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Diophantine Sets. Part II

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Summary. The article is the next in a series aiming to formalize the MDPR-theorem using the Mizar proof assistant [3], [6], [4]. We analyze four equations from the Diophantine standpoint that are crucial in the bounded quantifier theorem, that is used in one of the approaches to solve the problem.

Based on our previous work [1], we prove that the value of a given binomial coefficient and factorial can be determined by its arguments in a Diophantine way. Then we prove that two products

$$z = \prod_{i=1}^{x}(1 + i \cdot y), \qquad z = \prod_{i=1}^{x}(y + 1 - j), \tag{0.1}$$

where $y > x$ are Diophantine.

The formalization follows [10], Z. Adamowicz, P. Zbierski [2] as well as M. Davis [5].

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1. Product of Zero Based Finite Sequences

From now on i, j, n, n_1, n_2, m, k, l, u denote natural numbers, r, r_1, r_2 denote real numbers, x, y denote integers, a, b denote non trivial natural numbers, F denotes a finite 0-sequence, $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$ denote complex-valued finite 0-sequences, and c, c_1, c_2 denote complex numbers.

Let us consider c_1 and c_2. Let us note that $\langle c_1, c_2 \rangle$ is complex-valued.

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Let $\mathcal{F}$ be a finite 0-sequence. The functor $\prod \mathcal{F}$ yielding an element of $\mathbb{C}$ is defined by the term

(Def. 1) $\cdot_{\mathbb{C}} \odot \mathcal{F}$.

Now we state the propositions:

(1) If $\mathcal{F}$ is real-valued, then $\prod \mathcal{F} = \cdot_{\mathbb{R}} \odot \mathcal{F}$.

(2) If $\mathcal{F}$ is $\mathbb{Z}$-valued, then $\prod \mathcal{F} = \cdot_{\mathbb{Z}} \odot \mathcal{F}$.

(3) If $\mathcal{F}$ is natural-valued, then $\prod \mathcal{F} = \cdot_{\mathbb{N}} \odot \mathcal{F}$.

Let F be a real-valued finite 0-sequence. One can check that $\prod F$ is real.

Let F be a natural-valued finite 0-sequence. One can verify that $\prod F$ is natural.

Now we state the propositions:

(4) If $\mathcal{F} = \emptyset$, then $\prod \mathcal{F} = 1$.

(5) $\prod \langle c \rangle = c$.

(6) $\prod \langle c_1, c_2 \rangle = c_1 \cdot c_2$.

(7) $\prod (\mathcal{F}_1 \frown \mathcal{F}_2) = (\prod \mathcal{F}_1) \cdot (\prod \mathcal{F}_2)$.

(8) $c + \mathcal{F}_1 \frown \mathcal{F}_2 = (c + \mathcal{F}_1) \frown (c + \mathcal{F}_2)$.

PROOF: For every object x such that $x \in \operatorname{dom}(c + \mathcal{F}_1 \frown \mathcal{F}_2)$ holds $(c + \mathcal{F}_1 \frown \mathcal{F}_2)(x) = ((c + \mathcal{F}_1) \frown (c + \mathcal{F}_2))(x)$. $\square$

(9) $c_1 + \langle c_2 \rangle = \langle c_1 + c_2 \rangle$.

(10) Let us consider finite 0-sequences f_1, f_2, and n. Suppose $n \leqslant \operatorname{len} f_1$. Then $(f_1 \frown f_2)_{\restriction n} = f_{1\,\restriction n} \frown f_2$.

Let us consider n. One can verify that there exists a finite 0-sequence which is n-element and natural-valued and there exists a finite 0-sequence which is natural-valued and positive yielding.

Let R be a positive yielding binary relation and X be a set. Observe that $R \restriction X$ is positive yielding.

Let X be a positive yielding, real-valued finite 0-sequence. One can verify that $\prod X$ is positive.

Now we state the proposition:

(11) Let us consider a natural-valued, positive yielding finite 0-sequence X. If $i \in \operatorname{dom} X$, then $X(i) \leqslant \prod X$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every natural-valued, positive yielding finite 0-sequence X for every natural number i such that $\operatorname{len} X = \$_1$ and $i \in \operatorname{dom} X$ holds $X(i) \leqslant \prod X$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[n]$. $\square$

Let X be a natural-valued finite 0-sequence and n be a positive natural number. Let us observe that $n + X$ is positive yielding.

Now we state the proposition:

(12) Let us consider natural-valued finite 0-sequences X_1, X_2. Suppose $\operatorname{len} X_1 = \operatorname{len} X_2$ and for every n such that $n \in \operatorname{dom} X_1$ holds $X_1(n) \leqslant X_2(n)$. Then $\prod X_1 \leqslant \prod X_2$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every natural-valued finite 0-sequences X_1, X_2 such that $\operatorname{len} X_1 = \$_1 = \operatorname{len} X_2$ and for every n such that $n \in \operatorname{dom} X_1$ holds $X_1(n) \leqslant X_2(n)$ holds $\prod X_1 \leqslant \prod X_2$. $\mathcal{P}[0]$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[n]$. □

2. BINOMIAL IS DIOPHANTINE

Now we state the propositions:

(13) If $k \leqslant n$, then $\binom{n}{k} \leqslant n^k$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant n$, then $\binom{n}{\$_1} \leqslant n^{\$_1}$. If $\mathcal{P}[m]$, then $\mathcal{P}[m+1]$. $\mathcal{P}[m]$. □

(14) If $u > n^k$ and $n \geqslant k > i$, then $\binom{n}{i} \cdot (u^i) < \frac{u^k}{n}$. The theorem is a consequence of (13).

