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# Cross-Ratio in Real Vector Space

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**Summary.** Using Mizar [1], in the context of a real vector space, we introduce the concept of affine ratio of three aligned points (see [5]).

It is also equivalent to the notion of "Mesure algébrique"[1], to the opposite of the notion of Teilverhältnis[2] or to the opposite of the ordered length-ratio [9].

In the second part, we introduce the classic notion of "cross-ratio" of 4 points aligned in a real vector space.

Finally, we show that if the real vector space is the real line, the notion corresponds to the classical notion[3] [9]:

> The cross-ratio of a quadruple of distinct points on the real line with coordinates $x_1, x_2, x_3, x_4$ is given by:

$$(x_1, x_2; x_3, x_4) = \frac{x_3 - x_1}{x_3 - x_2} . \frac{x_4 - x_2}{x_4 - x_1}$$

In the Mizar Mathematical Library, the vector spaces were first defined by Kusak, Leończuk and Muzalewski in the article [6], while the actual real vector space was defined by Trybulec [10] and the complex vector space was defined by Endou [4]. Nakasho and Shidama have developed a solution to explore the notions introduced by different authors[4] [7]. The definitions can be directly linked in the HTMLized version of the Mizar library[5].

The study of the cross-ratio will continue within the framework of the Klein-Beltrami model [2], [3]. For a generalized cross-ratio, see Papadopoulos [8].

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[1]`https://fr.wikipedia.org/wiki/Mesure_algébrique`
[2]`https://de.wikipedia.org/wiki/Teilverhältnis`
[3]`https://en.wikipedia.org/wiki/Cross-ratio`
[4]`http://webmizar.cs.shinshu-u.ac.jp/mmlfe/current`
[5]Example: `RealLinearSpace` `http://mizar.org/version/current/html/rlvect_1.html#NM2`




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## 1. Preliminaries

Let $a$, $b$, $c$, $d$ be objects. Observe that $\langle a,b,c,d\rangle(1)$ reduces to $a$ and $\langle a,b,c,d\rangle(2)$ reduces to $b$ and $\langle a,b,c,d\rangle(3)$ reduces to $c$ and $\langle a,b,c,d\rangle(4)$ reduces to $d$.

Now we state the proposition:

(1) Let us consider objects $a$, $b$, $c$, $d$, $a'$, $b'$, $c'$, $d'$. Suppose $\langle a,b,c,d\rangle = \langle a', b', c', d'\rangle$. Then

(i) $a = a'$, and

(ii) $b = b'$, and

(iii) $c = c'$, and

(iv) $d = d'$.

Let $r$ be a real number. We say that $r$ is unit if and only if

(Def. 1) $r = 1$.

Let us observe that there exists a non zero real number which is non unit.

Let $r$ be a non unit, non zero real number. The functor $\mathrm{op1}(r)$ yielding a non unit, non zero real number is defined by the term

(Def. 2) $\frac{1}{r}$.

One can check that the functor is involutive.

The functor $\mathrm{op2}(r)$ yielding a non unit, non zero real number is defined by the term

(Def. 3) $1 - r$.

Let us observe that the functor is involutive.

From now on $a$, $b$, $r$ denote non unit, non zero real numbers.

Now we state the propositions:

(2) (i) $\mathrm{op2}(\mathrm{op1}(r)) = \frac{r-1}{r}$, and

(ii) $\mathrm{op1}(\mathrm{op2}(r)) = \frac{1}{1-r}$, and

(iii) $\mathrm{op1}(\mathrm{op2}(\mathrm{op1}(r))) = \frac{r}{r-1}$, and

(iv) $\mathrm{op2}(\mathrm{op1}(\mathrm{op2}(r))) = \frac{r}{r-1}$.

