{\mathbb{E}}$ and $\operatorname{rng} F_{\mathbb{V}}$, $\operatorname{rng} F_{\mathbb{E}}$ respectively, and the images of walks under F. Of course the inverse of F and the composition of two graph mappings are included as well.

Additionally, the ordering of a graph, which is just an enumeration of its vertices, has been introduced as yet another graph decorator. This decorator is planned as a tool to identify graphs with trees from [1]. Attributes describing if F preserves the weights, edge labels, vertex labels or the ordering have been added as well.

1. Preliminaries

Now we state the propositions:

(1) Let us consider functions A, B, C, D. Suppose $D \cdot A = C \restriction \operatorname{dom} A$. Then $(D \restriction \operatorname{dom} B) \cdot A = C \restriction \operatorname{dom}(B \cdot A)$.

Proof: Set $f = (D \restriction \operatorname{dom} B) \cdot A$. Set $g = C \restriction \operatorname{dom}(B \cdot A)$. For every object x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$. $\square$

(2) Let us consider a one-to-one function A, and functions C, D. Suppose $D \cdot A = C \restriction \operatorname{dom} A$. Then $C \cdot (A^{-1}) = D \restriction \operatorname{dom}(A^{-1})$.

Proof: For every object y, $y \in \operatorname{dom}(C \cdot (A^{-1}))$ iff $y \in \operatorname{dom}(D \restriction \operatorname{dom}(A^{-1}))$. For every object y such that $y \in \operatorname{dom}(C \cdot (A^{-1}))$ holds $(C \cdot (A^{-1}))(y) = (D \restriction \operatorname{dom}(A^{-1}))(y)$. $\square$

Let G be a non finite graph and X be a set. One can verify that

G.set(WeightSelector, X) is non finite and G.set(ELabelSelector, X) is non finite and G.set(VLabelSelector, X) is non finite.

Let G be a non loopless graph. One can check that G.set(WeightSelector, X) is non loopless and G.set(ELabelSelector, X) is non loopless and

G.set(VLabelSelector, X) is non loopless.

Let G be a non non-multi graph. Note that G.set(WeightSelector, X) is non non-multi and G.set(ELabelSelector, X) is non non-multi and

G.set(VLabelSelector, X) is non non-multi. Let G be a non non-directed-multi graph. Let us note that G.set(WeightSelector, X) is non non-directed-multi and G.set(ELabelSelector, X) is non non-directed-multi and

G.set(VLabelSelector, X) is non non-directed-multi.

Let G be a non connected graph. Observe that G.set(WeightSelector, X) is non connected and G.set(ELabelSelector, X) is non connected and

G.set(VLabelSelector, X) is non connected.

Let G be a non acyclic graph. Let us observe that G.set(WeightSelector, X) is non acyclic and G.set(ELabelSelector, X) is non acyclic and

G.set(VLabelSelector, X) is non acyclic. Let G be a graph. We say that G is elabel-full if and only if

(Def. 1) ELabelSelector $\in \operatorname{dom} G$ and there exists a many sorted set f indexed by the edges of G such that $G(\text{ELabelSelector}) = f$.

We say that G is vlabel-full if and only if

(Def. 2) VLabelSelector $\in \operatorname{dom} G$ and there exists a many sorted set f indexed by the vertices of G such that $G(\text{VLabelSelector}) = f$.

Let us observe that every graph which is elabel-full is also elabeled and every graph which is vlabel-full is also vlabeled.

Let G be an e-graph. We say that G is elabel-distinct if and only if

(Def. 3) the elabel of G is one-to-one.

Let G be a v-graph. We say that G is vlabel-distinct if and only if

(Def. 4) the vlabel of G is one-to-one.

Let G be a graph. Observe that G.set(ELabelSelector, $\mathrm{id}_{\text{the edges of } G}$) is elabel-full and elabel-distinct and G.set(VLabelSelector, $\mathrm{id}_{\text{the vertices of } G}$) is vlabel-full and vlabel-distinct and there exists an e-graph which is elabel-distinct and elabel-full and there exists a v-graph which is vlabel-distinct and vlabel-full.

Let G be an elabel-full graph. Let us observe that the elabel of G yields a many sorted set indexed by the edges of G. Let G be a vlabel-full graph. Observe that the vlabel of G yields a many sorted set indexed by the vertices of G. Let G be an elabel-distinct e-graph. Let us note that the elabel of G is one-to-one.

Let G be a vlabel-distinct v-graph. Observe that the vlabel of G is one-to-one. Let G be an elabel-full graph and X be a set. One can verify that

G.set(WeightSelector, X) is elabel-full and G.set(VLabelSelector, X) is elabel-full. Let G be a vlabel-full graph. One can check that G.set(WeightSelector, X) is vlabel-full and G.set(ELabelSelector, X) is vlabel-full.

Let G be an elabel-distinct e-graph. Note that G.set(WeightSelector, X) is elabel-distinct and G.set(VLabelSelector, X) is elabel-distinct.

Let G be a vlabel-distinct v-graph. Let us observe that G.set(WeightSelector, X) is vlabel-distinct and G.set(ELabelSelector, X) is vlabel-distinct and there exists an ev-graph which is elabel-full, elabel-distinct, vlabel-full, and vlabel-distinct.

Let G_1 be a w-graph, E be a set, and G_2 be a graph given by reversing directions of the edges E of G_1. Observe that G_2.set(WeightSelector, the weight of G_1) is weighted.

Let G_1 be an e-graph. One can verify that G_2.set(ELabelSelector, the elabel of G_1) is elabeled.

Let G_1 be a v-graph, V be a set, and G_2 be a graph given by reversing directions of the edges V of G_1. Observe that G_2.set(VLabelSelector, the vlabel of G_1) is vlabeled.

Let G_1 be an elabel-full graph, E be a set, and G_2 be a graph given by reversing directions of the edges E of G_1. Note that G_2.set(ELabelSelector, the elabel of G_1) is elabel-full.

Let G_1 be a vlabel-full graph, V be a set, and G_2 be a graph given by reversing directions of the edges V of G_1. Note that G_2.set(VLabelSelector, the vlabel of G_1) is vlabel-full. Let G_1 be an elabel-distinct e-graph, E be a set, and G_2 be a graph given by reversing directions of the edges E of G_1. Note that G_2.set(ELabelSelector, the elabel of G_1) is elabel-distinct. Let G_1 be a vlabel-distinct v-graph. Observe that G_2.set(VLabelSelector, the vlabel of G_1) is vlabel-distinct.

2. Ordering of a Graph

The functor OrderingSelector yielding an element of $\mathbb{N}$ is defined by the term

(Def. 5) 8.

Let G be a graph structure. We say that G is ordered if and only if

(Def. 6) OrderingSelector $\in \operatorname{dom} G$ and G(OrderingSelector) is an enumeration of the vertices of G.

Let G be a graph and X be a set. Note that G.set(OrderingSelector, X) is graph-like and G.set(OrderingSelector, X) is non plain.

Let G be a w-graph. One can verify that G.set(OrderingSelector, X) is weighted.

Let G be an e-graph. One can check that G.set(OrderingSelector, X) is elabeled.

Let G be a v-graph. Note that G.set(OrderingSelector, X) is vlabeled.

Let G be a graph and X be an enumeration of the vertices of G. Note that G.set(OrderingSelector, X) is ordered and there exists a graph structure which is graph-like, weighted, elabeled, vlabeled, and ordered.

Let G be an ordered graph. The ordering of G yielding an enumeration of the vertices of G is defined by the term

(Def. 7) G(OrderingSelector).

Now we state the proposition:

(3) Let us consider a graph G, and a set X.
Then $G \approx G$.set(OrderingSelector, X).

Let G be an elabel-full graph and X be a set. Let us note that G.set(OrderingSelector, X) is elabel-full.

Let G be a vlabel-full graph. Let us note that G.set(OrderingSelector, X) is vlabel-full.

Let G be an elabel-distinct e-graph. Let us note that G.set(OrderingSelector, X) is elabel-distinct.

Let G be a vlabel-distinct v-graph. Observe that G.set(OrderingSelector, X) is vlabel-distinct.

Let G be a finite graph. Let us observe that G.set(OrderingSelector, X) is finite.

Let G be a non finite graph. Let us observe that G.set(OrderingSelector, X) is non finite.

Let G be a loopless graph. Let us observe that G.set(OrderingSelector, X) is loopless.

Let G be a non loopless graph. Let us observe that G.set(OrderingSelector, X) is non loopless.

Let G be a trivial graph. Let us observe that G.set(OrderingSelector, X) is trivial.

Let G be a non trivial graph. Let us observe that G.set(OrderingSelector, X) is non trivial.

Let G be a non-multi graph. Let us observe that G.set(OrderingSelector, X) is non-multi.

Let G be a non non-multi graph. Let us observe that G.set(OrderingSelector, X) is non non-multi.

Let G be a non-directed-multi graph. Let us observe that G.set(OrderingSelector, X) is non-directed-multi.

Let G be a non non-directed-multi graph. Let us observe that G.set(OrderingSelector, X) is non non-directed-multi.

Let G be a connected graph. Let us observe that G.set(OrderingSelector, X) is connected.

Let G be a non connected graph. Let us note that G.set(OrderingSelector, X) is non connected.

Let G be an acyclic graph. Let us note that G.set(OrderingSelector, X) is acyclic.

Let G be a non acyclic graph. One can check that G.set(OrderingSelector, X) is non acyclic.

Let G be an edgeless graph. One can check that G.set(OrderingSelector, X) is edgeless.

Let G be a non edgeless graph. Let us observe that G.set(OrderingSelector, X) is non edgeless.

Let G be an ordered graph. Let us observe that G.set(WeightSelector, X) is ordered and G.set(ELabelSelector, X) is ordered and G.set(VLabelSelector, X) is ordered.

Let G_1 be an ordered graph and G_2 be a spanning subgraph of G_1. Note that G_2.set(OrderingSelector, the ordering of G_1) is ordered.

Let E be a set and G_2 be a graph given by reversing directions of the edges E of G_1. Let us observe that G_2.set(OrderingSelector, the ordering of G_1) is ordered.

3. Graph Mappings

Let G_1, G_2 be graphs. A partial graph mapping from G_1 to G_2 is an object defined by

(Def. 8) there exist functions f, g such that $it = \langle f, g \rangle$ and $\operatorname{dom} f \subseteq$ the vertices of G_1 and $\operatorname{rng} f \subseteq$ the vertices of G_2 and $\operatorname{dom} g \subseteq$ the edges of G_1 and $\operatorname{rng} g \subseteq$ the edges of G_2 and for every object e such that $e \in \operatorname{dom} g$ holds (the source of G_1)(e), (the target of G_1)(e) $\in \operatorname{dom} f$ and for every objects e, v, w such that $e \in \operatorname{dom} g$ and v, $w \in \operatorname{dom} f$ holds if e joins v and w in G_1, then $g(e)$ joins $f(v)$ and $f(w)$ in G_2.

Let us observe that every partial graph mapping from G_1 to G_2 is pair.

Let F be a partial graph mapping from G_1 to G_2. We introduce the notation $F_{\mathbb{V}}$ as a synonym of $(F)_{\mathbf{1}}$ and $F_{\mathbb{E}}$ as a synonym of $(F)_{\mathbf{2}}$.

One can check that $\langle F_{\mathbb{V}}, F_{\mathbb{E}} \rangle$ reduces to F.

One can verify that $F_{\mathbb{V}}$ is function-like and relation-like as a set and $F_{\mathbb{E}}$ is function-like and relation-like as a set and $F_{\mathbb{V}}$ is (the vertices of G_1)-defined and (the vertices of G_2)-valued as a function and $F_{\mathbb{E}}$ is (the edges of G_1)-defined and (the edges of G_2)-valued as a function.

Note that the functor $F_{\mathbb{V}}$ yields a partial function from the vertices of G_1 to the vertices of G_2. Observe that the functor $F_{\mathbb{E}}$ yields a partial function from the edges of G_1 to the edges of G_2. Now we state the proposition:

(4) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and objects e, v, w. Suppose $e \in \mathrm{dom}(F_{\mathbb{E}})$ and v, $w \in \mathrm{dom}(F_{\mathbb{V}})$. If e joins v and w in G_1, then $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ and $(F_{\mathbb{V}})(w)$ in G_2.

Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and an object e. Now we state the propositions:

(5) Suppose $e \in \mathrm{dom}(F_{\mathbb{E}})$. Then (the source of G_1)(e), (the target of G_1)(e) $\in \mathrm{dom}(F_{\mathbb{V}})$.

(6) Suppose $e \in \mathrm{rng}\, F_{\mathbb{E}}$. Then (the source of G_2)(e), (the target of G_2)(e) $\in \mathrm{rng}\, F_{\mathbb{V}}$. The theorem is a consequence of (5) and (4).