(15) If $u > n^k$ and $n \geqslant k$, then $\lfloor \frac{(u+1)^n}{u^k} \rfloor \bmod u = \binom{n}{k}$.

PROOF: Set $I = \langle \binom{n}{0} 1^0 u^n, \ldots, \binom{n}{n} 1^n u^0 \rangle$. Set $k_1 = k + 1$. Consider q being a finite sequence such that $I = (I \restriction k_1) \frown q$. Reconsider $I_1 = I$ as a finite sequence of elements of $\mathbb{N}$. Set $k_2 = k \mapsto \frac{u^k}{n}$. For every natural number i such that $i \in \operatorname{Seg} k$ holds $(I_1 \restriction k)(i) < k_2(i)$. Define $\mathcal{P}[\text{natural number}, \text{object}] \equiv \$_2 \in \mathbb{N}$ and for every natural number i such that $i = \$_2$ holds $q(\$_1) = u^k \cdot u \cdot i$. For every natural number j such that $j \in \operatorname{Seg} \operatorname{len} q$ there exists an object x such that $\mathcal{P}[j, x]$. Consider Q being a finite sequence such that $\operatorname{dom} Q = \operatorname{Seg} \operatorname{len} q$ and for every natural number j such that $j \in \operatorname{Seg} \operatorname{len} q$ holds $\mathcal{P}[j, Q(j)]$. $\operatorname{rng} Q \subseteq \mathbb{N}$. For every natural number i such that $1 \leqslant i \leqslant \operatorname{len} q$ holds $q(i) = (u^k \cdot u \cdot Q)(i)$. $\lfloor \frac{\sum I_1}{u^k} \rfloor = \binom{n}{k} + u \cdot (\sum Q)$. $\binom{n}{k} \leqslant n^k$. □

(16) Let us consider natural numbers x, y, z. Then $x \geqslant z$ and $y = \binom{x}{z}$ if and only if there exist natural numbers u, v, y_1, y_2, y_3 such that $y_1 = x^z$ and $y_2 = (u+1)^x$ and $y_3 = u^z$ and $u > y_1$ and $v = \lfloor \frac{y_2}{y_3} \rfloor$ and $y \equiv v \pmod{u}$ and $y < u$ and $x \geqslant z$.

PROOF: If $x \geqslant z$ and $y = \binom{x}{z}$, then there exist natural numbers u, v, y_1, y_2, y_3 such that $y_1 = x^z$ and $y_2 = (u+1)^x$ and $y_3 = u^z$ and $u > y_1$ and $v = \lfloor \frac{y_2}{y_3} \rfloor$ and $y \equiv v \pmod{u}$ and $y < u$ and $x \geqslant z$. $y \bmod u = \binom{x}{z}$. □

3. Factorial is Diophantine

Now we state the propositions:

(17) If $k > 0$ and $n > 2 \cdot k^{k+1}$, then $k! = \lfloor \frac{n^k}{\binom{n}{k}} \rfloor$.

(18) Let us consider natural numbers x, y. Then $y = x!$ if and only if there exist natural numbers n, y_1, y_2, y_3 such that $y_1 = 2 \cdot x^{x+1}$ and $y_2 = n^x$ and $y_3 = \binom{n}{x}$ and $n > y_1$ and $y = \lfloor \frac{y_2}{y_3} \rfloor$.
Proof: If $y = x!$, then there exist natural numbers n, y_1, y_2, y_3 such that $y_1 = 2 \cdot x^{x+1}$ and $y_2 = n^x$ and $y_3 = \binom{n}{x}$ and $n > y_1$ and $y = \lfloor \frac{y_2}{y_3} \rfloor$. □

4. Diophanticity of Selected Products

In the sequel x, y, x_1, u, w denote natural numbers.

Now we state the propositions:

(19) Let us consider natural numbers x_1, w, u. Suppose $x_1 \cdot w \equiv 1 \pmod u$. Let us consider a natural number x. Then $\prod(1 + x_1 \cdot (\mathrm{idseq}(x))) \equiv {x_1}^x \cdot (x!) \cdot \binom{w+x}{x} \pmod u$.
Proof: Consider b being an integer such that $u \cdot b = x_1 \cdot w - 1$. Define $\mathcal{P}[\text{natural number}] \equiv \prod(1 + x_1 \cdot (\mathrm{idseq}(\$_1))) \equiv {x_1}^{\$_1} \cdot (\$_1!) \cdot \binom{w+\$_1}{\$_1} \pmod u$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by [12, (43)]. $\mathcal{P}[n]$. □