(3) (i) $\mathrm{op2}(\mathrm{op1}(\mathrm{op2}(\mathrm{op1}(r)))) = \mathrm{op1}(\mathrm{op2}(r))$, and

(ii) $\mathrm{op1}(\mathrm{op2}(\mathrm{op1}(\mathrm{op2}(r)))) = \mathrm{op2}(\mathrm{op1}(r))$.

The theorem is a consequence of (2).

(4) $\frac{\mathrm{op1}(a)}{\mathrm{op1}(b)} = \frac{b}{a}$.

In the sequel $X$ denotes a non empty set and $x$ denotes a 4-tuple of $X$.

Now we state the propositions:

(5) $X^4 =$ the set of all $\langle d_1, d_2, d_3, d_4\rangle$ where $d_1, d_2, d_3, d_4$ are elements of $X$.

(6) Let us consider objects $a$, $b$, $c$, $d$. Suppose ($a = x(1)$ or $a = x(2)$ or $a = x(3)$ or $a = x(4)$) and ($b = x(1)$ or $b = x(2)$ or $b = x(3)$ or $b = x(4)$) and ($c = x(1)$ or $c = x(2)$ or $c = x(3)$ or $c = x(4)$) and ($d = x(1)$ or $d = x(2)$ or $d = x(3)$ or $d = x(4)$). Then $\langle a, b, c, d\rangle$ is a 4-tuple of $X$. The theorem is a consequence of (5).

Let $X$ be a non empty set and $x$ be a 4-tuple of $X$. The functors: $\sigma_{1342}(x)$, $\sigma_{1423}(x)$, $\sigma_{2143}(x)$, $\sigma_{2314}(x)$, and $\sigma_{2341}(x)$ yielding 4-tuples of $X$ are defined by terms

(Def. 4) $\langle x(1), x(3), x(4), x(2)\rangle$,

(Def. 5) $\langle x(1), x(4), x(2), x(3)\rangle$,

(Def. 6) $\langle x(2), x(1), x(4), x(3)\rangle$,

(Def. 7) $\langle x(2), x(3), x(1), x(4)\rangle$,

(Def. 8) $\langle x(2), x(3), x(4), x(1)\rangle$,

respectively. The functors: $\sigma_{2413}(x)$, $\sigma_{2431}(x)$, $\sigma_{3124}(x)$, $\sigma_{3142}(x)$, and $\sigma_{3241}(x)$ yielding 4-tuples of $X$ are defined by terms

(Def. 9) $\langle x(2), x(4), x(1), x(3)\rangle$,

(Def. 10) $\langle x(2), x(4), x(3), x(1)\rangle$,

(Def. 11) $\langle x(3), x(1), x(2), x(4)\rangle$,

(Def. 12) $\langle x(3), x(1), x(4), x(2)\rangle$,

(Def. 13) $\langle x(3), x(2), x(4), x(1)\rangle$,

respectively. The functors: $\sigma_{3412}(x)$, $\sigma_{3421}(x)$, $\sigma_{4123}(x)$, $\sigma_{4132}(x)$, and $\sigma_{4213}(x)$ yielding 4-tuples of $X$ are defined by terms

(Def. 14) $\langle x(3), x(4), x(1), x(2)\rangle$,

(Def. 15) $\langle x(3), x(4), x(2), x(1)\rangle$,

(Def. 16) $\langle x(4), x(1), x(2), x(3)\rangle$,

(Def. 17) $\langle x(4), x(1), x(3), x(2)\rangle$,

(Def. 18) $\langle x(4), x(2), x(1), x(3)\rangle$,

respectively. The functors: $\sigma_{4312}(x)$ and $\sigma_{4321}(x)$ yielding 4-tuples of $X$ are defined by terms

(Def. 19) $\langle x(4), x(3), x(1), x(2)\rangle$,

(Def. 20) $\langle x(4), x(3), x(2), x(1)\rangle$,

respectively. The functors: $\sigma_{\mathrm{id}}(x)$ and $\sigma_{12}(x)$ yielding 4-tuples of $X$ are defined by terms

(Def. 21) $\langle x(1), x(2), x(3), x(4)\rangle$,

(Def. 22) $\langle x(2), x(1), x(3), x(4)\rangle$, respectively. Observe that the functor is involutive.