(7) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Then

(i) $\mathrm{dom}(F_{\mathbb{E}}) \subseteq G_1$.edgesBetween($\mathrm{dom}(F_{\mathbb{V}})$), and

(ii) $\mathrm{rng}\, F_{\mathbb{E}} \subseteq G_2$.edgesBetween($\mathrm{rng}\, F_{\mathbb{V}}$).

Proof: For every object e such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ holds $e \in G_1$.edgesBetween($\mathrm{dom}(F_{\mathbb{V}})$). For every object e such that $e \in \mathrm{rng}\, F_{\mathbb{E}}$ holds $e \in G_2$.edgesBetween($\mathrm{rng}\, F_{\mathbb{V}}$). $\square$

(8) Let us consider graphs G_1, G_2, a partial function f from the vertices of G_1 to the vertices of G_2, and a partial function g from the edges of G_1 to the edges of G_2. Suppose for every object e such that $e \in \mathrm{dom}\, g$ holds (the source of G_1)(e), (the target of G_1)(e) $\in \mathrm{dom}\, f$ and for every objects e, v, w such that $e \in \mathrm{dom}\, g$ and v, $w \in \mathrm{dom}\, f$ holds if e joins v and w in G_1, then $g(e)$ joins $f(v)$ and $f(w)$ in G_2. Then $\langle f, g \rangle$ is a partial graph mapping from G_1 to G_2.

Let us consider graphs G_1, G_2, G_3, G_4 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(9) If $G_1 \approx G_3$ and $G_2 \approx G_4$, then F is a partial graph mapping from G_3 to G_4. The theorem is a consequence of (5), (4), and (8).

(10) Suppose there exist sets E_1, E_2 such that G_3 is a graph given by reversing directions of the edges E_1 of G_1 and G_4 is a graph given by reversing directions of the edges E_2 of G_2. Then F is a partial graph mapping from G_3 to G_4. The theorem is a consequence of (5), (4), and (8).

Let G be a graph. The functor id_G yielding a partial graph mapping from G to G is defined by the term

(Def. 9) $\langle \mathrm{id}_\alpha, \mathrm{id}_\beta \rangle$, where α is the vertices of G and β is the edges of G.

Now we state the propositions:

(11) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. Then

(i) $\mathrm{id}_{G_1} = \mathrm{id}_{G_2}$, and

(ii) id_{G_1} is a partial graph mapping from G_1 to G_2.

The theorem is a consequence of (9).

(12) Let us consider a graph G_1, a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then

(i) $\mathrm{id}_{G_1} = \mathrm{id}_{G_2}$, and

(ii) id_{G_1} is a partial graph mapping from G_1 to G_2.

Proof: There exist sets E_1, E_2 such that G_1 is a graph given by reversing directions of the edges E_1 of G_1 and G_2 is a graph given by reversing directions of the edges E_2 of G_1. $\square$

Let G_1, G_2 be graphs and F be a partial graph mapping from G_1 to G_2. We say that F is empty if and only if

(Def. 10) $\mathrm{dom}(F_{\mathbb{V}})$ is empty.

We say that F is total if and only if

(Def. 11) $\mathrm{dom}(F_{\mathbb{V}}) =$ the vertices of G_1 and $\mathrm{dom}(F_{\mathbb{E}}) =$ the edges of G_1.

We say that F is onto if and only if

(Def. 12) $\mathrm{rng}\, F_{\mathbb{V}} =$ the vertices of G_2 and $\mathrm{rng}\, F_{\mathbb{E}} =$ the edges of G_2.

We say that F is one-to-one if and only if

(Def. 13) $F_{\mathbb{V}}$ is one-to-one and $F_{\mathbb{E}}$ is one-to-one.

We say that F is directed if and only if

(Def. 14) for every objects e, v, w such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ holds if e joins v to w in G_1, then $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$ in G_2.

We say that F is semi-continuous if and only if

(Def. 15) for every objects e, v, w such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ holds if $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ and $(F_{\mathbb{V}})(w)$ in G_2, then e joins v and w in G_1.

We say that F is continuous if and only if

(Def. 16) for every objects $\tilde{e}$, v, w such that $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ and $\tilde{e}$ joins $(F_{\mathbb{V}})(v)$ and $(F_{\mathbb{V}})(w)$ in G_2 there exists an object e such that e joins v and w in G_1 and $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $(F_{\mathbb{E}})(e) = \tilde{e}$.

We say that F is semi-directed-continuous if and only if

(Def. 17) for every objects e, v, w such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ holds if $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$ in G_2, then e joins v to w in G_1.

We say that F is directed-continuous if and only if

(Def. 18) for every objects $\tilde{e}$, v, w such that $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ and $\tilde{e}$ joins $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$ in G_2 there exists an object e such that e joins v to w in G_1 and $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $(F_{\mathbb{E}})(e) = \tilde{e}$.

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(13) F is directed if and only if for every object e such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ holds (the source of G_2)$((F_{\mathbb{E}})(e)) = (F_{\mathbb{V}})$((the source of G_1)(e)) and (the target of G_2)$((F_{\mathbb{E}})(e)) = (F_{\mathbb{V}})$((the target of G_1)(e)). The theorem is a consequence of (5).

(14) F is directed if and only if (the source of G_2) $\cdot (F_{\mathbb{E}}) = (F_{\mathbb{V}}) \cdot$ ((the source of G_1)$\restriction \mathrm{dom}(F_{\mathbb{E}})$) and (the target of G_2) $\cdot (F_{\mathbb{E}}) = (F_{\mathbb{V}}) \cdot$ ((the target of G_1)$\restriction \mathrm{dom}(F_{\mathbb{E}})$). The theorem is a consequence of (13) and (5).

(15) F is semi-continuous if and only if for every objects e, v, w such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ holds e joins v and w in G_1 iff $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ and $(F_{\mathbb{V}})(w)$ in G_2.

(16) F is semi-directed-continuous if and only if for every objects e, v, w such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ holds e joins v to w in G_1 iff $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$ in G_2.

Proof: If F is semi-directed-continuous, then for every objects e, v, w such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ holds e joins v to w in G_1 iff $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$ in G_2. $\square$

Let G_1, G_2 be graphs. Note that there exists a partial graph mapping from G_1 to G_2 which is empty, one-to-one, directed-continuous, directed, continuous, semi-directed-continuous, and semi-continuous and there exists a partial graph mapping from G_1 to G_2 which is non empty, one-to-one, directed, semi-directed-continuous, and semi-continuous.

Let F be an empty partial graph mapping from G_1 to G_2. One can verify that $F_{\mathbb{V}}$ is empty as a set and $F_{\mathbb{E}}$ is empty as a set.

Let F be a non empty partial graph mapping from G_1 to G_2. One can verify that $F_{\mathbb{V}}$ is non empty as a set.

Let F be a one-to-one partial graph mapping from G_1 to G_2. One can verify that $F_{\mathbb{V}}$ is one-to-one as a function and $F_{\mathbb{E}}$ is one-to-one as a function.

Now we state the propositions:

(17) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. If $F_{\mathbb{V}}$ is one-to-one, then F is semi-continuous. The theorem is a consequence of (5) and (4).

(18) Let us consider graphs G_1, G_2, and a directed partial graph mapping F from G_1 to G_2. If $F_{\mathbb{V}}$ is one-to-one, then F is semi-directed-continuous. The theorem is a consequence of (5).

(19) Let us consider graphs G_1, G_2, and a semi-continuous partial graph mapping F from G_1 to G_2. Suppose $\mathrm{rng}\, F_{\mathbb{E}} =$ the edges of G_2. Then F is continuous.

(20) Let us consider graphs G_1, G_2, and a semi-directed-continuous partial graph mapping F from G_1 to G_2. Suppose $\operatorname{rng} F_{\mathbb{E}} =$ the edges of G_2. Then F is directed-continuous.

(21) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose $F_{\mathbb{V}}$ is one-to-one and $\operatorname{rng} F_{\mathbb{E}} =$ the edges of G_2. Then F is continuous. The theorem is a consequence of (17) and (19).

(22) Let us consider graphs G_1, G_2, and a directed partial graph mapping F from G_1 to G_2. Suppose $F_{\mathbb{V}}$ is one-to-one and $\operatorname{rng} F_{\mathbb{E}} =$ the edges of G_2. Then F is directed-continuous. The theorem is a consequence of (18) and (20).

(23) Let us consider graphs G_1, G_2, and a continuous partial graph mapping F from G_1 to G_2. If $F_{\mathbb{E}}$ is one-to-one, then F is semi-continuous.

Let us consider graphs G_1, G_2 and a directed-continuous partial graph mapping F from G_1 to G_2. Now we state the propositions:

(24) If $F_{\mathbb{E}}$ is one-to-one, then F is semi-directed-continuous.

(25) If $F_{\mathbb{E}}$ is one-to-one, then F is directed. The theorem is a consequence of (4).

(26) Let us consider graphs G_1, G_2, a semi-continuous partial graph mapping F from G_1 to G_2, and objects v_1, v_2. Suppose $v_1, v_2 \in \operatorname{dom}(F_{\mathbb{V}})$ and $(F_{\mathbb{V}})(v_1) = (F_{\mathbb{V}})(v_2)$ and there exist objects e, w such that $e \in \operatorname{dom}(F_{\mathbb{E}})$ and $w \in \operatorname{dom}(F_{\mathbb{V}})$ and $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v_1)$ and $(F_{\mathbb{V}})(w)$ in G_2. Then $v_1 = v_2$.

(27) Let us consider graphs G_1, G_2, and a semi-continuous partial graph mapping F from G_1 to G_2. Suppose for every object v such that $v \in \operatorname{dom}(F_{\mathbb{V}})$ there exist objects e, w such that $e \in \operatorname{dom}(F_{\mathbb{E}})$ and $w \in \operatorname{dom}(F_{\mathbb{V}})$ and $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ and $(F_{\mathbb{V}})(w)$ in G_2. Then $F_{\mathbb{V}}$ is one-to-one. The theorem is a consequence of (26).

(28) Let us consider graphs G_1, G_2, a semi-directed-continuous partial graph mapping F from G_1 to G_2, and objects v_1, v_2. Suppose $v_1, v_2 \in \operatorname{dom}(F_{\mathbb{V}})$ and $(F_{\mathbb{V}})(v_1) = (F_{\mathbb{V}})(v_2)$ and there exist objects e, w such that $e \in \operatorname{dom}(F_{\mathbb{E}})$ and $w \in \operatorname{dom}(F_{\mathbb{V}})$ and $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v_1)$ to $(F_{\mathbb{V}})(w)$ in G_2. Then $v_1 = v_2$.

(29) Let us consider graphs G_1, G_2, and a semi-directed-continuous partial graph mapping F from G_1 to G_2. Suppose for every object v such that $v \in \operatorname{dom}(F_{\mathbb{V}})$ there exist objects e, w such that $e \in \operatorname{dom}(F_{\mathbb{E}})$ and $w \in \operatorname{dom}(F_{\mathbb{V}})$ and $(F_{\mathbb{E}})(e)$ joins $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$ in G_2. Then $F_{\mathbb{V}}$ is one-to-one. The theorem is a consequence of (28).

Let G_1, G_2 be graphs. One can verify that every partial graph mapping from

G_1 to G_2 which is one-to-one is also semi-continuous and every partial graph mapping from G_1 to G_2 which is one-to-one and directed is also semi-directed-continuous and every partial graph mapping from G_1 to G_2 which is one-to-one and onto is also continuous and every partial graph mapping from G_1 to G_2 which is directed, one-to-one, and onto is also directed-continuous.

Every partial graph mapping from G_1 to G_2 which is semi-continuous and onto is also continuous and every partial graph mapping from G_1 to G_2 which is semi-directed-continuous is also directed and semi-continuous and every partial graph mapping from G_1 to G_2 which is semi-directed-continuous and onto is also directed-continuous and every partial graph mapping from G_1 to G_2 which is directed-continuous is also continuous.

Every partial graph mapping from G_1 to G_2 which is directed-continuous and one-to-one is also directed and semi-directed-continuous and every partial graph mapping from G_1 to G_2 which is empty is also one-to-one, directed-continuous, directed, and continuous and every partial graph mapping from G_1 to G_2 which is total is also non empty and every partial graph mapping from G_1 to G_2 which is onto is also non empty.

Let G be a graph. One can verify that id_G is total, non empty, onto, one-to-one, and directed-continuous.

Let us consider graphs G_1, G_2, a partial function f from the vertices of G_1 to the vertices of G_2, and a partial function g from the edges of G_1 to the edges of G_2. Now we state the propositions:

(30) Suppose for every object e such that $e \in \operatorname{dom} g$ holds (the source of $G_1)(e)$, (the target of $G_1)(e) \in \operatorname{dom} f$ and for every objects e, v, w such that $e \in \operatorname{dom} g$ and $v, w \in \operatorname{dom} f$ holds if e joins v to w in G_1, then $g(e)$ joins $f(v)$ to $f(w)$ in G_2. Then $\langle f, g\rangle$ is a directed partial graph mapping from G_1 to G_2. The theorem is a consequence of (8).