(20) Let us consider natural numbers x, y, x_1. Suppose $x_1 \geqslant 1$. Then $y = \prod(1 + x_1 \cdot (\mathrm{idseq}(x)))$ if and only if there exist natural numbers u, w, y_1, y_2, y_3, y_4, y_5 such that $u > y$ and $x_1 \cdot w \equiv 1 \pmod u$ and $y_1 = {x_1}^x$ and $y_2 = x!$ and $y_3 = \binom{w+x}{x}$ and $y_1 \cdot y_2 \cdot y_3 \equiv y \pmod u$ and $y_4 = 1 + x_1 \cdot x$ and $y_5 = {y_4}^x$ and $u > y_5$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv (1 + x_1 \cdot \$_1)^{\$_1} \geqslant \prod(1+x_1 \cdot (\mathrm{idseq}(\$_1)))$. If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$. $\mathcal{P}[n]$. If $y = \prod(1+x_1 \cdot (\mathrm{idseq}(x)))$, then there exist natural numbers u, w, y_1, y_2, y_3, y_4, y_5 such that $u > y$ and $x_1 \cdot w \equiv 1 \pmod u$ and $y_1 = {x_1}^x$ and $y_2 = x!$ and $y_3 = \binom{w+x}{x}$ and $y_1 \cdot y_2 \cdot y_3 \equiv y \pmod u$ and $y_4 = 1+x_1 \cdot x$ and $y_5 = {y_4}^x$ and $u > y_5$ by [8, (16)]. Set $U = {x_1}^x \cdot (x!) \cdot \binom{w+x}{x}$. $\prod(1 + x_1 \cdot (\mathrm{idseq}(x))) \equiv U \pmod u$. □

(21) $c_1 + n \mapsto c_2 = n \mapsto (c_1 + c_2)$.

(22) Let us consider natural numbers x, y, x_1. If $x_1 = 0$, then $y = \prod(1 + x_1 \cdot (\mathrm{idseq}(x)))$ iff $y = 1$. The theorem is a consequence of (21).

(23) If $n \geqslant k$, then $\prod(n + 1 + -\mathrm{idseq}(k)) = k! \cdot \binom{n}{k}$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant n$, then $\prod(n+1+-\mathrm{idseq}(\$_1)) = \$_1! \cdot \binom{n}{\$_1}$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [7, (3), (2)]. $\mathcal{P}[i]$. □

(24) Let us consider natural numbers y, x_1, x_2. Then $y = \prod(x_2 + 1 + -\mathrm{idseq}(x_1))$ and $x_2 > x_1$ if and only if $y = x_1! \cdot \binom{x_2}{x_1}$ and $x_2 > x_1$.

5. Selected Subsets of Zero Based Finite Sequences of $\mathbb{N}$ as Diophantine Sets

From now on n, m, k denote natural numbers, p, q denote n-element finite 0-sequences of $\mathbb{N}$, i_1, i_2, i_3, i_4, i_5, i_6 denote elements of n, and a, b, d, f denote integers.

Now we state the propositions:

(25) Let us consider natural numbers a, b, i_1, i_2, and i_3. Then $\{p : p(i_1) = (a \cdot p(i_2) + b) \cdot p(i_3)\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.
Proof: Define $\mathcal{R}(\text{natural number}, \text{natural number}, \text{natural number}) = a \cdot \$_1 + b$. Define $\mathcal{P}_1[\text{natural number}, \text{natural number}, \text{natural object}, \text{natural number}, \text{natural number}, \text{natural number}] \equiv 1 \cdot \$_1 = 1 \cdot \$_3 \cdot \$_2$. For every n, i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}_1[p(i_1), p(i_2), \mathcal{R}(p(i_3), p(i_4), p(i_5)), p(i_3), p(i_4), p(i_5)]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{Q}_1[\text{finite 0-sequence of } \mathbb{N}] \equiv \mathcal{P}_1[\$_1(i_1), \$_1(i_3), a \cdot \$_1(i_2) + b, \$_1(i_3), \$_1(i_3), \$_1(i_3)]$. Define $\mathcal{Q}_2[\text{finite 0-sequence of } \mathbb{N}] \equiv \$_1(i_1) = (a \cdot \$_1(i_2) + b) \cdot \$_1(i_3)$. $\{p : \mathcal{Q}_1[p]\} = \{q : \mathcal{Q}_2[q]\}$. □

(26) $\{p : p(i_1) = a \cdot p(i_2) \cdot p(i_3)\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.
Proof: Define $\mathcal{Q}_1[\text{finite 0-sequence of } \mathbb{N}] \equiv 1 \cdot \$_1(i_1) = a \cdot \$_1(i_2) \cdot \$_1(i_3)$. Define $\mathcal{Q}_2[\text{finite 0-sequence of } \mathbb{N}] \equiv \$_1(i_1) = a \cdot \$_1(i_2) \cdot \$_1(i_3)$. $\{p : \mathcal{Q}_1[p]\} = \{q : \mathcal{Q}_2[q]\}$. □

(27) Let us consider a Diophantine subset A of the n-xtuples of $\mathbb{N}$, and natural numbers k, n_4. Suppose $k + n_4 = n$. Then $\{p_{\lfloor n_4} : p \in A\}$ is a Diophantine subset of the k-xtuples of $\mathbb{N}$.
Proof: Consider n_3 being a natural number, p_1 being a $\mathbb{Z}$-valued polynomial of $n + n_3$,$\mathbb{R}_{\mathrm{F}}$ such that for every object s, $s \in A$ iff there exists an n-element finite 0-sequence x of $\mathbb{N}$ and there exists an n_3-element finite 0-sequence y of $\mathbb{N}$ such that $s = x$ and $\mathrm{eval}(p_1, {}^{@}(x \frown y)) = 0$. Reconsider $I = \mathrm{id}_{n+n_3}$ as a finite 0-sequence. Set $I_1 = I \restriction n_4$. Set $I_2 = (I \restriction n)_{\lfloor n_4}$. Set $I_3 = I_{\lfloor n}$. Reconsider $J = (I_2 \frown I_1) \frown I_3$ as a function from $n + n_3$ into $n + n_3$. Set $h =$ the p_1 permuted by J^{-1}. Reconsider $H = h$ as a polynomial of $k + (n_4 + n_3)$,$\mathbb{R}_{\mathrm{F}}$. Set $Y = \{p_{\lfloor n_4} : p \in A\}$. $Y \subseteq$ the k-xtuples of $\mathbb{N}$. For every object s, $s \in Y$ iff there exists a k-element finite 0-sequence x of $\mathbb{N}$ and there exists an $(n_4 + n_3)$-element finite 0-sequence y of $\mathbb{N}$ such that $s = x$ and $\mathrm{eval}(H, {}^{@}(x \frown y)) = 0$ by [9, (25)], [11, (27)]. □