The functors: $\sigma_{13}(x)$ and $\sigma_{14}(x)$ yielding 4-tuples of $X$ are defined by terms

(Def. 23) $\langle x(3), x(2), x(1), x(4)\rangle$,

(Def. 24) $\langle x(4), x(2), x(3), x(1)\rangle$,

respectively. One can check that the functor is involutive.

The functor $\sigma_{23}(x)$ yielding a 4-tuple of $X$ is defined by the term

(Def. 25) $\langle x(1), x(3), x(2), x(4)\rangle$.

Note that the functor is involutive.

The functors: $\sigma_{24}(x)$ and $\sigma_{34}(x)$ yielding 4-tuples of $X$ are defined by terms

(Def. 26) $\langle x(1), x(4), x(3), x(2)\rangle$,

(Def. 27) $\langle x(1), x(2), x(4), x(3)\rangle$,

respectively. Let us observe that the functor is involutive.

Note that $\sigma_{\rm id}(x)$ reduces to $x$.

We introduce the notation $\sigma_{1234}(x)$ as a synonym of $\sigma_{\rm id}(x)$ and $\sigma_{2134}(x)$ as a synonym of $\sigma_{12}(x)$ and $\sigma_{3214}(x)$ as a synonym of $\sigma_{13}(x)$. And $\sigma_{4231}(x)$ as a synonym of $\sigma_{14}(x)$ and $\sigma_{1324}(x)$ as a synonym of $\sigma_{23}(x)$ and $\sigma_{1432}(x)$ as a synonym of $\sigma_{24}(x)$ and $\sigma_{1243}(x)$ as a synonym of $\sigma_{34}(x)$.

Now we state the propositions:

(7) (i) $\sigma_{12}(\sigma_{13}(x)) = \sigma_{13}(\sigma_{23}(x))$, and

(ii) $\sigma_{12}(\sigma_{14}(x)) = \sigma_{14}(\sigma_{24}(x))$, and

(iii) $\sigma_{12}(\sigma_{23}(x)) = \sigma_{13}(\sigma_{12}(x))$, and

(iv) $\sigma_{12}(\sigma_{24}(x)) = \sigma_{14}(\sigma_{12}(x))$, and

(v) $\sigma_{12}(\sigma_{34}(x)) = \sigma_{34}(\sigma_{12}(x))$, and

(vi) $\sigma_{13}(\sigma_{12}(x)) = \sigma_{23}(\sigma_{13}(x))$, and

(vii) $\sigma_{13}(\sigma_{14}(x)) = \sigma_{34}(\sigma_{13}(x))$, and

(viii) $\sigma_{13}(\sigma_{23}(x)) = \sigma_{12}(\sigma_{13}(x))$, and

(ix) $\sigma_{13}(\sigma_{24}(x)) = \sigma_{13}(\sigma_{24}(x))$, and

(x) $\sigma_{13}(\sigma_{34}(x)) = \sigma_{14}(\sigma_{13}(x))$, and

(xi) $\sigma_{23}(\sigma_{12}(x)) = \sigma_{13}(\sigma_{23}(x))$, and

(xii) $\sigma_{23}(\sigma_{13}(x)) = \sigma_{12}(\sigma_{23}(x))$, and

(xiii) $\sigma_{23}(\sigma_{14}(x)) = \sigma_{14}(\sigma_{23}(x))$, and

(xiv) $\sigma_{23}(\sigma_{24}(x)) = \sigma_{34}(\sigma_{23}(x))$, and

(xv) $\sigma_{23}(\sigma_{34}(x)) = \sigma_{24}(\sigma_{23}(x))$, and

(xvi) $\sigma_{24}(\sigma_{12}(x)) = \sigma_{14}(\sigma_{24}(x))$, and

(xvii) $\sigma_{24}(\sigma_{13}(x)) = \sigma_{13}(\sigma_{24}(x))$, and

(xviii) $\sigma_{24}(\sigma_{14}(x)) = \sigma_{12}(\sigma_{24}(x))$, and

(xix) $\sigma_{24}(\sigma_{23}(x)) = \sigma_{34}(\sigma_{24}(x))$, and

(xx) $\sigma_{24}(\sigma_{34}(x)) = \sigma_{23}(\sigma_{24}(x))$, and

(xxi) $\sigma_{34}(\sigma_{12}(x)) = \sigma_{12}(\sigma_{34}(x))$, and

(xxii) $\sigma_{34}(\sigma_{13}(x)) = \sigma_{14}(\sigma_{34}(x))$, and

(xxiii) $\sigma_{34}(\sigma_{14}(x)) = \sigma_{13}(\sigma_{34}(x))$, and

(xxiv) $\sigma_{34}(\sigma_{23}(x)) = \sigma_{24}(\sigma_{34}(x))$, and

(xxv) $\sigma_{34}(\sigma_{24}(x)) = \sigma_{23}(\sigma_{34}(x))$.

(8) (i) $\sigma_{1342}(x) = \sigma_{34}(\sigma_{23}(x))$, and

(ii) $\sigma_{1423}(x) = \sigma_{34}(\sigma_{24}(x))$, and

(iii) $\sigma_{2143}(x) = \sigma_{12}(\sigma_{34}(x))$, and

(iv) $\sigma_{2314}(x) = \sigma_{23}(\sigma_{12}(x))$, and

(v) $\sigma_{2341}(x) = \sigma_{34}(\sigma_{23}(\sigma_{12}(x)))$, and

(vi) $\sigma_{2413}(x) = \sigma_{34}(\sigma_{24}(\sigma_{12}(x)))$, and

(vii) $\sigma_{2431}(x) = \sigma_{24}(\sigma_{12}(x))$, and

(viii) $\sigma_{3124}(x) = \sigma_{23}(\sigma_{13}(x))$, and

(ix) $\sigma_{3142}(x) = \sigma_{24}(\sigma_{34}(\sigma_{13}(x)))$, and

(x) $\sigma_{3241}(x) = \sigma_{34}(\sigma_{13}(x))$, and

(xi) $\sigma_{3412}(x) = \sigma_{24}(\sigma_{13}(x))$, and

(xii) $\sigma_{3421}(x) = \sigma_{24}(\sigma_{23}(\sigma_{13}(x)))$, and

(xiii) $\sigma_{4123}(x) = \sigma_{23}(\sigma_{34}(\sigma_{14}(x)))$, and

(xiv) $\sigma_{4132}(x) = \sigma_{24}(\sigma_{14}(x))$, and

(xv) $\sigma_{4213}(x) = \sigma_{34}(\sigma_{14}(x))$, and

(xvi) $\sigma_{4312}(x) = \sigma_{23}(\sigma_{24}(\sigma_{14}(x)))$, and

(xvii) $\sigma_{4321}(x) = \sigma_{23}(\sigma_{14}(x))$.

(9) (i) $\sigma_{13}(\sigma_{23}(\sigma_{13}(x))) = \sigma_{12}(x)$, and

(ii) $\sigma_{12}(\sigma_{34}(\sigma_{23}(\sigma_{13}(x)))) = \sigma_{34}(\sigma_{23}(x))$, and

(iii) $\sigma_{23}(\sigma_{24}(\sigma_{14}(\sigma_{23}(\sigma_{13}(x))))) = \sigma_{14}(x)$.