(31) Suppose for every object e such that $e \in \operatorname{dom} g$ holds (the source of $G_1)(e)$, (the target of $G_1)(e) \in \operatorname{dom} f$ and for every objects e, v, w such that $e \in \operatorname{dom} g$ and $v, w \in \operatorname{dom} f$ holds e joins v and w in G_1 iff $g(e)$ joins $f(v)$ and $f(w)$ in G_2. Then $\langle f, g\rangle$ is a semi-continuous partial graph mapping from G_1 to G_2. The theorem is a consequence of (8).

(32) Suppose for every object e such that $e \in \operatorname{dom} g$ holds (the source of $G_1)(e)$, (the target of $G_1)(e) \in \operatorname{dom} f$ and for every objects e, v, w such that $e \in \operatorname{dom} g$ and $v, w \in \operatorname{dom} f$ holds e joins v to w in G_1 iff $g(e)$ joins $f(v)$ to $f(w)$ in G_2. Then $\langle f, g\rangle$ is a semi-directed-continuous partial graph mapping from G_1 to G_2. The theorem is a consequence of (8).

(33) Let us consider graphs G_1, G_2. Then $\langle\emptyset, \emptyset\rangle$ is an empty, one-to-one, directed-continuous partial graph mapping from G_1 to G_2.

(34) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is total. Let us consider a vertex v of G_1. Then $(F_{\mathbb{V}})(v)$ is a vertex of G_2.

(35) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is total. Then

(i) if G_2 is loopless, then G_1 is loopless, and

(ii) if G_2 is edgeless, then G_1 is edgeless.

The theorem is a consequence of (4).

(36) Let us consider graphs G_1, G_2, and a continuous partial graph mapping F from G_1 to G_2. Suppose $\operatorname{rng} F_{\mathbb{V}} =$ the vertices of G_2. If G_1 is loopless, then G_2 is loopless.

Proof: For every object v, there exists no object e such that e joins v and v in G_2. $\square$

(37) Let us consider graphs G_1, G_2, and a semi-continuous partial graph mapping F from G_1 to G_2. If F is onto, then if G_1 is loopless, then G_2 is loopless.

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(38) If $\operatorname{rng} F_{\mathbb{E}} =$ the edges of G_2, then if G_1 is edgeless, then G_2 is edgeless.

(39) If F is onto, then if G_1 is edgeless, then G_2 is edgeless.

(40) Let us consider a graph G_1, a non-multi graph G_2, and partial graph mappings F_1, F_2 from G_1 to G_2. Suppose $F_{1\mathbb{V}} = F_{2\mathbb{V}}$ and $\operatorname{dom}(F_{1\mathbb{E}}) = \operatorname{dom}(F_{2\mathbb{E}})$. Then $F_1 = F_2$. The theorem is a consequence of (5) and (4).

(41) Let us consider a graph G_1, a non-directed-multi graph G_2, and directed partial graph mappings F_1, F_2 from G_1 to G_2. Suppose $F_{1\mathbb{V}} = F_{2\mathbb{V}}$ and $\operatorname{dom}(F_{1\mathbb{E}}) = \operatorname{dom}(F_{2\mathbb{E}})$. Then $F_1 = F_2$. The theorem is a consequence of (5).

(42) Let us consider a non-multi graph G_1, a graph G_2, and a semi-continuous partial graph mapping F from G_1 to G_2. Then $F_{\mathbb{E}}$ is one-to-one. The theorem is a consequence of (5) and (4).

(43) Let us consider a non-multi graph G_1, a graph G_2, and a partial graph mapping F from G_1 to G_2. If $F_{\mathbb{V}}$ is one-to-one, then $F_{\mathbb{E}}$ is one-to-one. The theorem is a consequence of (5) and (4).

(44) Let us consider a non-directed-multi graph G_1, a graph G_2, and a directed partial graph mapping F from G_1 to G_2. If $F_{\mathbb{V}}$ is one-to-one, then $F_{\mathbb{E}}$ is one-to-one. The theorem is a consequence of (5).

Let G_1 be a graph and G_2 be a loopless graph. Observe that every partial graph mapping from G_1 to G_2 which is directed and semi-continuous is also semi-directed-continuous and every partial graph mapping from G_1 to G_2 which is directed and continuous is also directed-continuous.

Let G_1 be a trivial graph and G_2 be a graph. Observe that every partial graph mapping from G_1 to G_2 is directed and every partial graph mapping from G_1 to G_2 which is semi-continuous is also semi-directed-continuous and every partial graph mapping from G_1 to G_2 which is continuous is also directed-continuous.

Let G_1 be a trivial, non-directed-multi graph. Note that every partial graph mapping from G_1 to G_2 is one-to-one.

Let G_1 be a trivial, edgeless graph. Observe that every partial graph mapping from G_1 to G_2 which is non empty is also total.

Let G_1 be a graph and G_2 be a trivial, edgeless graph. Note that every partial graph mapping from G_1 to G_2 which is non empty is also onto and every partial graph mapping from G_1 to G_2 is semi-continuous and continuous.

Let G_1, G_2 be graphs and F be a partial graph mapping from G_1 to G_2. We say that F is weak subgraph embedding if and only if

(Def. 19) F is total and one-to-one.

We say that F is strong subgraph embedding if and only if

(Def. 20) F is total, one-to-one, and continuous.

We say that F is isomorphism if and only if

(Def. 21) F is total, one-to-one, and onto.

We say that F is directed-isomorphism if and only if

(Def. 22) F is directed, total, one-to-one, and onto.

One can check that every partial graph mapping from G_1 to G_2 which is weak subgraph embedding is also total, non empty, one-to-one, and semi-continuous and every partial graph mapping from G_1 to G_2 which is total and one-to-one is also weak subgraph embedding and every partial graph mapping from G_1 to G_2 which is strong subgraph embedding is also total, non empty, one-to-one, continuous, and weak subgraph embedding and every partial graph mapping from G_1 to G_2 which is total, one-to-one, and continuous is also strong subgraph embedding.

Every partial graph mapping from G_1 to G_2 which is weak subgraph embedding and continuous is also strong subgraph embedding and every partial graph mapping from G_1 to G_2 which is isomorphism is also onto, semi-continuous, continuous, total, non empty, one-to-one, weak subgraph embedding, and strong subgraph embedding and every partial graph mapping from G_1 to G_2 which is total, one-to-one, onto, and continuous is also isomorphism and every partial graph mapping from G_1 to G_2 which is strong subgraph embedding and onto is also isomorphism.

Every partial graph mapping from G_1 to G_2 which is weak subgraph embedding, continuous, and onto is also isomorphism and every partial graph mapping from G_1 to G_2 which is directed-isomorphism is also directed, isomorphism, continuous, total, non empty, semi-directed-continuous, semi-continuous, one-to-one, weak subgraph embedding, and strong subgraph embedding and every partial graph mapping from G_1 to G_2 which is directed and isomorphism is also directed-continuous and directed-isomorphism.

Let G be a graph. Let us note that id_G is weak subgraph embedding, strong subgraph embedding, isomorphism, and directed-isomorphism and there exists a partial graph mapping from G to G which is weak subgraph embedding, strong subgraph embedding, isomorphism, and directed-isomorphism.

Now we state the propositions:

(45) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is weak subgraph embedding. Then

(i) $G_1.\mathrm{order}() \subseteq G_2.\mathrm{order}()$, and

(ii) $G_1.\mathrm{size}() \subseteq G_2.\mathrm{size}()$.

(46) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and subsets X, Y of the vertices of G_1. Suppose F is weak subgraph embedding. Then $\overline{\overline{G_1.\mathrm{edgesBetween}(X,Y)}} \subseteq \overline{\overline{G_2.\mathrm{edgesBetween}((F_{\mathbb{V}})^{\circ}X,(F_{\mathbb{V}})^{\circ}Y)}}$.
Proof: Set $f = F_{\mathbb{E}}{\restriction}G_1.\mathrm{edgesBetween}(X,Y)$. For every object y such that $y \in \mathrm{rng}\, f$ holds $y \in G_2.\mathrm{edgesBetween}((F_{\mathbb{V}})^{\circ}X,(F_{\mathbb{V}})^{\circ}Y)$. $\square$

(47) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a subset X of the vertices of G_1. Suppose F is weak subgraph embedding. Then $\overline{\overline{G_1.\mathrm{edgesBetween}(X)}} \subseteq \overline{\overline{G_2.\mathrm{edgesBetween}((F_{\mathbb{V}})^{\circ}X)}}$.
Proof: Set $f = F_{\mathbb{E}}{\restriction}G_1.\mathrm{edgesBetween}(X)$. For every object y such that $y \in \mathrm{rng}\, f$ holds $y \in G_2.\mathrm{edgesBetween}((F_{\mathbb{V}})^{\circ}X)$. $\square$

(48) Let us consider graphs G_1, G_2, a directed partial graph mapping F from G_1 to G_2, and subsets X, Y of the vertices of G_1. Suppose F is weak subgraph embedding. Then $\overline{\overline{G_1.\mathrm{edgesDBetween}(X,Y)}} \subseteq$ $\overline{\overline{G_2.\mathrm{edgesDBetween}((F_{\mathbb{V}})^{\circ}X,(F_{\mathbb{V}})^{\circ}Y)}}$.
Proof: Set $f = F_{\mathbb{E}}{\restriction}G_1.\mathrm{edgesDBetween}(X,Y)$. For every object y such that $y \in \mathrm{rng}\, f$ holds $y \in G_2.\mathrm{edgesDBetween}((F_{\mathbb{V}})^{\circ}X,(F_{\mathbb{V}})^{\circ}Y)$. $\square$

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(49) Suppose F is weak subgraph embedding. Then

(i) if G_2 is trivial, then G_1 is trivial, and

(ii) if G_2 is non-multi, then G_1 is non-multi, and

(iii) if G_2 is simple, then G_1 is simple, and

(iv) if G_2 is finite, then G_1 is finite.

PROOF: If G_2 is non-multi, then G_1 is non-multi. G_1.order() $\subseteq$ G_2.order() and G_1.size() $\subseteq$ G_2.size(). $\square$

(50) Suppose F is directed and weak subgraph embedding. Then

(i) if G_2 is non-directed-multi, then G_1 is non-directed-multi, and

(ii) if G_2 is directed-simple, then G_1 is directed-simple.

PROOF: If G_2 is non-directed-multi, then G_1 is non-directed-multi. G_1 is loopless and non-directed-multi. $\square$

(51) Let us consider finite graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is strong subgraph embedding and G_1.order() $=$ G_2.order() and G_1.size() $=$ G_2.size(). Then F is isomorphism.

(52) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is strong subgraph embedding. If G_2 is complete, then G_1 is complete.

Let G_1, G_2 be graphs. We say that G_2 is G_1-isomorphic if and only if

(Def. 23) there exists a partial graph mapping F from G_1 to G_2 such that F is isomorphism.

We say that G_2 is G_1-directed-isomorphic if and only if

(Def. 24) there exists a partial graph mapping F from G_1 to G_2 such that F is directed-isomorphism.

Let G be a graph. Note that every graph which is G-directed-isomorphic is also G-isomorphic and there exists a graph which is G-directed-isomorphic and G-isomorphic.

Now we state the proposition:

(53) Every graph is directed-isomorphic and isomorphic to itself.

Let G_1 be a graph and G_2 be a G_1-isomorphic graph. Let us observe that there exists a partial graph mapping from G_1 to G_2 which is isomorphism, strong subgraph embedding, weak subgraph embedding, total, non empty, one-to-one, onto, semi-continuous, and continuous.

An isomorphism between G_1 and G_2 is an isomorphism partial graph mapping from G_1 to G_2. Let G_2 be a G_1-directed-isomorphic graph. One can verify that there exists a partial graph mapping from G_1 to G_2 which is isomorphism, strong subgraph embedding, weak subgraph embedding, total, non empty, one-to-one, onto, directed, semi-directed-continuous, and directed-continuous.

A directed isomorphism of G_1 and G_2 is a directed-isomorphism partial graph mapping from G_1 to G_2. Let G_1, G_2 be w-graphs and F be a partial graph mapping from G_1 to G_2. We say that F preserves weight if and only if

(Def. 25) (the weight of G_2) $\cdot$ $(F_{\mathbb{E}})$ = (the weight of G_1)$\restriction$dom$(F_{\mathbb{E}})$.

Let G_1, G_2 be e-graphs. We say that F preserves elabel if and only if

(Def. 26) (the elabel of G_2) $\cdot$ $(F_{\mathbb{E}})$ = (the elabel of G_1)$\restriction$dom$(F_{\mathbb{E}})$.

Let G_1, G_2 be v-graphs. We say that F preserves vlabel if and only if

(Def. 27) (the vlabel of G_2) $\cdot$ $(F_{\mathbb{V}})$ = (the vlabel of G_1)$\restriction$dom$(F_{\mathbb{V}})$.

Let G_1, G_2 be ordered graphs. We say that F preserves ordering if and only if

(Def. 28) (the ordering of G_2) $\cdot$ $(F_{\mathbb{V}})$ = the ordering of $G_1\restriction$dom$(F_{\mathbb{V}})$.