(28) Let us consider integers a, b, a natural number c, i_1, i_2, and i_3. Then $\{p : a \cdot p(i_1) = \lfloor \frac{b \cdot p(i_2)}{c \cdot p(i_3)} \rfloor$ and $c \cdot p(i_3) \neq 0\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.

PROOF: Define $\mathcal{F}_2$(natural number, natural number, natural number) $= c \cdot \$_3 + a \cdot c \cdot \$_1 \cdot \$_3$. For every n, i_1, i_2, i_3, i_4, and d, $\{p : \mathcal{F}_2(p(i_1), p(i_2), p(i_3)) = d \cdot p(i_4)\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{P}_2$[natural number, natural number, integer] $\equiv b \cdot \$_1 + 0 < \$_3$. For every n, i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}_2[p(i_1), p(i_2), \mathcal{F}_2(p(i_3), p(i_4), p(i_5))]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{P}_3$[natural number, natural number, integer] $\equiv b \cdot \$_1 \geqslant \$_3 + 0$. Define $\mathcal{F}_3$(natural number, natural number, natural number) $= a \cdot c \cdot \$_1 \cdot \$_3$. For every n, i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}_3[p(i_1), p(i_2), \mathcal{F}_3(p(i_3), p(i_4), p(i_5))]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.

Define $\mathcal{Q}_1$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_2[\$_1(i_2), \$_1(i_2), \mathcal{F}_2(\$_1(i_1), \$_1(i_1), \$_1(i_3))]$. Define $\mathcal{Q}_2$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_3[\$_1(i_2), \$_1(i_2), \mathcal{F}_3(\$_1(i_1), \$_1(i_1), \$_1(i_3))]$. Define $\mathcal{Q}_{12}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{Q}_1[\$_1]$ and $\mathcal{Q}_2[\$_1]$. Define $\mathcal{Q}_3$[finite 0-sequence of $\mathbb{N}$] $\equiv c \cdot \$_1(i_3) \neq 0 \cdot \$_1(i_3) + 0$. Define $\mathcal{Q}_{123}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{Q}_{12}[\$_1]$ and $\mathcal{Q}_3[\$_1]$. Define $\mathcal{T}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) = \lfloor \frac{b \cdot \$_1(i_2)}{c \cdot \$_1(i_3)} \rfloor$ and $c \cdot \$_1(i_3) \neq 0$. $\{p : \mathcal{Q}_1[p]$ and $\mathcal{Q}_2[p]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. $\{p : \mathcal{Q}_{12}[p]$ and $\mathcal{Q}_3[p]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. For every p, $\mathcal{T}[p]$ iff $\mathcal{Q}_{123}[p]$. $\{p : \mathcal{T}[p]\} = \{q : \mathcal{Q}_{123}[q]\}$. $\square$

Let us consider i_1, i_2, and i_3. Now we state the propositions:

(29) If $n \neq 0$, then $\{p : p(i_1) \geqslant p(i_3)$ and $p(i_2) = \binom{p(i_1)}{p(i_3)}\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.

PROOF: Set $n_6 = n + 6$. Define $\mathcal{R}$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(i_1) \geqslant \$_1(i_3)$ and $\$_1(i_2) = \binom{\$_1(i_1)}{\$_1(i_3)}$. Set $RR = \{p : \mathcal{R}[p]\}$. Reconsider $X = i_1$, $Y = i_2$, $Z = i_3$, $U = n$, $V = n + 1$, $Y_1 = n + 2$, $Y_2 = n + 3$, $Y_3 = n + 4$, $U_1 = n + 5$ as an element of $n + 6$. Define $\mathcal{P}_1$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(Y_1) = \$_1(X)^{\$_1(Z)}$. Define $\mathcal{P}_2$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(Y_2) = \$_1(U_1)^{\$_1(X)}$. Define $\mathcal{P}_3$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(Y_3) = \$_1(U)^{\$_1(Z)}$. Define $\mathcal{P}_4$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(U) > 1 \cdot \$_1(Y_1) + 0$. Define $\mathcal{P}_5$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(V) = \lfloor \frac{1 \cdot \$_1(Y_2)}{1 \cdot \$_1(Y_3)} \rfloor$ and $1 \cdot \$_1(Y_3) \neq 0$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N} : \mathcal{P}_5[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$.