(10) (i) $\sigma_{23}(\sigma_{14}(\sigma_{34}(x))) = \sigma_{24}(\sigma_{23}(\sigma_{13}(x)))$, and

(ii) $\sigma_{34}(\sigma_{24}(\sigma_{12}(x))) = \sigma_{24}(\sigma_{13}(\sigma_{23}(x)))$, and

(iii) $\sigma_{24}(\sigma_{34}(\sigma_{13}(x))) = \sigma_{12}(\sigma_{34}(\sigma_{23}(x)))$.

## 2. Affine Ratio

In the sequel $V$ denotes a real linear space and $A$, $B$, $C$, $P$, $Q$, $R$, $S$ denote elements of $V$.

Now we state the proposition:

(11) $P$, $Q$ and $Q$ are collinear.

Let $V$ be a real linear space and $A$, $B$, $C$ be elements of $V$. Assume $A \neq C$ and $A$, $B$ and $C$ are collinear. The functor $\mathrm{AffineRatio}(A, B, C)$ yielding a real number is defined by

(Def. 28) $B - A = it \cdot (C - A)$.

Now we state the propositions:

(12) If $A \neq C$ and $A$, $B$ and $C$ are collinear, then $A - B = (\mathrm{AffineRatio}(A, B, C)) \cdot (A - C)$.

(13) If $A \neq C$ and $A$, $B$ and $C$ are collinear, then $\mathrm{AffineRatio}(A, B, C) = 0$ iff $A = B$.

(14) If $A \neq C$ and $A$, $B$ and $C$ are collinear, then $\mathrm{AffineRatio}(A, B, C) = 1$ iff $B = C$.

(15) Let us consider real numbers $a$, $b$. If $P \neq Q$ and $a \cdot (P - Q) = b \cdot (P - Q)$, then $a = b$.

(16) If $P$, $Q$ and $R$ are collinear and $P \neq R$ and $P \neq Q$, then $\mathrm{AffineRatio}(P, R, Q) = \frac{1}{\mathrm{AffineRatio}(P,Q,R)}$. The theorem is a consequence of (15).

(17) Suppose $P$, $Q$ and $R$ are collinear and $P \neq R$ and $Q \neq R$ and $P \neq Q$. Then $\mathrm{AffineRatio}(Q, P, R) = \frac{\mathrm{AffineRatio}(P,Q,R)}{\mathrm{AffineRatio}(P,Q,R)-1}$. The theorem is a consequence of (13) and (14).

(18) If $P$, $Q$ and $R$ are collinear and $P \neq R$, then $\mathrm{AffineRatio}(R, Q, P) = 1 - \mathrm{AffineRatio}(P, Q, R)$. The theorem is a consequence of (15).

(19) If $P$, $Q$ and $R$ are collinear and $P \neq R$ and $P \neq Q$, then $\mathrm{AffineRatio}(Q, R, P) = \frac{\mathrm{AffineRatio}(P,Q,R)-1}{\mathrm{AffineRatio}(P,Q,R)}$. The theorem is a consequence of (13) and (15).

(20) If $P$, $Q$ and $R$ are collinear and $P \neq R$ and $Q \neq R$, then $\mathrm{AffineRatio}(R, P, Q) = \frac{1}{1-\mathrm{AffineRatio}(P,Q,R)}$. The theorem is a consequence of (14) and (15).

(21) Let us consider a real number $r$. Suppose $P$, $Q$ and $R$ are collinear and $P \neq R$ and $Q \neq R$ and $P \neq Q$ and $r = \mathrm{AffineRatio}(P, Q, R)$. Then

(i) $\mathrm{AffineRatio}(P, R, Q) = \frac{1}{r}$, and

(ii) $\mathrm{AffineRatio}(Q, P, R) = \frac{r}{r-1}$, and

(iii) $\mathrm{AffineRatio}(Q, R, P) = \frac{r-1}{r}$, and

(iv) $\mathrm{AffineRatio}(R, P, Q) = \frac{1}{1-r}$, and

(v) AffineRatio$(R, Q, P) = 1 - r$.