Let G be a w-graph. Note that id_G preserves weight.

Let G be an e-graph. Let us note that id_G preserves elabel.

Let G be a v-graph. Observe that id_G preserves vlabel.

Let G be an ordered graph. Let us observe that id_G preserves ordering.

Let G_1, G_2 be graphs and F be a partial graph mapping from G_1 to G_2. The functor dom F yielding a subgraph of G_1 induced by dom$(F_{\mathbb{V}})$ and dom$(F_{\mathbb{E}})$ is defined by the term

(Def. 29) the plain subgraph of G_1 induced by dom$(F_{\mathbb{V}})$ and dom$(F_{\mathbb{E}})$.

The functor rng F yielding a subgraph of G_2 induced by rng $F_{\mathbb{V}}$ and rng $F_{\mathbb{E}}$ is defined by the term

(Def. 30) the plain subgraph of G_2 induced by rng $F_{\mathbb{V}}$ and rng $F_{\mathbb{E}}$.

One can verify that dom F is plain and rng F is plain.

Let us consider graphs G_1, G_2 and a non empty partial graph mapping F from G_1 to G_2. Now we state the propositions:

(54) (i) the vertices of dom F = dom$(F_{\mathbb{V}})$, and

(ii) the edges of dom F = dom$(F_{\mathbb{E}})$, and

(iii) the vertices of rng F = rng $F_{\mathbb{V}}$, and

(iv) the edges of rng F = rng $F_{\mathbb{E}}$.

The theorem is a consequence of (7).

(55) F is total if and only if dom $F \approx G_1$. The theorem is a consequence of (54).

(56) F is onto if and only if rng $F \approx G_2$. The theorem is a consequence of (54).

Let G_1, G_2 be graphs, H be a subgraph of G_1, and F be a partial graph mapping from G_1 to G_2. The functor $F\restriction H$ yielding a partial graph mapping from H to G_2 is defined by the term

(Def. 31) $\langle F_{\mathbb{V}}\restriction$(the vertices of H), $F_{\mathbb{E}}\restriction$(the edges of H)$\rangle$.

Now we state the propositions:

(57) Let us consider graphs G_1, G_2, a subgraph H of G_1, and a partial graph mapping F from G_1 to G_2. Then

(i) if F is empty, then $F{\restriction}H$ is empty, and

(ii) if F is total, then $F{\restriction}H$ is total, and

(iii) if F is one-to-one, then $F{\restriction}H$ is one-to-one, and

(iv) if F is weak subgraph embedding, then $F{\restriction}H$ is weak subgraph embedding, and

(v) if F is semi-continuous, then $F{\restriction}H$ is semi-continuous, and

(vi) if F is not onto, then $F{\restriction}H$ is not onto, and

(vii) if F is directed, then $F{\restriction}H$ is directed, and

(viii) if F is semi-directed-continuous, then $F{\restriction}H$ is semi-directed-continuous.

Proof: If F is total, then $F{\restriction}H$ is total. If F is semi-continuous, then $F{\restriction}H$ is semi-continuous. If $F{\restriction}H$ is onto, then F is onto. If F is directed, then $F{\restriction}H$ is directed. If F is semi-directed-continuous, then $F{\restriction}H$ is semi-directed-continuous. □

(58) Let us consider graphs G_1, G_2, a set V, a subgraph H of G_1 induced by V, and a partial graph mapping F from G_1 to G_2. Then

(i) if F is continuous, then $F{\restriction}H$ is continuous, and

(ii) if F is strong subgraph embedding, then $F{\restriction}H$ is strong subgraph embedding, and

(iii) if F is directed-continuous, then $F{\restriction}H$ is directed-continuous.

The theorem is a consequence of (57).

Let G_1, G_2 be graphs, H be a subgraph of G_1, and F be an empty partial graph mapping from G_1 to G_2. Let us observe that $F{\restriction}H$ is empty.

Let F be a one-to-one partial graph mapping from G_1 to G_2. Let us observe that $F{\restriction}H$ is one-to-one.

Let F be a semi-continuous partial graph mapping from G_1 to G_2. Observe that $F{\restriction}H$ is semi-continuous.

Let V be a set, H be a subgraph of G_1 induced by V, and F be a continuous partial graph mapping from G_1 to G_2. Let us observe that $F{\restriction}H$ is continuous.

Let H be a subgraph of G_1 and F be a directed partial graph mapping from G_1 to G_2. Note that $F{\restriction}H$ is directed.

Let F be a semi-directed-continuous partial graph mapping from G_1 to G_2. One can check that $F{\restriction}H$ is semi-directed-continuous.

Let V be a set, H be a subgraph of G_1 induced by V, and F be a directed-continuous partial graph mapping from G_1 to G_2. Note that $F{\restriction}H$ is directed-continuous.

Let F be a non empty partial graph mapping from G_1 to G_2. One can verify that $F \restriction \operatorname{dom} F$ is total.

Now we state the propositions:

(59) Let us consider graphs G_1, G_2, a subgraph H of G_1, and a partial graph mapping F from G_1 to G_2. Then

(i) $\operatorname{dom}((F \restriction H)_{\mathbb{V}}) = \operatorname{dom}(F_{\mathbb{V}}) \cap$ (the vertices of H), and

(ii) $\operatorname{dom}((F \restriction H)_{\mathbb{E}}) = \operatorname{dom}(F_{\mathbb{E}}) \cap$ (the edges of H).

(60) Let us consider w-graphs G_1, G_2, a w-subgraph H of G_1, and a partial graph mapping F from G_1 to G_2. If F preserves weight, then $F \restriction H$ preserves weight. The theorem is a consequence of (59).

(61) Let us consider e-graphs G_1, G_2, an e-subgraph H of G_1, and a partial graph mapping F from G_1 to G_2. If F preserves elabel, then $F \restriction H$ preserves elabel. The theorem is a consequence of (59).

(62) Let us consider v-graphs G_1, G_2, a v-subgraph H of G_1, and a partial graph mapping F from G_1 to G_2. If F preserves vlabel, then $F \restriction H$ preserves vlabel. The theorem is a consequence of (59).

Let G_1, G_2 be graphs, H be a subgraph of G_2, and F be a partial graph mapping from G_1 to G_2. The functor $H \upharpoonleft F$ yielding a partial graph mapping from G_1 to H is defined by the term

(Def. 32) $\langle$(the vertices of $H) \upharpoonleft F_{\mathbb{V}}$, (the edges of $H) \upharpoonleft F_{\mathbb{E}}\rangle$.

Now we state the proposition:

(63) Let us consider graphs G_1, G_2, a subgraph H of G_2, and a partial graph mapping F from G_1 to G_2. Then

(i) if F is empty, then $H \upharpoonleft F$ is empty, and

(ii) if F is one-to-one, then $H \upharpoonleft F$ is one-to-one, and

(iii) if F is onto, then $H \upharpoonleft F$ is onto, and

(iv) if F is not total, then $H \upharpoonleft F$ is not total, and

(v) if F is directed, then $H \upharpoonleft F$ is directed, and

(vi) if F is semi-continuous, then $H \upharpoonleft F$ is semi-continuous, and

(vii) if F is continuous, then $H \upharpoonleft F$ is continuous, and

(viii) if F is semi-directed-continuous, then $H \upharpoonleft F$ is semi-directed-continuous, and

(ix) if F is directed-continuous, then $H \upharpoonleft F$ is directed-continuous.

Proof: If F is onto, then $H \upharpoonleft F$ is onto. If F is directed, then $H \upharpoonleft F$ is directed. If F is semi-continuous, then $H \upharpoonleft F$ is semi-continuous. If F is continuous, then $H \upharpoonleft F$ is continuous. If F is semi-directed-continuous, then

$H \upharpoonleft F$ is semi-directed-continuous. If F is directed-continuous, then $H \upharpoonleft F$ is directed-continuous. □

Let G_1, G_2 be graphs, H be a subgraph of G_2, and F be an empty partial graph mapping from G_1 to G_2. One can verify that $H \upharpoonleft F$ is empty.

Let F be a one-to-one partial graph mapping from G_1 to G_2. Let us observe that $H \upharpoonleft F$ is one-to-one.

Let F be a semi-continuous partial graph mapping from G_1 to G_2. Observe that $H \upharpoonleft F$ is semi-continuous.

Let F be a continuous partial graph mapping from G_1 to G_2. Let us note that $H \upharpoonleft F$ is continuous.

Let F be a directed partial graph mapping from G_1 to G_2. Note that $H \upharpoonleft F$ is directed.

Let F be a semi-directed-continuous partial graph mapping from G_1 to G_2. One can check that $H \upharpoonleft F$ is semi-directed-continuous.

Let F be a directed-continuous partial graph mapping from G_1 to G_2. One can verify that $H \upharpoonleft F$ is directed-continuous.

Let F be a non empty partial graph mapping from G_1 to G_2. Observe that $\operatorname{rng} F \upharpoonleft F$ is onto.

Now we state the propositions:

(64) Let us consider graphs G_1, G_2, a subgraph H of G_2, and a partial graph mapping F from G_1 to G_2. Then

(i) $\operatorname{rng}(H \upharpoonleft F)_{\mathbb{V}} = \operatorname{rng} F_{\mathbb{V}} \cap (\text{the vertices of } H)$, and

(ii) $\operatorname{rng}(H \upharpoonleft F)_{\mathbb{E}} = \operatorname{rng} F_{\mathbb{E}} \cap (\text{the edges of } H)$.

(65) Let us consider w-graphs G_1, G_2, a w-subgraph H of G_2, and a partial graph mapping F from G_1 to G_2. If F preserves weight, then $H \upharpoonleft F$ preserves weight.

(66) Let us consider e-graphs G_1, G_2, an e-subgraph H of G_2, and a partial graph mapping F from G_1 to G_2. If F preserves elabel, then $H \upharpoonleft F$ preserves elabel.

(67) Let us consider v-graphs G_1, G_2, a v-subgraph H of G_2, and a partial graph mapping F from G_1 to G_2. If F preserves vlabel, then $H \upharpoonleft F$ preserves vlabel.

(68) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, a subgraph H_1 of G_1, and a subgraph H_2 of G_2. Then $(H_2 \upharpoonleft F) \restriction H_1 = H_2 \upharpoonleft (F \restriction H_1)$.

Let G_1, G_2 be graphs and F be a one-to-one partial graph mapping from G_1 to G_2. The functor F^{-1} yielding a partial graph mapping from G_2 to G_1 is defined by the term

(Def. 33) $\langle (F_{\mathbb{V}})^{-1}, (F_{\mathbb{E}})^{-1} \rangle$.

One can verify that F^{-1} is one-to-one and semi-continuous.

Let F be an empty, one-to-one partial graph mapping from G_1 to G_2. One can verify that F^{-1} is empty.

Let F be a non empty, one-to-one partial graph mapping from G_1 to G_2. Let us note that F^{-1} is non empty.

Let F be a one-to-one, semi-directed-continuous partial graph mapping from G_1 to G_2. One can verify that F^{-1} is semi-directed-continuous.

Let us consider graphs G_1, G_2 and a one-to-one partial graph mapping F from G_1 to G_2. Now we state the propositions:

(69) (i) $F^{-1}{}_{\mathbb{V}} = (F_{\mathbb{V}})^{-1}$, and

(ii) $F^{-1}{}_{\mathbb{E}} = (F_{\mathbb{E}})^{-1}$.

(70) $(F^{-1})^{-1} = F$.

(71) F is total if and only if F^{-1} is onto.

(72) F is onto if and only if F^{-1} is total.

(73) If F is total and continuous, then F^{-1} is continuous.

(74) If F is total and directed-continuous, then F^{-1} is directed-continuous.

(75) F is isomorphism if and only if F^{-1} is isomorphism.

(76) Let us consider w-graphs G_1, G_2, and a one-to-one partial graph mapping F from G_1 to G_2. Then F preserves weight if and only if F^{-1} preserves weight. The theorem is a consequence of (2) and (70).

(77) Let us consider e-graphs G_1, G_2, and a one-to-one partial graph mapping F from G_1 to G_2. Then F preserves elabel if and only if F^{-1} preserves elabel. The theorem is a consequence of (2) and (70).

(78) Let us consider v-graphs G_1, G_2, and a one-to-one partial graph mapping F from G_1 to G_2. Then F preserves vlabel if and only if F^{-1} preserves vlabel. The theorem is a consequence of (2) and (70).

(79) Let us consider graphs G_1, G_2, and a one-to-one partial graph mapping F from G_1 to G_2. Suppose F is onto. Let us consider a vertex v of G_2. Then $(F^{-1}{}_{\mathbb{V}})(v)$ is a vertex of G_1.

(80) Let us consider a graph G. Then $(\mathrm{id}_G)^{-1} = \mathrm{id}_G$.