Define $\mathcal{P}_6$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(Y) \equiv 1 \cdot \$_1(V) \pmod{1 \cdot \$_1(U)}$. Define $\mathcal{P}_7$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(U) > 1 \cdot \$_1(Y) + 0$. Define $\mathcal{P}_8$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(X) \geqslant 1 \cdot \$_1(Z) + 0$. Define $\mathcal{P}_9$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(U_1) = 1 \cdot \$_1(U) + 1$. Define $\mathcal{P}_{12}$[finite 0-sequence of $\mathbb{N}$] $\equiv$

$\mathcal{P}_1[\$_1]$ and $\mathcal{P}_2[\$_1]$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_{12}[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$. Define $\mathcal{P}_{123}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_{12}[\$_1]$ and $\mathcal{P}_3[\$_1]$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_{123}[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$. Define $\mathcal{P}_{1234}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_{123}[\$_1]$ and $\mathcal{P}_4[\$_1]$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_{1234}[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$. Define $\mathcal{P}_{12345}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_{1234}[\$_1]$ and $\mathcal{P}_5[\$_1]$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_{12345}[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$.

Define $\mathcal{P}_{123456}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_{12345}[\$_1]$ and $\mathcal{P}_6[\$_1]$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_{123456}[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$. Define $\mathcal{P}_{1234567}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_{123456}[\$_1]$ and $\mathcal{P}_7[\$_1]$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_{1234567}[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$. Define $\mathcal{P}_{12345678}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_{1234567}[\$_1]$ and $\mathcal{P}_8[\$_1]$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_{12345678}[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$. Define $\mathcal{P}_{123456789}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_{12345678}[\$_1]$ and $\mathcal{P}_9[\$_1]$. Set $PP = \{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_{123456789}[p]\}$. PP is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$. Reconsider $PP_n = \{p{\restriction}n$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $p \in PP\}$ as a Diophantine subset of the n-xtuples of $\mathbb{N}$. $PP_n \subseteq RR$. $RR \subseteq PP_n$. $\square$

(30) $\{p : p(i_1) \geqslant p(i_3)$ and $p(i_2) = \binom{p(i_1)}{p(i_3)}\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. The theorem is a consequence of (29).

Let us consider i_1 and i_2. Now we state the propositions:

(31) If $n \neq 0$, then $\{p : p(i_1) = p(i_2)!\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.

Proof: Set $n_6 = n+6$. Define $\mathcal{R}$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(i_1) = \$_1(i_2)!$. Set $RR = \{p : \mathcal{R}[p]\}$. Reconsider $Y = i_1$, $X = i_2$, $N = n$, $Y_1 = n+1$, $Y_2 = n+2$, $Y_3 = n+3$, $X_1 = n+4$, $X_2 = n+5$ as an element of $n+6$. Define $\mathcal{P}_1$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(Y_1) = \$_1(X_2)^{\$_1(X_1)}$. Define $\mathcal{P}_2$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(Y_2) = \$_1(N)^{\$_1(X)}$. Define $\mathcal{P}_3$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(N) \geqslant \$_1(X)$ and $\$_1(Y_3) = \binom{\$_1(N)}{\$_1(X)}$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_3[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$. Define $\mathcal{P}_4$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(Y) = \lfloor \frac{1 \cdot \$_1(Y_2)}{1 \cdot \$_1(Y_3)} \rfloor$ and $1 \cdot \$_1(Y_3) \neq 0$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_4[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$. Define $\mathcal{P}_5$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(X_2) = 2 \cdot \$_1(X) + 0$. Define $\mathcal{P}_6$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(X_1) = 1 \cdot \$_1(X) + 1$. Define $\mathcal{P}_7$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(N) >$

$1 \cdot \$_1(Y_1) + 0$. Define $\mathcal{P}_{12}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_1[\$_1]$ and $\mathcal{P}_2[\$_1]$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_{12}[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$.

Define $\mathcal{P}_{123}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_{12}[\$_1]$ and $\mathcal{P}_3[\$_1]$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_{123}[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$. Define $\mathcal{P}_{1234}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_{123}[\$_1]$ and $\mathcal{P}_4[\$_1]$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_{1234}[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$. Define $\mathcal{P}_{12345}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_{1234}[\$_1]$ and $\mathcal{P}_5[\$_1]$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_{12345}[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$. Define $\mathcal{P}_{123456}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_{12345}[\$_1]$ and $\mathcal{P}_6[\$_1]$. $\{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_{123456}[p]\}$ is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$. Define $\mathcal{P}_{1234567}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_{123456}[\$_1]$ and $\mathcal{P}_7[\$_1]$. Set $PP = \{p$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $\mathcal{P}_{1234567}[p]\}$. PP is a Diophantine subset of the n_6-xtuples of $\mathbb{N}$. Reconsider $PP_n = \{p{\restriction}n$, where p is an n_6-element finite 0-sequence of $\mathbb{N}$: $p \in PP\}$ as a Diophantine subset of the n-xtuples of $\mathbb{N}$. $PP_n \subseteq RR$. $RR \subseteq PP_n$. □

(32) $\{p : p(i_1) = p(i_2)!\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. The theorem is a consequence of (31).

(33) $\{p : 1 + (p(i_1) + 1) \cdot (p(i_2)!) = p(i_3)\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.

Proof: Define $\mathcal{R}$(natural number, natural number, natural number) $= 1 \cdot \$_1 + -1$. Define $\mathcal{P}_1$[natural number, natural number, integer] $\equiv 1 \cdot \$_1 \cdot \$_2 = \$_3$. For every i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}_1[p(i_1), p(i_2), \mathcal{R}(p(i_3), p(i_4), p(i_5))]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{F}_2$(natural number, natural number, natural number) $= \$_1!$. For every i_1, i_2, i_3, and i_4, $\{p : \mathcal{F}_2(p(i_1), p(i_2), p(i_3)) = p(i_4)\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{P}_2$[natural number, natural number, natural object, natural number, natural number, natural number] $\equiv 1 \cdot \$_1 \cdot \$_3 = 1 \cdot \$_2 - 1$.