(22) Let us consider a non zero real number $a$. Suppose $P$, $Q$ and $R$ are collinear and $P \neq R$. Then AffineRatio$(P, Q, R) =$ AffineRatio$(a \cdot P, a \cdot Q, a \cdot R)$.

(23) Let us consider elements $x$, $y$ of $\mathcal{R}^1$, and 1-tuples $p$, $q$ of $\mathbb{R}$. If $p = x$ and $q = y$, then $x + y = p + q$.

Let us consider elements $x$, $y$ of $\mathcal{E}^1_{\mathrm{T}}$ and 1-tuples $p$, $q$ of $\mathbb{R}$. Now we state the propositions:

(24) If $p = x$ and $q = y$, then $x + y = p + q$.

(25) If $p = x$ and $q = y$, then $x - y = p - q$.

(26) Let us consider an element $x$ of $\mathcal{E}^1_{\mathrm{T}}$, and a 1-tuple $p$ of $\mathbb{R}$. If $p = x$, then $-x = -p$.

(27) Let us consider a real linear space $T$, elements $x$, $y$ of $T$, and 1-tuples $p$, $q$ of $\mathbb{R}$. If $T = \mathcal{E}^1_{\mathrm{T}}$ and $p = x$ and $q = y$, then $x + y = p + q$.

(28) Let us consider a 1-tuple $p$ of $\mathbb{R}$. Then $-p$ is a 1-tuple of $\mathbb{R}$.

(29) Let us consider a real linear space $T$, an element $x$ of $T$, and a 1-tuple $p$ of $\mathbb{R}$. If $T = \mathcal{E}^1_{\mathrm{T}}$ and $p = x$, then $-p = -x$. The theorem is a consequence of (27).

(30) Let us consider a real linear space $T$, an element $x$ of $T$, and an element $p$ of $\mathcal{E}^1_{\mathrm{T}}$. If $T = \mathcal{E}^1_{\mathrm{T}}$ and $p = x$, then $-p = -x$. The theorem is a consequence of (29).

(31) Let us consider a real linear space $T$, elements $x$, $y$ of $T$, and 1-tuples $p$, $q$ of $\mathbb{R}$. If $T = \mathcal{E}^1_{\mathrm{T}}$ and $p = x$ and $q = y$, then $x - y = p - q$. The theorem is a consequence of (28) and (29).

(32) Let us consider a real linear space $T$, elements $x$, $y$ of $T$, and elements $p$, $q$ of $\mathcal{E}^1_{\mathrm{T}}$. If $T = \mathcal{E}^1_{\mathrm{T}}$ and $p = x$ and $q = y$, then $x + y = p + q$. The theorem is a consequence of (27).

(33) Let us consider a set $D$, and an element $d$ of $D$. Then $\mathrm{Seg}\, 1 \longmapsto d = \langle d \rangle$.

(34) Let us consider real numbers $a$, $r$. Then $(\cdot_{\mathbb{R}})^\circ(\mathrm{Seg}\, 1 \longmapsto a, \langle r \rangle) = \langle a \cdot r \rangle$. The theorem is a consequence of (33).

Let us consider a real number $a$ and a 1-tuple $p$ of $\mathbb{R}$. Now we state the propositions:

(35) $(\cdot_{\mathbb{R}})^\circ(\mathrm{dom}\, p \longmapsto a, p) = a \cdot p$. The theorem is a consequence of (34).

(36) $(\cdot_{\mathbb{R}})^\circ(\mathrm{dom}\, p \longmapsto a, p) = a \cdot p$.

(37) Let us consider a real linear space $T$, elements $x$, $y$ of $T$, a real number $a$, and 1-tuples $p$, $q$ of $\mathbb{R}$. If $T = \mathcal{E}^1_{\mathrm{T}}$ and $p = x$ and $q = y$ and $x = a \cdot y$, then $p = a \cdot q$. The theorem is a consequence of (35).