(81) Let us consider graphs G_1, G_2, and a non empty, one-to-one partial graph mapping F from G_1 to G_2. Then

(i) $\mathrm{dom}\, F = \mathrm{rng}\, F^{-1}$, and

(ii) $\mathrm{rng}\, F = \mathrm{dom}(F^{-1})$.

The theorem is a consequence of (54).

(82) Let us consider graphs G_1, G_2, a one-to-one partial graph mapping F from G_1 to G_2, and a subgraph H of G_1. Then $(F \upharpoonright H)^{-1} = H \upharpoonleft F^{-1}$.

(83) Let us consider graphs G_1, G_2, a one-to-one partial graph mapping F from G_1 to G_2, and a subgraph H of G_2. Then $(H \upharpoonleft F)^{-1} = F^{-1} \upharpoonright H$. The theorem is a consequence of (82) and (70).

(84) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is isomorphism. Then

   (i) $G_1.\text{order}() = G_2.\text{order}()$, and

   (ii) $G_1.\text{size}() = G_2.\text{size}()$.

   The theorem is a consequence of (45) and (75).

(85) Let us consider finite graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Suppose F is strong subgraph embedding. If there exists a partial graph mapping F_0 from G_1 to G_2 such that F_0 is isomorphism, then F is isomorphism. The theorem is a consequence of (84) and (51).

(86) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and subsets X, Y of the vertices of G_1. Suppose F is isomorphism. Then $\overline{\overline{G_1.\text{edgesBetween}(X, Y)}} = \overline{\overline{G_2.\text{edgesBetween}((F_{\mathbb{V}})^{\circ}X, (F_{\mathbb{V}})^{\circ}Y)}}$. The theorem is a consequence of (46) and (75).

(87) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a subset X of the vertices of G_1. Suppose F is isomorphism. Then $\overline{\overline{G_1.\text{edgesBetween}(X)}} = \overline{\overline{G_2.\text{edgesBetween}((F_{\mathbb{V}})^{\circ}X)}}$. The theorem is a consequence of (47) and (75).

(88) Let us consider graphs G_1, G_2, a directed partial graph mapping F from G_1 to G_2, and subsets X, Y of the vertices of G_1. Suppose F is isomorphism. Then $\overline{\overline{G_1.\text{edgesDBetween}(X, Y)}} = \overline{\overline{G_2.\text{edgesDBetween}((F_{\mathbb{V}})^{\circ}X, (F_{\mathbb{V}})^{\circ}Y)}}$. The theorem is a consequence of (48) and (75).

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(89) Suppose F is isomorphism. Then

   (i) G_1 is trivial iff G_2 is trivial, and

   (ii) G_1 is loopless iff G_2 is loopless, and

   (iii) G_1 is edgeless iff G_2 is edgeless, and

   (iv) G_1 is non-multi iff G_2 is non-multi, and

   (v) G_1 is simple iff G_2 is simple, and

   (vi) G_1 is finite iff G_2 is finite, and

(vii) G_1 is complete iff G_2 is complete.

The theorem is a consequence of (75), (35), (49), and (52).

(90) Suppose F is directed-continuous and isomorphism. Then

(i) G_1 is non-directed-multi iff G_2 is non-directed-multi, and

(ii) G_1 is directed-simple iff G_2 is directed-simple.

The theorem is a consequence of (74), (75), and (50).

(91) Let us consider graphs G_1, G_2, and a non empty, one-to-one partial graph mapping F from G_1 to G_2. Then $\overline{\overline{\operatorname{dom} F.\mathrm{loops}()}} = \overline{\overline{\operatorname{rng} F.\mathrm{loops}()}}$. The theorem is a consequence of (81).

Let us consider graphs G_1, G_2 and a one-to-one partial graph mapping F from G_1 to G_2. Now we state the propositions:

(92) If F is total, then $\overline{\overline{G_1.\mathrm{loops}()}} \subseteq \overline{\overline{G_2.\mathrm{loops}()}}$. The theorem is a consequence of (55).

(93) If F is onto, then $\overline{\overline{G_2.\mathrm{loops}()}} \subseteq \overline{\overline{G_1.\mathrm{loops}()}}$. The theorem is a consequence of (72) and (92).

(94) If F is isomorphism, then $\overline{\overline{G_1.\mathrm{loops}()}} = \overline{\overline{G_2.\mathrm{loops}()}}$. The theorem is a consequence of (92) and (93).

(95) Let us consider a graph G_1, and a G_1-isomorphic graph G_2. Then G_1 is G_2-isomorphic. The theorem is a consequence of (75).

(96) Let us consider a graph G_1, and a G_1-directed-isomorphic graph G_2. Then G_1 is G_2-directed-isomorphic. The theorem is a consequence of (71) and (72).

Let us consider a graph G_1, a G_1-isomorphic graph G_2, a G_2-isomorphic graph G_3, and an isomorphism F between G_1 and G_2. Now we state the propositions:

(97) Suppose there exists a set E such that G_3 is a graph given by reversing directions of the edges E of G_1. Then F^{-1} is an isomorphism between G_2 and G_3.

Proof: Reconsider $F_2 = F^{-1}$ as a partial graph mapping from G_2 to G_3. F_2 is total. F_2 is onto. □

(98) If $G_1 \approx G_3$, then F^{-1} is an isomorphism between G_2 and G_3. The theorem is a consequence of (97).

(99) Let us consider a graph G_1, a G_1-directed-isomorphic graph G_2, a G_2-directed-isomorphic graph G_3, and a directed isomorphism F of G_1 and G_2. Suppose $G_1 \approx G_3$. Then F^{-1} is a directed isomorphism of G_2 and G_3.

Proof: Reconsider $F_2 = F^{-1}$ as a partial graph mapping from G_2 to G_3. F_2 is total. F_2 is onto. □

Let G_1, G_2, G_3 be graphs, F_1 be a partial graph mapping from G_1 to G_2, and F_2 be a partial graph mapping from G_2 to G_3. The functor $F_2 \cdot F_1$ yielding a partial graph mapping from G_1 to G_3 is defined by the term

(Def. 34) $\langle (F_{2\mathbb{V}}) \cdot (F_{1\mathbb{V}}), (F_{2\mathbb{E}}) \cdot (F_{1\mathbb{E}}) \rangle$.

Let us consider graphs G_1, G_2, G_3, a partial graph mapping F_1 from G_1 to G_2, and a partial graph mapping F_2 from G_2 to G_3. Now we state the propositions:

(100) (i) $F_2 \cdot F_{1\mathbb{V}} = (F_{2\mathbb{V}}) \cdot (F_{1\mathbb{V}})$, and

(ii) $F_2 \cdot F_{1\mathbb{E}} = (F_{2\mathbb{E}}) \cdot (F_{1\mathbb{E}})$.

(101) If $F_2 \cdot F_1$ is onto, then F_2 is onto.

(102) If $F_2 \cdot F_1$ is total, then F_1 is total.

Let G_1, G_2, G_3 be graphs, F_1 be a one-to-one partial graph mapping from G_1 to G_2, and F_2 be a one-to-one partial graph mapping from G_2 to G_3. Observe that $F_2 \cdot F_1$ is one-to-one.

Let F_1 be a semi-continuous partial graph mapping from G_1 to G_2 and F_2 be a semi-continuous partial graph mapping from G_2 to G_3. Let us observe that $F_2 \cdot F_1$ is semi-continuous.

Let F_1 be a continuous partial graph mapping from G_1 to G_2 and F_2 be a continuous partial graph mapping from G_2 to G_3. One can check that $F_2 \cdot F_1$ is continuous.

Let F_1 be a directed partial graph mapping from G_1 to G_2 and F_2 be a directed partial graph mapping from G_2 to G_3. One can check that $F_2 \cdot F_1$ is directed.

Let F_1 be a semi-directed-continuous partial graph mapping from G_1 to G_2 and F_2 be a semi-directed-continuous partial graph mapping from G_2 to G_3. Note that $F_2 \cdot F_1$ is semi-directed-continuous.

Let F_1 be a directed-continuous partial graph mapping from G_1 to G_2 and F_2 be a directed-continuous partial graph mapping from G_2 to G_3. Observe that $F_2 \cdot F_1$ is directed-continuous.

Let F_1 be an empty partial graph mapping from G_1 to G_2 and F_2 be a partial graph mapping from G_2 to G_3. Observe that $F_2 \cdot F_1$ is empty.

Let F_1 be a partial graph mapping from G_1 to G_2 and F_2 be an empty partial graph mapping from G_2 to G_3. Let us observe that $F_2 \cdot F_1$ is empty.

Let us consider graphs G_1, G_2, G_3, a partial graph mapping F_1 from G_1 to G_2, and a partial graph mapping F_2 from G_2 to G_3. Now we state the propositions:

(103) Suppose F_1 is total and $\operatorname{rng} F_{1\mathbb{V}} \subseteq \operatorname{dom}(F_{2\mathbb{V}})$ and $\operatorname{rng} F_{1\mathbb{E}} \subseteq \operatorname{dom}(F_{2\mathbb{E}})$. Then $F_2 \cdot F_1$ is total.

(104) If F_1 is total and F_2 is total, then $F_2 \cdot F_1$ is total. The theorem is a consequence of (103).

(105) Suppose F_2 is onto and $\mathrm{dom}(F_{2\mathbb{V}}) \subseteq \mathrm{rng}\, F_{1\mathbb{V}}$ and $\mathrm{dom}(F_{2\mathbb{E}}) \subseteq \mathrm{rng}\, F_{1\mathbb{E}}$. Then $F_2 \cdot F_1$ is onto.

(106) If F_1 is onto and F_2 is onto, then $F_2 \cdot F_1$ is onto. The theorem is a consequence of (105).

(107) If F_1 is weak subgraph embedding and F_2 is weak subgraph embedding, then $F_2 \cdot F_1$ is weak subgraph embedding.

(108) If F_1 is strong subgraph embedding and F_2 is strong subgraph embedding, then $F_2 \cdot F_1$ is strong subgraph embedding.

(109) If F_1 is isomorphism and F_2 is isomorphism, then $F_2 \cdot F_1$ is isomorphism.

(110) If F_1 is directed-isomorphism and F_2 is directed-isomorphism, then $F_2 \cdot F_1$ is directed-isomorphism. The theorem is a consequence of (109).

(111) Let us consider w-graphs G_1, G_2, G_3, a partial graph mapping F_1 from G_1 to G_2, and a partial graph mapping F_2 from G_2 to G_3. Suppose F_1 preserves weight and F_2 preserves weight. Then $F_2 \cdot F_1$ preserves weight. The theorem is a consequence of (1).

(112) Let us consider e-graphs G_1, G_2, G_3, a partial graph mapping F_1 from G_1 to G_2, and a partial graph mapping F_2 from G_2 to G_3. Suppose F_1 preserves elabel and F_2 preserves elabel. Then $F_2 \cdot F_1$ preserves elabel. The theorem is a consequence of (1).

(113) Let us consider v-graphs G_1, G_2, G_3, a partial graph mapping F_1 from G_1 to G_2, and a partial graph mapping F_2 from G_2 to G_3. Suppose F_1 preserves vlabel and F_2 preserves vlabel. Then $F_2 \cdot F_1$ preserves vlabel. The theorem is a consequence of (1).

(114) Let us consider graphs G_1, G_2, G_3, G_4, a partial graph mapping F_1 from G_1 to G_2, a partial graph mapping F_2 from G_2 to G_3, and a partial graph mapping F_3 from G_3 to G_4. Then $F_3 \cdot (F_2 \cdot F_1) = (F_3 \cdot F_2) \cdot F_1$.

(115) Let us consider graphs G_1, G_2, and a one-to-one partial graph mapping F from G_1 to G_2. Suppose F is isomorphism. Then

  (i) $F \cdot (F^{-1}) = \mathrm{id}_{G_2}$, and

  (ii) $F^{-1} \cdot F = \mathrm{id}_{G_1}$.

(116) Let us consider graphs G_1, G_2, and a partial graph mapping F from G_1 to G_2. Then

  (i) $F \cdot (\mathrm{id}_{G_1}) = F$, and

  (ii) $\mathrm{id}_{G_2} \cdot F = F$.

(117) Let us consider graphs G_1, G_2, G_3, a partial graph mapping F_1 from G_1 to G_2, a partial graph mapping F_2 from G_2 to G_3, and a subgraph H of G_1. Then $F_2 \cdot (F_1 \restriction H) = (F_2 \cdot F_1) \restriction H$.

(118) Let us consider graphs G_1, G_2, G_3, a partial graph mapping F_1 from G_1 to G_2, a partial graph mapping F_2 from G_2 to G_3, and a subgraph H of G_3. Then $(H \upharpoonleft F_2) \cdot F_1 = H \upharpoonleft (F_2 \cdot F_1)$.

Let G_1 be a graph and G_2 be a G_1-isomorphic graph. Let us note that every graph which is G_2-isomorphic is also G_1-isomorphic.

Let G_2 be a G_1-directed-isomorphic graph. Note that every graph which is G_2-directed-isomorphic is also G_1-directed-isomorphic.