For every i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}_2[p(i_1), p(i_2), \mathcal{F}_2(p(i_3), p(i_4), p(i_5)), p(i_3), p(i_4), p(i_5)]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{P}_3$[natural number, natural number, natural object, natural number, natural number, natural number] $\equiv 1 \cdot \$_3 \cdot (\$_1!) = 1 \cdot \$_2 - 1$. Define $\mathcal{F}_3$(natural number, natural number, natural number) $= 1 \cdot \$_1 + 1$. For every n, i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}_3[p(i_1), p(i_2), \mathcal{F}_3(p(i_3), p(i_4), p(i_5)), p(i_3), p(i_4), p(i_5)]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{Q}_1$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}_3[\$_1(i_2), \$_1(i_3), 1 \cdot \$_1(i_1)+1, \$_1(i_3), \$_1(i_3), \$_1(i_3)]$. Define $\mathcal{Q}_2$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 + (\$_1(i_1) + 1) \cdot (\$_1(i_2)!) = \$_1(i_3)$. $\{p : \mathcal{Q}_1[p]\} = \{q : \mathcal{Q}_2[q]\}$. □

Let us consider i_1, i_2, and i_3. Now we state the propositions:

(34) If $n \neq 0$, then $\{p : p(i_3) = \prod(1 + p(i_1) \cdot (\mathrm{idseq}(p(i_2))))$ and $p(i_1) \geqslant 1\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.

PROOF: Set $n_{12} = n+13$. Define $\mathcal{R}$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(i_3) = \prod(1+ \$_1(i_1) \cdot (\mathrm{idseq}(\$_1(i_2))))$ and $\$_1(i_1) \geqslant 1$. Set $RR = \{p : \mathcal{R}[p]\}$. Reconsider $X_1 = i_1$, $X = i_2$, $Y = i_3$, $U = n$, $W = n+1$, $Y_1 = n+2$, $Y_2 = n+3$, $Y_3 = n+4$, $Y_4 = n+5$, $Y_5 = n+6$, $X_3 = n+7$, $W_1 = n+8$, $Y_6 = n+9$, $Y_7 = n+10$, $X_4 = n+11$, $O = n+12$ as an element of n_{12}. Define $\mathcal{Q}$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(X_1) \geqslant 0 \cdot \$_1(Y) + 1$. Define $\mathcal{P}_1$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(U) > 1 \cdot \$_1(Y) + 0$. Define $\mathcal{P}_2$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(X_3) = 1 \cdot \$_1(X_1) \cdot \$_1(W)$.

Define $\mathcal{P}_3$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(O) = 1$. Define $\mathcal{P}_4$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(X_3) \equiv 1 \cdot \$_1(O) \pmod{1 \cdot \$_1(U)}$. Define $\mathcal{P}_5$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(Y_1) = \$_1(X_1)^{\$_1(X)}$. Define $\mathcal{P}_6$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(Y_2) = \$_1(X)!$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N} : \mathcal{P}_6[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{P}_7$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(W_1) = 1 \cdot \$_1(W) + 1 \cdot \$_1(X) + 0$. Define $\mathcal{P}_8$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(W_1) \geqslant \$_1(X)$ and $\$_1(Y_3) = \binom{\$_1(W_1)}{\$_1(X)}$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N} : \mathcal{P}_8[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{P}_9$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(Y_6) = 1 \cdot \$_1(Y_1) \cdot \$_1(Y_2)$. Define $\mathcal{PA}$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(Y_7) = 1 \cdot \$_1(Y_6) \cdot \$_1(Y_3)$. Define $\mathcal{PB}$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(Y_7) \equiv 1 \cdot \$_1(Y) \pmod{1 \cdot \$_1(U)}$. Define $\mathcal{PC}$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(X_4) = 1 \cdot \$_1(X_1) \cdot \$_1(X)$. Define $\mathcal{PD}$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(Y_4) = 1 \cdot \$_1(X_4) + 1$. Define $\mathcal{PE}$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(Y_5) = \$_1(Y_4)^{\$_1(X)}$. Define $\mathcal{PF}$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(U) > 1 \cdot \$_1(Y_5) + 0$.

Define $\mathcal{C}_1$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{Q}[\$_1]$ and $\mathcal{P}_1[\$_1]$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N} : \mathcal{C}_1[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{C}_2$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{C}_1[\$_1]$ and $\mathcal{P}_2[\$_1]$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N} : \mathcal{C}_2[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{C}_3$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{C}_2[\$_1]$ and $\mathcal{P}_3[\$_1]$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N} : \mathcal{C}_3[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{C}_4$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{C}_3[\$_1]$ and $\mathcal{P}_4[\$_1]$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N} : \mathcal{C}_4[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{C}_5$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{C}_4[\$_1]$ and $\mathcal{P}_5[\$_1]$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N} : \mathcal{C}_5[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{C}_6$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{C}_5[\$_1]$ and $\mathcal{P}_6[\$_1]$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N} : \mathcal{C}_6[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{C}_7$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{C}_6[\$_1]$ and $\mathcal{P}_7[\$_1]$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N}$: $\mathcal{C}_7[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{C}_8$[finite 0-sequence of $\mathbb{N}] \equiv \mathcal{C}_7[\$_1]$ and $\mathcal{P}_8[\$_1]$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N}$: $\mathcal{C}_8[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{C}_9$[finite 0-sequence of $\mathbb{N}] \equiv \mathcal{C}_8[\$_1]$ and $\mathcal{P}_9[\$_1]$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N}$: $\mathcal{C}_9[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$.