(38) Let us consider a real linear space $T$, elements $x$, $y$ of $T$, a real number $a$, and elements $p$, $q$ of $\mathcal{E}^1_{\mathrm{T}}$. If $T = \mathcal{E}^1_{\mathrm{T}}$ and $p = x$ and $q = y$, then if $x = a \cdot y$, then $p = a \cdot q$. The theorem is a consequence of (37).

(39) Let us consider a real linear space $T$, elements $x$, $y$ of $T$, and elements $p$, $q$ of $\mathcal{E}^1_{\mathrm{T}}$. If $T = \mathcal{E}^1_{\mathrm{T}}$ and $p = x$ and $q = y$, then $x - y = p - q$. The theorem is a consequence of (30) and (32).

(40) Let us consider 1-tuples $p$, $q$ of $\mathbb{R}$, and a real number $r$. Suppose $p = r \cdot q$ and $p \neq \langle 0 \rangle$. Then there exist real numbers $a$, $b$ such that

(i) $p = \langle a \rangle$, and

(ii) $q = \langle b \rangle$, and

(iii) $r = \frac{a}{b}$.

(41) Let us consider elements $x$, $y$, $z$ of $\mathcal{E}^1_{\mathrm{T}}$. Then $x$, $y$ and $z$ are collinear.

Let us consider a real linear space $T$ and elements $x$, $y$, $z$ of $T$. Now we state the propositions:

(42) If $T = \mathcal{E}^1_{\mathrm{T}}$, then $x$, $y$ and $z$ are collinear.

(43) Suppose $T = \mathcal{E}^1_{\mathrm{T}}$. Then suppose $z \neq x$ and $y \neq x$. Then there exist real numbers $a$, $b$, $c$ such that

(i) $x = \langle a \rangle$, and

(ii) $y = \langle b \rangle$, and

(iii) $z = \langle c \rangle$, and

(iv) $\mathrm{AffineRatio}(x, y, z) = \frac{b-a}{c-a}$.

The theorem is a consequence of (31), (41), (37), and (40).

Now we state the propositions:

(44) Let us consider an element $x$ of $\mathcal{E}^1_{\mathrm{T}}$, and real numbers $a$, $r$. If $x = \langle a \rangle$, then $r \cdot x = \langle r \cdot a \rangle$.

(45) Let us consider elements $x$, $y$ of $\mathcal{E}^1_{\mathrm{T}}$, and real numbers $a$, $b$, $r$. If $x = \langle a \rangle$ and $y = \langle b \rangle$, then $x = r \cdot y$ iff $a = r \cdot b$. The theorem is a consequence of (44).

(46) Let us consider elements $x$, $y$ of $\mathcal{E}^1_{\mathrm{T}}$, and real numbers $a$, $b$. If $x = \langle a \rangle$ and $y = \langle b \rangle$, then $x - y = \langle a - b \rangle$.

(47) Let us consider a real linear space $V$, elements $x$, $y$ of $\mathbb{R}_{\mathrm{F}}$, and elements $x'$, $y'$ of $V$. If $V = \mathbb{R}_{\mathrm{F}}$ and $x = x'$ and $y = y'$, then $x + y = x' + y'$.

Let us consider a real linear space $V$ and elements $P$, $Q$, $R$ of $V$. Now we state the propositions:

(48) If $P$, $Q$ and $R$ are collinear and $P \neq R$ and $Q \neq R$ and $P \neq Q$, then $\mathrm{AffineRatio}(P, Q, R) \neq 0$ and $\mathrm{AffineRatio}(P, Q, R) \neq 1$.