4. Walks Induced by Graph Mappings

Let G_1, G_2 be graphs, F be a partial graph mapping from G_1 to G_2, and W_1 be a walk of G_1. We say that W_1 is F-defined if and only if

(Def. 35) W_1.vertices() $\subseteq \operatorname{dom}(F_{\mathbb{V}})$ and W_1.edges() $\subseteq \operatorname{dom}(F_{\mathbb{E}})$.

Let W_2 be a walk of G_2. We say that W_2 is F-valued if and only if

(Def. 36) W_2.vertices() $\subseteq \operatorname{rng} F_{\mathbb{V}}$ and W_2.edges() $\subseteq \operatorname{rng} F_{\mathbb{E}}$.

Let F be a non empty partial graph mapping from G_1 to G_2. Observe that there exists a walk of G_1 which is F-defined and trivial and there exists a walk of G_2 which is F-valued and trivial.

Let us consider graphs G_1, G_2 and an empty partial graph mapping F from G_1 to G_2. Now we state the propositions:

(119) Every walk of G_1 is not F-defined.

(120) Every walk of G_2 is not F-valued.

(121) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a walk W_1 of G_1. If F is total, then W_1 is F-defined.

(122) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a walk W_2 of G_2. If F is onto, then W_2 is F-valued.

Let G_1, G_2 be graphs and F be a one-to-one partial graph mapping from G_1 to G_2. Observe that every walk of G_1 which is F-defined is also (F^{-1})-valued and every walk of G_2 which is F-valued is also (F^{-1})-defined.

Let F be a non empty partial graph mapping from G_1 to G_2 and W_1 be an F-defined walk of G_1. The functor $F^{\circ}W_1$ yielding a walk of G_2 is defined by

(Def. 37) $(F_{\mathbb{V}}) \cdot (W_1$.vertexSeq()) $=$ *it*.vertexSeq() and $(F_{\mathbb{E}}) \cdot (W_1$.edgeSeq()) $=$ *it*.edgeSeq().

Note that $F^{\circ}W_1$ is F-valued.

Let us observe that the functor $F^{\circ}W_1$ yields an F-valued walk of G_2. Let F be a non empty, one-to-one partial graph mapping from G_1 to G_2 and W_2 be an F-valued walk of G_2. The functor $F^{-1}(W_2)$ yielding an F-defined walk of G_1 is defined by the term

(Def. 38) $(F^{-1})^{\circ}W_2$.

Let us observe that the functor $F^{-1}(W_2)$ is defined by

(Def. 39) $(F_{\mathbb{V}}) \cdot (it.\mathrm{vertexSeq}()) = W_2.\mathrm{vertexSeq}()$ and $(F_{\mathbb{E}}) \cdot (it.\mathrm{edgeSeq}()) = W_2.\mathrm{edgeSeq}()$.

Now we state the propositions:

(123) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and an F-defined walk W_1 of G_1. Then $F^{-1}(F^{\circ}W_1) = W_1$.

(124) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and an F-valued walk W_2 of G_2. Then $F^{\circ}(F^{-1}(W_2)) = W_2$.

(125) Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, and an F-defined walk W_1 of G_1. Then

(i) $W_1.\mathrm{length}() = (F^{\circ}W_1).\mathrm{length}()$, and

(ii) $\mathrm{len}\, W_1 = \mathrm{len}(F^{\circ}W_1)$.

(126) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and an F-valued walk W_2 of G_2. Then

(i) $W_2.\mathrm{length}() = (F^{-1}(W_2)).\mathrm{length}()$, and

(ii) $\mathrm{len}\, W_2 = \mathrm{len}(F^{-1}(W_2))$.

(127) Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, and an F-defined walk W_1 of G_1. Then

(i) $(F_{\mathbb{V}})(W_1.\mathrm{first}()) = (F^{\circ}W_1).\mathrm{first}()$, and

(ii) $(F_{\mathbb{V}})(W_1.\mathrm{last}()) = (F^{\circ}W_1).\mathrm{last}()$.

(128) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and an F-valued walk W_2 of G_2. Then

(i) $((F_{\mathbb{V}})^{-1})(W_2.\mathrm{first}()) = (F^{-1}(W_2)).\mathrm{first}()$, and

(ii) $((F_{\mathbb{V}})^{-1})(W_2.\mathrm{last}()) = (F^{-1}(W_2)).\mathrm{last}()$.

(129) Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, an F-defined walk W_1 of G_1, and an odd element n of $\mathbb{N}$. If $n \leqslant \mathrm{len}\, W_1$, then $(F_{\mathbb{V}})(W_1(n)) = (F^{\circ}W_1)(n)$. The theorem is a consequence of (125).

(130) Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, an F-defined walk W_1 of G_1, and an even element n of $\mathbb{N}$. Suppose $1 \leqslant n \leqslant \operatorname{len} W_1$. Then $(F_{\mathbb{E}})(W_1(n)) = (F^{\circ} W_1)(n)$. The theorem is a consequence of (125).

Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, an F-defined walk W_1 of G_1, and objects v, w. Now we state the propositions:

(131) If W_1 is walk from v to w, then $v, w \in \operatorname{dom}(F_{\mathbb{V}})$.

(132) If W_1 is walk from v to w, then $F^{\circ} W_1$ is walk from $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$. The theorem is a consequence of (129) and (125).

(133) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, an F-defined walk W_1 of G_1, and objects v, w. Then W_1 is walk from v to w if and only if $v, w \in \operatorname{dom}(F_{\mathbb{V}})$ and $F^{\circ} W_1$ is walk from $(F_{\mathbb{V}})(v)$ to $(F_{\mathbb{V}})(w)$. The theorem is a consequence of (131), (132), and (123).

(134) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and an F-defined walk W_1 of G_1. Suppose $(F_{\mathbb{V}})(W_1.\mathrm{first}()) = (F_{\mathbb{V}})(W_1.\mathrm{last}())$. Then $W_1.\mathrm{first}() = W_1.\mathrm{last}()$. The theorem is a consequence of (4).

Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, and an F-defined walk W_1 of G_1. Now we state the propositions:

(135) $(F^{\circ} W_1).\mathrm{vertices}() = (F_{\mathbb{V}})^{\circ}(W_1.\mathrm{vertices}())$.
PROOF: For every object y, $y \in \operatorname{rng}(F_{\mathbb{V}}) \cdot (W_1.\mathrm{vertexSeq}())$ iff $y \in (F_{\mathbb{V}})^{\circ}(W_1.\mathrm{vertices}())$. $\square$

(136) $(F^{\circ} W_1).\mathrm{edges}() = (F_{\mathbb{E}})^{\circ}(W_1.\mathrm{edges}())$.
PROOF: For every object y, $y \in \operatorname{rng}(F_{\mathbb{E}}) \cdot (W_1.\mathrm{edgeSeq}())$ iff $y \in (F_{\mathbb{E}})^{\circ}(W_1.\mathrm{edges}())$. $\square$

(137) (i) if W_1 is trivial, then $F^{\circ} W_1$ is trivial, and

(ii) if W_1 is closed, then $F^{\circ} W_1$ is closed, and

(iii) if $F^{\circ} W_1$ is trail-like, then W_1 is trail-like, and

(iv) if $F^{\circ} W_1$ is path-like, then W_1 is path-like.

PROOF: If $F^{\circ} W_1$ is trail-like, then W_1 is trail-like. For every odd elements m, n of $\mathbb{N}$ such that $m < n \leqslant \operatorname{len} W_1$ holds if $W_1(m) = W_1(n)$, then $m = 1$ and $n = \operatorname{len} W_1$. $\square$

(138) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and an F-defined walk W_1 of G_1. Then

(i) W_1 is trivial iff $F^{\circ} W_1$ is trivial, and

(ii) W_1 is closed iff $F^\circ W_1$ is closed, and

(iii) W_1 is trail-like iff $F^\circ W_1$ is trail-like, and

(iv) W_1 is path-like iff $F^\circ W_1$ is path-like, and

(v) W_1 is circuit-like iff $F^\circ W_1$ is circuit-like, and

(vi) W_1 is cycle-like iff $F^\circ W_1$ is cycle-like.

The theorem is a consequence of (123) and (137).

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(139) If F is strong subgraph embedding, then if G_2 is acyclic, then G_1 is acyclic. The theorem is a consequence of (121) and (138).

(140) Suppose F is isomorphism. Then

(i) G_1 is acyclic iff G_2 is acyclic, and

(ii) G_1 is chordal iff G_2 is chordal, and

(iii) G_1 is connected iff G_2 is connected.

Proof: F^{-1} is isomorphism and semi-continuous. For every vertices u, v of G_1, there exists a walk W_1 of G_1 such that W_1 is walk from u to v. □

5. Graph Mappings and Graph Modes

Let us consider graphs G_1, G_2, sets E_1, E_2, a graph G_3 given by reversing directions of the edges E_1 of G_1, a graph G_4 given by reversing directions of the edges E_2 of G_2, and a partial graph mapping F_0 from G_1 to G_2. Now we state the propositions:

(141) There exists a partial graph mapping F from G_3 to G_4 such that

(i) $F = F_0$, and

(ii) if F_0 is not empty, then F is not empty, and

(iii) if F_0 is total, then F is total, and

(iv) if F_0 is onto, then F is onto, and

(v) if F_0 is one-to-one, then F is one-to-one, and

(vi) if F_0 is semi-continuous, then F is semi-continuous, and

(vii) if F_0 is continuous, then F is continuous.

Proof: Reconsider $F = F_0$ as a partial graph mapping from G_3 to G_4. If F_0 is semi-continuous, then F is semi-continuous. If F_0 is continuous, then F is continuous by [13, (9)]. □

(142) There exists a partial graph mapping F from G_3 to G_4 such that

(i) $F = F_0$, and

(ii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(iii) if F_0 is strong subgraph embedding, then F is strong subgraph embedding, and

(iv) if F_0 is isomorphism, then F is isomorphism.

The theorem is a consequence of (141).

(143) Let us consider a graph G_1, a G_1-isomorphic graph G_2, sets E_1, E_2, and a graph G_3 given by reversing directions of the edges E_1 of G_1. Then every graph given by reversing directions of the edges E_2 of G_2 is G_3-isomorphic. The theorem is a consequence of (142).

Let us consider graphs G_3, G_4, sets V_1, V_2, a supergraph G_1 of G_3 extended by the vertices from V_1, a supergraph G_2 of G_4 extended by the vertices from V_2, a partial graph mapping F_0 from G_3 to G_4, and a one-to-one function f. Now we state the propositions:

(144) Suppose $\mathrm{dom}\, f = V_1 \setminus$ (the vertices of G_3) and $\mathrm{rng}\, f = V_2 \setminus$ (the vertices of G_4). Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} {+}{\cdot} f,\ F_{0\mathbb{E}} \rangle$, and

(ii) if F_0 is not empty, then F is not empty, and

(iii) if F_0 is total, then F is total, and

(iv) if F_0 is onto, then F is onto, and

(v) if F_0 is one-to-one, then F is one-to-one, and

(vi) if F_0 is directed, then F is directed, and

(vii) if F_0 is semi-continuous, then F is semi-continuous, and

(viii) if F_0 is continuous, then F is continuous, and

(ix) if F_0 is semi-directed-continuous, then F is semi-directed-continuous, and

(x) if F_0 is directed-continuous, then F is directed-continuous.

Proof: Set $h = F_{0\mathbb{V}} {+}{\cdot} f$. Reconsider $g = F_{0\mathbb{E}}$ as a partial function from the edges of G_1 to the edges of G_2. Reconsider $F = \langle h, g \rangle$ as a partial graph mapping from G_1 to G_2. If F_0 is total, then F is total. If F_0 is onto, then F is onto. If F_0 is directed, then F is directed. If F_0 is semi-continuous, then F is semi-continuous. If F_0 is continuous, then F is continuous. If F_0 is semi-directed-continuous, then F is semi-directed-continuous. If F_0 is directed-continuous, then F is directed-continuous. $\square$

(145) Suppose $\operatorname{dom} f = V_1 \setminus$ (the vertices of G_3) and $\operatorname{rng} f = V_2 \setminus$ (the vertices of G_4). Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} +\cdot f, F_{0\mathbb{E}} \rangle$, and

(ii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(iii) if F_0 is strong subgraph embedding, then F is strong subgraph embedding, and

(iv) if F_0 is isomorphism, then F is isomorphism, and

(v) if F_0 is directed-isomorphism, then F is directed-isomorphism.

The theorem is a consequence of (144).

(146) Let us consider a graph G_3, a G_3-isomorphic graph G_4, sets V_1, V_2, a supergraph G_1 of G_3 extended by the vertices from V_1, and a supergraph G_2 of G_4 extended by the vertices from V_2. Suppose $\overline{\overline{V_1 \setminus \alpha}} = \overline{\overline{V_2 \setminus \beta}}$. Then G_2 is G_1-isomorphic, where α is the vertices of G_3 and β is the vertices of G_4. The theorem is a consequence of (145).