Define $\mathcal{CA}$[finite 0-sequence of $\mathbb{N}] \equiv \mathcal{C}_9[\$_1]$ and $\mathcal{PA}[\$_1]$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N}$: $\mathcal{CA}[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{CB}$[finite 0-sequence of $\mathbb{N}] \equiv \mathcal{CA}[\$_1]$ and $\mathcal{PB}[\$_1]$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N}$: $\mathcal{CB}[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{CC}$[finite 0-sequence of $\mathbb{N}] \equiv \mathcal{CB}[\$_1]$ and $\mathcal{PC}[\$_1]$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N}$: $\mathcal{CC}[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{CD}$[finite 0-sequence of $\mathbb{N}] \equiv \mathcal{CC}[\$_1]$ and $\mathcal{PD}[\$_1]$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N}$: $\mathcal{CD}[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{CE}$[finite 0-sequence of $\mathbb{N}] \equiv \mathcal{CD}[\$_1]$ and $\mathcal{PE}[\$_1]$. $\{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N}$: $\mathcal{CE}[p]\}$ is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Define $\mathcal{CF}$[finite 0-sequence of $\mathbb{N}] \equiv \mathcal{CE}[\$_1]$ and $\mathcal{PF}[\$_1]$. Set $PP = \{p$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N}$: $\mathcal{CF}[p]\}$. PP is a Diophantine subset of the n_{12}-xtuples of $\mathbb{N}$. Reconsider $PP_n = \{p{\restriction}n$, where p is an n_{12}-element finite 0-sequence of $\mathbb{N}$: $p \in PP\}$ as a Diophantine subset of the n-xtuples of $\mathbb{N}$. $PP_n \subseteq RR$. $RR \subseteq PP_n$. $\square$

(35) $\{p : p(i_3) = \prod(1 + p(i_1) \cdot (\text{idseq}(p(i_2))))$ and $p(i_1) \geqslant 1\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. The theorem is a consequence of (34).

(36) $\{p : p(i_3) = \prod(1 + p(i_1)! \cdot (\text{idseq}(1 + p(i_2))))\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.

PROOF: Define $\mathcal{R}$(natural number, natural number, natural number) $= \$_1!$. For every i_1, i_2, i_3, and i_4, $\{p : \mathcal{R}(p(i_1), p(i_2), p(i_3)) = p(i_4)\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{P}_1$[natural number, natural number, natural object, natural number, natural number, natural number] $\equiv \$_1 = \prod(1 + \$_3 \cdot (\text{idseq}(\$_2)))$ and $\$_3 \geqslant 1$. For every i_1, i_2, i_3, i_4, i_5, and i_6, $\{p : \mathcal{P}_1[p(i_1), p(i_2), p(i_3), p(i_4), p(i_5), p(i_6)]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.

For every i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}_1[p(i_1), p(i_2), \mathcal{R}(p(i_3), p(i_4), p(i_5)), p(i_3), p(i_4), p(i_5)]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{F}_2$(natural number, natural number, natural number) $= 1 \cdot \$_1 + 1$. Define $\mathcal{P}_2$[natural number, natural number, natural object, natural number, natural number, natural number] $\equiv \$_1 = \prod(1 + \$_2! \cdot (\text{idseq}(\$_3)))$ and $\$_2! \geqslant 1$. For every i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}_2[p(i_1), p(i_2), \mathcal{F}_2(p(i_3), p(i_4), p(i_5)), p(i_3), p(i_4), p(i_5)]\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{Q}_1$[finite

0-sequence of $\mathbb{N}] \equiv \mathcal{P}_2[\$_1(i_3), \$_1(i_1), 1 \cdot \$_1(i_2) + 1, 1 \cdot \$_1(i_3), \$_1(i_3), \$_1(i_3)]$. Define $\mathcal{Q}_2$[finite 0-sequence of $\mathbb{N}] \equiv \$_1(i_3) = \prod(1 + \$_1(i_1)! \cdot (\mathrm{idseq}(1 + \$_1(i_2))))$. $\{p : \mathcal{Q}_1[p]\} = \{q : \mathcal{Q}_2[q]\}$. $\square$

Let us consider i_1, i_2, and i_3. Now we state the propositions:

(37) If $n \neq 0$, then $\{p : p(i_3) = \prod(p(i_2)+1+-\mathrm{idseq}(p(i_1)))$ and $p(i_2) > p(i_1)\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.