(147) Let us consider a graph G_3, a G_3-directed-isomorphic graph G_4, sets V_1, V_2, a supergraph G_1 of G_3 extended by the vertices from V_1, and a supergraph G_2 of G_4 extended by the vertices from V_2. Suppose $\overline{\overline{V_1 \setminus \alpha}} = \overline{\overline{V_2 \setminus \beta}}$. Then G_2 is G_1-directed-isomorphic, where α is the vertices of G_3 and β is the vertices of G_4. The theorem is a consequence of (145).

Let us consider graphs G_3, G_4, objects v_1, v_2, a supergraph G_1 of G_3 extended by v_1, a supergraph G_2 of G_4 extended by v_2, and a partial graph mapping F_0 from G_3 to G_4. Now we state the propositions:

(148) Suppose $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} \rangle$, and

(ii) if F_0 is total, then F is total, and

(iii) if F_0 is onto, then F is onto, and

(iv) if F_0 is one-to-one, then F is one-to-one, and

(v) if F_0 is directed, then F is directed, and

(vi) if F_0 is semi-continuous, then F is semi-continuous, and

(vii) if F_0 is continuous, then F is continuous, and

(viii) if F_0 is semi-directed-continuous, then F is semi-directed-continuous, and

(ix) if F_0 is directed-continuous, then F is directed-continuous.

The theorem is a consequence of (144).

(149) Suppose $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} + \cdot (v_1 \mapsto v_2), F_{0\mathbb{E}} \rangle$, and

(ii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(iii) if F_0 is strong subgraph embedding, then F is strong subgraph embedding, and

(iv) if F_0 is isomorphism, then F is isomorphism, and

(v) if F_0 is directed-isomorphism, then F is directed-isomorphism.

The theorem is a consequence of (148).

(150) Let us consider a graph G_3, a G_3-isomorphic graph G_4, objects v_1, v_2, a supergraph G_1 of G_3 extended by v_1, and a supergraph G_2 of G_4 extended by v_2. Suppose $v_1 \in$ the vertices of G_3 iff $v_2 \in$ the vertices of G_4. Then G_2 is G_1-isomorphic. The theorem is a consequence of (146).

(151) Let us consider a graph G_3, a G_3-directed-isomorphic graph G_4, objects v_1, v_2, a supergraph G_1 of G_3 extended by v_1, and a supergraph G_2 of G_4 extended by v_2. Suppose $v_1 \in$ the vertices of G_3 iff $v_2 \in$ the vertices of G_4. Then G_2 is G_1-directed-isomorphic. The theorem is a consequence of (147).

Let us consider graphs G_3, G_4, vertices v_1, v_3 of G_3, vertices v_2, v_4 of G_4, objects e_1, e_2, a supergraph G_1 of G_3 extended by e_1 between vertices v_1 and v_3, a supergraph G_2 of G_4 extended by e_2 between vertices v_2 and v_4, and a partial graph mapping F_0 from G_3 to G_4. Now we state the propositions:

(152) Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1, v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $((F_{0\mathbb{V}})(v_1) = v_2$ and $(F_{0\mathbb{V}})(v_3) = v_4$ or $(F_{0\mathbb{V}})(v_1) = v_4$ and $(F_{0\mathbb{V}})(v_3) = v_2)$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}}, F_{0\mathbb{E}} + \cdot (e_1 \mapsto e_2) \rangle$, and

(ii) if F_0 is total, then F is total, and

(iii) if F_0 is onto, then F is onto, and

(iv) if F_0 is one-to-one, then F is one-to-one.

The theorem is a consequence of (5), (4), and (8).

(153) Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1, v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $((F_{0\mathbb{V}})(v_1) = v_2$ and $(F_{0\mathbb{V}})(v_3) = v_4$ or $(F_{0\mathbb{V}})(v_1) = v_4$ and

$(F_{0\mathbb{V}})(v_3) = v_2)$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}}, F_{0\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2)\rangle$, and

(ii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(iii) if F_0 is isomorphism, then F is isomorphism.

The theorem is a consequence of (152).

(154) Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and v_1, $v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $(F_{0\mathbb{V}})(v_1) = v_2$ and $(F_{0\mathbb{V}})(v_3) = v_4$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}}, F_{0\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2)\rangle$, and

(ii) if F_0 is directed, then F is directed, and

(iii) if F_0 is directed-isomorphism, then F is directed-isomorphism.

Proof: Consider F being a partial graph mapping from G_1 to G_2 such that $F = \langle F_{0\mathbb{V}}, F_{0\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2)\rangle$ and if F_0 is total, then F is total and if F_0 is onto, then F is onto and if F_0 is one-to-one, then F is one-to-one. If F_0 is directed, then F is directed by [15, (16)], [12, (71),(70),(106)]. □

Let us consider graphs G_3, G_4, a vertex v_3 of G_3, a vertex v_4 of G_4, objects e_1, e_2, v_1, v_2, a supergraph G_1 of G_3 extended by v_1, v_3 and e_1 between them, a supergraph G_2 of G_4 extended by v_2, v_4 and e_2 between them, and a partial graph mapping F_0 from G_3 to G_4. Now we state the propositions:

(155) Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4 and $v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $(F_{0\mathbb{V}})(v_3) = v_4$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2)\rangle$, and

(ii) if F_0 is total, then F is total, and

(iii) if F_0 is onto, then F is onto, and

(iv) if F_0 is one-to-one, then F is one-to-one, and

(v) if F_0 is directed, then F is directed.

Proof: Consider G_5 being a supergraph of G_3 extended by v_1 such that G_1 is a supergraph of G_5 extended by e_1 between vertices v_1 and v_3. Consider G_6 being a supergraph of G_4 extended by v_2 such that G_2 is a supergraph of G_6 extended by e_2 between vertices v_2 and v_4.

Consider F_1 being a partial graph mapping from G_5 to G_6 such that $F_1 = \langle F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}}\rangle$ and if F_0 is total, then F_1 is total and if F_0

is onto, then F_1 is onto and if F_0 is one-to-one, then F_1 is one-to-one and if F_0 is directed, then F_1 is directed and if F_0 is semi-continuous, then F_1 is semi-continuous and if F_0 is continuous, then F_1 is continuous and if F_0 is semi-directed-continuous, then F_1 is semi-directed-continuous and if F_0 is directed-continuous, then F_1 is directed-continuous. v_1, $v_3 \in \mathrm{dom}(F_{1\mathbb{V}})$ and $(F_{1\mathbb{V}})(v_1) = v_2$ and $(F_{1\mathbb{V}})(v_3) = v_4$.

Consider F_2 being a partial graph mapping from G_1 to G_2 such that $F_2 = \langle F_{1\mathbb{V}}, F_{1\mathbb{E}} {+\cdot} (e_1 \dot{\longmapsto} e_2)\rangle$ and if F_1 is total, then F_2 is total and if F_1 is onto, then F_2 is onto and if F_1 is one-to-one, then F_2 is one-to-one. Consider F_3 being a partial graph mapping from G_1 to G_2 such that $F_3 = \langle F_{1\mathbb{V}}, F_{1\mathbb{E}} {+\cdot} (e_1 \dot{\longmapsto} e_2)\rangle$ and if F_1 is directed, then F_3 is directed and if F_1 is directed-isomorphism, then F_3 is directed-isomorphism. $\square$

(156) Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4 and $v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $(F_{0\mathbb{V}})(v_3) = v_4$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} {+\cdot} (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} {+\cdot} (e_1 \dot{\longmapsto} e_2)\rangle$, and

(ii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(iii) if F_0 is isomorphism, then F is isomorphism, and

(iv) if F_0 is directed-isomorphism, then F is directed-isomorphism.

The theorem is a consequence of (155).

Let us consider graphs G_3, G_4, a vertex v_3 of G_3, a vertex v_4 of G_4, objects e_1, e_2, v_1, v_2, a supergraph G_1 of G_3 extended by v_3, v_1 and e_1 between them, a supergraph G_2 of G_4 extended by v_4, v_2 and e_2 between them, and a partial graph mapping F_0 from G_3 to G_4. Now we state the propositions:

(157) Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4 and $v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $(F_{0\mathbb{V}})(v_3) = v_4$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} {+\cdot} (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} {+\cdot} (e_1 \dot{\longmapsto} e_2)\rangle$, and

(ii) if F_0 is total, then F is total, and

(iii) if F_0 is onto, then F is onto, and

(iv) if F_0 is one-to-one, then F is one-to-one, and

(v) if F_0 is directed, then F is directed.

PROOF: Consider G_5 being a supergraph of G_3 extended by v_1 such that G_1 is a supergraph of G_5 extended by e_1 between vertices v_3 and v_1.

Consider G_6 being a supergraph of G_4 extended by v_2 such that G_2 is a supergraph of G_6 extended by e_2 between vertices v_4 and v_2.

Consider F_1 being a partial graph mapping from G_5 to G_6 such that $F_1 = \langle F_{0\mathbb{V}} +\cdot (v_1 \stackrel{\cdot}{\longmapsto} v_2), F_{0\mathbb{E}}\rangle$ and if F_0 is total, then F_1 is total and if F_0 is onto, then F_1 is onto and if F_0 is one-to-one, then F_1 is one-to-one and if F_0 is directed, then F_1 is directed and if F_0 is semi-continuous, then F_1 is semi-continuous and if F_0 is continuous, then F_1 is continuous and if F_0 is semi-directed-continuous, then F_1 is semi-directed-continuous and if F_0 is directed-continuous, then F_1 is directed-continuous. $v_1, v_3 \in \mathrm{dom}(F_{1\mathbb{V}})$ and $(F_{1\mathbb{V}})(v_1) = v_2$ and $(F_{1\mathbb{V}})(v_3) = v_4$.

Consider F_2 being a partial graph mapping from G_1 to G_2 such that $F_2 = \langle F_{1\mathbb{V}}, F_{1\mathbb{E}} +\cdot (e_1 \stackrel{\cdot}{\longmapsto} e_2)\rangle$ and if F_1 is total, then F_2 is total and if F_1 is onto, then F_2 is onto and if F_1 is one-to-one, then F_2 is one-to-one. Consider F_3 being a partial graph mapping from G_1 to G_2 such that $F_3 = \langle F_{1\mathbb{V}}, F_{1\mathbb{E}} +\cdot (e_1 \stackrel{\cdot}{\longmapsto} e_2)\rangle$ and if F_1 is directed, then F_3 is directed and if F_1 is directed-isomorphism, then F_3 is directed-isomorphism. □

(158) Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4 and $v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $(F_{0\mathbb{V}})(v_3) = v_4$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} +\cdot (v_1 \stackrel{\cdot}{\longmapsto} v_2), F_{0\mathbb{E}} +\cdot (e_1 \stackrel{\cdot}{\longmapsto} e_2)\rangle$, and

(ii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(iii) if F_0 is isomorphism, then F is isomorphism, and

(iv) if F_0 is directed-isomorphism, then F is directed-isomorphism.

The theorem is a consequence of (157).

(159) Let us consider graphs G_3, G_4, a vertex v_3 of G_3, a vertex v_4 of G_4, objects e_1, e_2, v_1, v_2, a supergraph G_1 of G_3 extended by v_1, v_3 and e_1 between them, a supergraph G_2 of G_4 extended by v_4, v_2 and e_2 between them, and a partial graph mapping F_0 from G_3 to G_4. Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4 and $v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $(F_{0\mathbb{V}})(v_3) = v_4$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} +\cdot (v_1 \stackrel{\cdot}{\longmapsto} v_2), F_{0\mathbb{E}} +\cdot (e_1 \stackrel{\cdot}{\longmapsto} e_2)\rangle$, and

(ii) if F_0 is total, then F is total, and

(iii) if F_0 is onto, then F is onto, and

(iv) if F_0 is one-to-one, then F is one-to-one, and

(v) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(vi) if F_0 is isomorphism, then F is isomorphism.

PROOF: Consider G_5 being a supergraph of G_3 extended by v_1 such that G_1 is a supergraph of G_5 extended by e_1 between vertices v_1 and v_3. Consider G_6 being a supergraph of G_4 extended by v_2 such that G_2 is a supergraph of G_6 extended by e_2 between vertices v_4 and v_2.

Consider F_1 being a partial graph mapping from G_5 to G_6 such that $F_1 = \langle F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} \rangle$ and if F_0 is total, then F_1 is total and if F_0 is onto, then F_1 is onto and if F_0 is one-to-one, then F_1 is one-to-one and if F_0 is directed, then F_1 is directed and if F_0 is semi-continuous, then F_1 is semi-continuous and if F_0 is continuous, then F_1 is continuous and if F_0 is semi-directed-continuous, then F_1 is semi-directed-continuous and if F_0 is directed-continuous, then F_1 is directed-continuous. $v_1, v_3 \in \mathrm{dom}(F_{1\mathbb{V}})$ and $(F_{1\mathbb{V}})(v_1) = v_2$ and $(F_{1\mathbb{V}})(v_3) = v_4$.

Consider F_2 being a partial graph mapping from G_1 to G_2 such that $F_2 = \langle F_{1\mathbb{V}}, F_{1\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2) \rangle$ and if F_1 is total, then F_2 is total and if F_1 is onto, then F_2 is onto and if F_1 is one-to-one, then F_2 is one-to-one. □

(160) Let us consider graphs G_3, G_4, a vertex v_3 of G_3, a vertex v_4 of G_4, objects e_1, e_2, v_1, v_2, a supergraph G_1 of G_3 extended by v_3, v_1 and e_1 between them, a supergraph G_2 of G_4 extended by v_2, v_4 and e_2 between them, and a partial graph mapping F_0 from G_3 to G_4. Suppose $e_1 \notin$ the edges of G_3 and $e_2 \notin$ the edges of G_4 and $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4 and $v_3 \in \mathrm{dom}(F_{0\mathbb{V}})$ and $(F_{0\mathbb{V}})(v_3) = v_4$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F = \langle F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2) \rangle$, and

(ii) if F_0 is total, then F is total, and

(iii) if F_0 is onto, then F is onto, and

(iv) if F_0 is one-to-one, then F is one-to-one, and

(v) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(vi) if F_0 is isomorphism, then F is isomorphism.

PROOF: Consider G_5 being a supergraph of G_3 extended by v_1 such that G_1 is a supergraph of G_5 extended by e_1 between vertices v_3 and v_1. Consider G_6 being a supergraph of G_4 extended by v_2 such that G_2 is a supergraph of G_6 extended by e_2 between vertices v_2 and v_4.

Consider F_1 being a partial graph mapping from G_5 to G_6 such that $F_1 = \langle F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2), F_{0\mathbb{E}} \rangle$ and if F_0 is total, then F_1 is total and if F_0

is onto, then F_1 is onto and if F_0 is one-to-one, then F_1 is one-to-one and if F_0 is directed, then F_1 is directed and if F_0 is semi-continuous, then F_1 is semi-continuous and if F_0 is continuous, then F_1 is continuous and if F_0 is semi-directed-continuous, then F_1 is semi-directed-continuous and if F_0 is directed-continuous, then F_1 is directed-continuous. v_1, $v_3 \in \mathrm{dom}(F_{1\mathbb{V}})$ and $(F_{1\mathbb{V}})(v_1) = v_2$ and $(F_{1\mathbb{V}})(v_3) = v_4$.

Consider F_2 being a partial graph mapping from G_1 to G_2 such that $F_2 = \langle F_{1\mathbb{V}}, F_{1\mathbb{E}} +\cdot (e_1 \dot{\longmapsto} e_2) \rangle$ and if F_1 is total, then F_2 is total and if F_1 is onto, then F_2 is onto and if F_1 is one-to-one, then F_2 is one-to-one. $\square$

(161) Let us consider a graph G, an object v, a set V, and supergraphs G_1, G_2 of G extended by vertex v and edges between v and V of G. Then G_2 is G_1-isomorphic. The theorem is a consequence of (8), (53), and (143).

(162) Let us consider graphs G_3, G_4, objects v_1, v_2, sets V_1, V_2, a supergraph G_1 of G_3 extended by vertex v_1 and edges between v_1 and V_1 of G_3, a supergraph G_2 of G_4 extended by vertex v_2 and edges between v_2 and V_2 of G_4, and a partial graph mapping F_0 from G_3 to G_4. Suppose $V_1 \subseteq$ the vertices of G_3 and $V_2 \subseteq$ the vertices of G_4 and $v_1 \notin$ the vertices of G_3 and $v_2 \notin$ the vertices of G_4 and $F_{0\mathbb{V}} \restriction V_1$ is one-to-one and $\mathrm{dom}(F_{0\mathbb{V}} \restriction V_1) = V_1$ and $\mathrm{rng}(F_{0\mathbb{V}} \restriction V_1) = V_2$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F_{\mathbb{V}} = F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2)$, and

(ii) $F_{\mathbb{E}} \restriction \mathrm{dom}(F_{0\mathbb{E}}) = F_{0\mathbb{E}}$, and

(iii) if F_0 is total, then F is total, and

(iv) if F_0 is onto, then F is onto, and

(v) if F_0 is one-to-one, then F is one-to-one, and

(vi) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(vii) if F_0 is isomorphism, then F is isomorphism.

Proof: $V_1 \subseteq \mathrm{dom}(F_{0\mathbb{V}})$. Set $f = F_{0\mathbb{V}} +\cdot (v_1 \dot{\longmapsto} v_2)$. Consider h_1 being a function from V_1 into G_1.edgesBetween(V_1, $\{v_1\}$) such that h_1 is one-to-one and onto and for every object w such that $w \in V_1$ holds $h_1(w)$ joins w and v_1 in G_1. Consider h_2 being a function from V_2 into G_2.edgesBetween(V_2, $\{v_2\}$) such that h_2 is one-to-one and onto and for every object w such that $w \in V_2$ holds $h_2(w)$ joins w and v_2 in G_2. Set $g = F_{0\mathbb{E}} +\cdot h_2 \cdot (F_{0\mathbb{V}}) \cdot ({h_1}^{-1})$.

$\mathrm{dom}(F_{0\mathbb{E}})$ misses $\mathrm{dom}(h_2 \cdot (F_{0\mathbb{V}}) \cdot ({h_1}^{-1}))$. $\mathrm{rng}\, F_{0\mathbb{E}}$ misses $\mathrm{rng}\, h_2 \cdot (F_{0\mathbb{V}}) \cdot ({h_1}^{-1})$. Consider E_1 being a set such that $\overline{\overline{V_1}} = \overline{\overline{E_1}}$ and E_1 misses the edges of G_3 and the edges of $G_1 =$ (the edges of G_3) $\cup\, E_1$ and for every object w_1 such that $w_1 \in V_1$ there exists an object e_1 such that $e_1 \in E_1$ and e_1

joins w_1 and v_1 in G_1 and for every object $\tilde{e}$ such that $\tilde{e}$ joins w_1 and v_1 in G_1 holds $e_1 = \tilde{e}$.

Consider E_2 being a set such that $\overline{\overline{V_2}} = \overline{\overline{E_2}}$ and E_2 misses the edges of G_4 and the edges of $G_2 =$ (the edges of G_4) $\cup E_2$ and for every object w_2 such that $w_2 \in V_2$ there exists an object e_2 such that $e_2 \in E_2$ and e_2 joins w_2 and v_2 in G_2 and for every object $\tilde{e}$ such that $\tilde{e}$ joins w_2 and v_2 in G_2 holds $e_2 = \tilde{e}$. Reconsider $F = \langle f, g\rangle$ as a partial graph mapping from G_1 to G_2. If F_0 is total, then F is total. If F_0 is onto, then F is onto. □

(163) Let us consider a graph G_3, a G_3-isomorphic graph G_4, objects v_1, v_2, a supergraph G_1 of G_3 extended by vertex v_1 and edges between v_1 and the vertices of G_3, and a supergraph G_2 of G_4 extended by vertex v_2 and edges between v_2 and the vertices of G_4. Suppose $v_1 \in$ the vertices of G_3 iff $v_2 \in$ the vertices of G_4. Then G_2 is G_1-isomorphic. The theorem is a consequence of (162) and (143).

Let us consider graphs G_1, G_2, a subgraph G_3 of G_1 with loops removed, a subgraph G_4 of G_2 with loops removed, and a one-to-one partial graph mapping F_0 from G_1 to G_2. Now we state the propositions:

(164) There exists a one-to-one partial graph mapping F from G_3 to G_4 such that

(i) $F = F_0 \restriction G_3$, and

(ii) if F_0 is total, then F is total, and

(iii) if F_0 is onto, then F is onto, and

(iv) if F_0 is directed, then F is directed, and

(v) if F_0 is semi-directed-continuous, then F is semi-directed-continuous.

Proof: Reconsider $F = G_4 \upharpoonleft (F_0 \restriction G_3)$ as a one-to-one partial graph mapping from G_3 to G_4. If F_0 is total, then F is total. If F_0 is onto, then F is onto. □

(165) There exists a one-to-one partial graph mapping F from G_3 to G_4 such that

(i) $F = F_0 \restriction G_3$, and

(ii) if F_0 is weak subgraph embedding, then F is weak subgraph embedding, and

(iii) if F_0 is isomorphism, then F is isomorphism, and

(iv) if F_0 is directed-isomorphism, then F is directed-isomorphism.

The theorem is a consequence of (164).

(166) Let us consider a graph G_1, a G_1-isomorphic graph G_2, and a subgraph G_3 of G_1 with loops removed. Then every subgraph of G_2 with loops removed is G_3-isomorphic. The theorem is a consequence of (165).

(167) Let us consider a graph G_1, a G_1-directed-isomorphic graph G_2, and a subgraph G_3 of G_1 with loops removed. Then every subgraph of G_2 with loops removed is G_3-directed-isomorphic. The theorem is a consequence of (165).

(168) Let us consider a graph G_1, a G_1-isomorphic graph G_2, and a subgraph G_3 of G_1 with parallel edges removed. Then every subgraph of G_2 with parallel edges removed is G_3-isomorphic.

PROOF: Consider G being a partial graph mapping from G_1 to G_2 such that G is isomorphism. Consider E_1 being a representative selection of the parallel edges of G_1 such that G_3 is a subgraph of G_1 induced by the vertices of G_1 and E_1.

Consider E_2 being a representative selection of the parallel edges of G_2 such that G_4 is a subgraph of G_2 induced by the vertices of G_2 and E_2. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 \in E_2$ and $\langle \$_1, \$_2 \rangle \in \text{EdgeParEqRel}(G_2)$. For every objects x, y_1, y_2 such that $x \in$ the edges of G_2 and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that $x \in$ the edges of G_2 there exists an object y such that $\mathcal{P}[x, y]$.

Consider h being a function such that $\operatorname{dom} h =$ the edges of G_2 and for every object x such that $x \in$ the edges of G_2 holds $\mathcal{P}[x, h(x)]$. $\square$

(169) Let us consider a graph G_1, and subgraphs G_2, G_3 of G_1 with parallel edges removed. Then G_3 is G_2-isomorphic. The theorem is a consequence of (53) and (168).

(170) Let us consider a graph G_1, a G_1-directed-isomorphic graph G_2, and a subgraph G_3 of G_1 with directed-parallel edges removed. Then every subgraph of G_2 with directed-parallel edges removed is G_3-directed-isomorphic.

PROOF: Consider G being a partial graph mapping from G_1 to G_2 such that G is directed-isomorphism. Consider E_1 being a representative selection of the directed-parallel edges of G_1 such that G_3 is a subgraph of G_1 induced by the vertices of G_1 and E_1.

Consider E_2 being a representative selection of the directed-parallel edges of G_2 such that G_4 is a subgraph of G_2 induced by the vertices of G_2 and E_2. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 \in E_2$ and $\langle \$_1, \$_2 \rangle \in \text{DEdgeParEqRel}(G_2)$. For every objects x, y_1, y_2 such that $x \in$ the edges of G_2 and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that $x \in$ the edges of G_2 there exists an object y such that $\mathcal{P}[x, y]$.

Consider h being a function such that $\operatorname{dom} h =$ the edges of G_2 and for every object x such that $x \in$ the edges of G_2 holds $\mathcal{P}[x, h(x)]$. $\square$

(171) Let us consider a graph G_1, and subgraphs G_2, G_3 of G_1 with directed-parallel edges removed. Then G_3 is G_2-directed-isomorphic. The theorem is a consequence of (53) and (170).

(172) Let us consider a graph G_1, a G_1-isomorphic graph G_2, and a simple graph G_3 of G_1. Then every simple graph of G_2 is G_3-isomorphic. The theorem is a consequence of (166) and (168).

(173) Let us consider a graph G_1, and simple graphs G_2, G_3 of G_1. Then G_3 is G_2-isomorphic. The theorem is a consequence of (53) and (172).

(174) Let us consider a graph G_1, a G_1-directed-isomorphic graph G_2, and a directed-simple graph G_3 of G_1. Then every directed-simple graph of G_2 is G_3-directed-isomorphic. The theorem is a consequence of (167) and (170).

(175) Let us consider a graph G_1, and directed-simple graphs G_2, G_3 of G_1. Then G_3 is G_2-directed-isomorphic. The theorem is a consequence of (53) and (174).

(176) Let us consider trivial, loopless graphs G_1, G_2, and a non empty partial graph mapping F from G_1 to G_2. Then

  (i) F is directed-isomorphism, and

  (ii) $F = \langle \text{the vertex of } G_1 \dot{\longmapsto} \text{the vertex of } G_2, \emptyset \rangle$.

(177) Let us consider trivial graphs G_1, G_2. Suppose G_1.size() $= G_2$.size(). Then there exists a partial graph mapping F from G_1 to G_2 such that F is directed-isomorphism. The theorem is a consequence of (31).

(178) Let us consider trivial, loopless graphs G_1, G_2. Then G_2 is G_1-directed-isomorphic and G_1-isomorphic.

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Accepted August 29, 2019