Proof: Set $n_2 = n + 2$. Define $\mathcal{R}$[finite 0-sequence of $\mathbb{N}] \equiv \$_1(i_3) = \prod(\$_1(i_2) + 1 + -\mathrm{idseq}(\$_1(i_1)))$ and $\$_1(i_2) > \$_1(i_1)$. Set $RR = \{p : \mathcal{R}[p]\}$. Reconsider $Y = i_3$, $X_2 = i_2$, $X_1 = i_1$, $C = n$, $F = n + 1$ as an element of n_2. Define $\mathcal{P}_1$[finite 0-sequence of $\mathbb{N}] \equiv \$_1(X_2) \geqslant \$_1(X_1)$ and $\$_1(C) = \binom{\$_1(X_2)}{\$_1(X_1)}$. $\{p$, where p is an n_2-element finite 0-sequence of $\mathbb{N} : \mathcal{P}_1[p]\}$ is a Diophantine subset of the n_2-xtuples of $\mathbb{N}$. Define $\mathcal{P}_2$[finite 0-sequence of $\mathbb{N}] \equiv \$_1(F) = \$_1(X_1)!$. $\{p$, where p is an n_2-element finite 0-sequence of $\mathbb{N} : \mathcal{P}_2[p]\}$ is a Diophantine subset of the n_2-xtuples of $\mathbb{N}$. Define $\mathcal{P}_3$[finite 0-sequence of $\mathbb{N}] \equiv 1 \cdot \$_1(X_2) > 1 \cdot \$_1(X_1) + 0$. Define $\mathcal{P}_4$[finite 0-sequence of $\mathbb{N}] \equiv 1 \cdot \$_1(Y) = 1 \cdot \$_1(F) \cdot \$_1(C)$.

Define $\mathcal{P}_{12}$[finite 0-sequence of $\mathbb{N}] \equiv \mathcal{P}_1[\$_1]$ and $\mathcal{P}_2[\$_1]$. $\{p$, where p is an n_2-element finite 0-sequence of $\mathbb{N} : \mathcal{P}_{12}[p]\}$ is a Diophantine subset of the n_2-xtuples of $\mathbb{N}$. Define $\mathcal{P}_{123}$[finite 0-sequence of $\mathbb{N}] \equiv \mathcal{P}_{12}[\$_1]$ and $\mathcal{P}_3[\$_1]$. $\{p$, where p is an n_2-element finite 0-sequence of $\mathbb{N} : \mathcal{P}_{123}[p]\}$ is a Diophantine subset of the n_2-xtuples of $\mathbb{N}$. Define $\mathcal{P}_{1234}$[finite 0-sequence of $\mathbb{N}] \equiv \mathcal{P}_{123}[\$_1]$ and $\mathcal{P}_4[\$_1]$. Set $PP = \{p$, where p is an n_2-element finite 0-sequence of $\mathbb{N} : \mathcal{P}_{1234}[p]\}$. PP is a Diophantine subset of the n_2-xtuples of $\mathbb{N}$. Reconsider $PP_n = \{p{\restriction}n$, where p is an n_2-element finite 0-sequence of $\mathbb{N} : p \in PP\}$ as a Diophantine subset of the n-xtuples of $\mathbb{N}$. $PP_n \subseteq RR$. $RR \subseteq PP_n$. $\square$

(38) $\{p : p(i_3) = \prod(p(i_2) + 1 + -\mathrm{idseq}(p(i_1)))$ and $p(i_2) > p(i_1)\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. The theorem is a consequence of (37).

(39) $\{p : p(i_1) = \prod(i + p_{\lfloor n_1} {\restriction} n_2)\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every n such that $\$_1 + n_1 \leqslant n$ for every i_1, $\{p : p(i_1) = \prod(i + p_{\lfloor n_1} {\restriction} \$_1)\}$ is a Diophantine subset of the n-xtuples of $\mathbb{N}$. $\mathcal{P}[0]$. If $\mathcal{P}[m]$, then $\mathcal{P}[m + 1]$. $\mathcal{P}[m]$. $\square$

References

[1] Marcin Acewicz and Karol Pąk. Basic Diophantine relations. *Formalized Mathematics*, 26(**2**):175–181, 2018. doi:10.2478/forma-2018-0015.

[2] Zofia Adamowicz and Paweł Zbierski. *Logic of Mathematics: A Modern Course of Classical*

Logic. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley-Interscience, 1997.

[3] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[4] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, and Karol Pąk. The role of the Mizar Mathematical Library for interactive proof development in Mizar. *Journal of Automated Reasoning*, 61(1):9–32, 2018. doi:10.1007/s10817-017-9440-6.

[5] Martin Davis. Hilbert's tenth problem is unsolvable. *The American Mathematical Monthly, Mathematical Association of America*, 80(3):233–269, 1973. doi:10.2307/2318447.

[6] Adam Grabowski, Artur Korniłowicz, and Adam Naumowicz. Four decades of Mizar. *Journal of Automated Reasoning*, 55(3):191–198, 2015. doi:10.1007/s10817-015-9345-1.

[7] Artur Korniłowicz and Karol Pąk. Basel problem – preliminaries. *Formalized Mathematics*, 25(**2**):141–147, 2017. doi:10.1515/forma-2017-0013.

[8] Xiquan Liang, Li Yan, and Junjie Zhao. Linear congruence relation and complete residue systems. *Formalized Mathematics*, 15(**4**):181–187, 2007. doi:10.2478/v10037-007-0022-7.

[9] Karol Pąk. Diophantine sets. Preliminaries. *Formalized Mathematics*, 26(**1**):81–90, 2018. doi:10.2478/forma-2018-0007.

[10] Craig Alan Smorynski. *Logical Number Theory I, An Introduction*. Universitext. Springer-Verlag Berlin Heidelberg, 1991. ISBN 978-3-642-75462-3.

[11] Tetsuya Tsunetou, Grzegorz Bancerek, and Yatsuka Nakamura. Zero-based finite sequences. *Formalized Mathematics*, 9(**4**):825–829, 2001.

[12] Rafał Ziobro. On subnomials. *Formalized Mathematics*, 24(**4**):261–273, 2016. doi:10.1515/forma-2016-0022.

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