(49) Suppose $P$, $Q$ and $R$ are collinear and $P \neq R$ and $Q \neq R$ and $P \neq Q$. Then there exists a non unit, non zero real number $r$ such that

(i) $r = \mathrm{AffineRatio}(P, Q, R)$, and

(ii) $\mathrm{AffineRatio}(P, R, Q) = \mathrm{op1}(r)$, and

(iii) $\mathrm{AffineRatio}(Q, P, R) = \mathrm{op1}(\mathrm{op2}(\mathrm{op1}(r)))$, and

(iv) $\mathrm{AffineRatio}(Q, R, P) = \mathrm{op2}(\mathrm{op1}(r))$, and

(v) $\mathrm{AffineRatio}(R, P, Q) = \mathrm{op1}(\mathrm{op2}(r))$, and

(vi) $\mathrm{AffineRatio}(R, Q, P) = \mathrm{op2}(r)$.

The theorem is a consequence of (13), (14), (16), (17), (18), (19), (20), and (2).

## 3. Cross-Ratio

Now we state the propositions:

(50) Let us consider a non empty set $X$, a 4-tuple $x$ of $X$, and elements $P$, $Q$, $R$, $S$ of $X$. Suppose $x = \langle P, Q, R, S \rangle$. Then

(i) $\sigma_{1234}(x) = \langle P, Q, R, S \rangle$, and

(ii) $\sigma_{1243}(x) = \langle P, Q, S, R \rangle$, and

(iii) $\sigma_{1324}(x) = \langle P, R, Q, S \rangle$, and

(iv) $\sigma_{1342}(x) = \langle P, R, S, Q \rangle$, and

(v) $\sigma_{1423}(x) = \langle P, S, Q, R \rangle$, and

(vi) $\sigma_{1432}(x) = \langle P, S, R, Q \rangle$, and

(vii) $\sigma_{2134}(x) = \langle Q, P, R, S \rangle$, and

(viii) $\sigma_{2143}(x) = \langle Q, P, S, R \rangle$, and

(ix) $\sigma_{2314}(x) = \langle Q, R, P, S \rangle$, and

(x) $\sigma_{2341}(x) = \langle Q, R, S, P \rangle$, and

(xi) $\sigma_{2413}(x) = \langle Q, S, P, R \rangle$, and

(xii) $\sigma_{2431}(x) = \langle Q, S, R, P \rangle$, and

(xiii) $\sigma_{3124}(x) = \langle R, P, Q, S \rangle$, and

(xiv) $\sigma_{3142}(x) = \langle R, P, S, Q \rangle$, and

(xv) $\sigma_{3214}(x) = \langle R, Q, P, S \rangle$, and

(xvi) $\sigma_{3241}(x) = \langle R, Q, S, P \rangle$, and

(xvii) $\sigma_{3412}(x) = \langle R, S, P, Q \rangle$, and

(xviii) $\sigma_{3421}(x) = \langle R, S, Q, P\rangle$, and

(xix) $\sigma_{4123}(x) = \langle S, P, Q, R\rangle$, and

(xx) $\sigma_{4132}(x) = \langle S, P, R, Q\rangle$, and

(xxi) $\sigma_{4213}(x) = \langle S, Q, P, R\rangle$, and

(xxii) $\sigma_{4231}(x) = \langle S, Q, R, P\rangle$, and

(xxiii) $\sigma_{4312}(x) = \langle S, R, P, Q\rangle$, and

(xxiv) $\sigma_{4321}(x) = \langle S, R, Q, P\rangle$.

(51) Let us consider a non empty set $X$, and a 4-tuple $x$ of $X$. Then

(i) $\sigma_{1324}(\sigma_{1243}(x)) = \sigma_{1423}(x)$, and

(ii) $\sigma_{2143}(\sigma_{1243}(x)) = \sigma_{2134}(x)$, and

(iii) $\sigma_{3412}(\sigma_{1243}(x)) = \sigma_{4312}(x)$, and

(iv) $\sigma_{4321}(\sigma_{1243}(x)) = \sigma_{3421}(x)$, and

(v) $\sigma_{3412}(\sigma_{1324}(x)) = \sigma_{2413}(x)$, and

(vi) $\sigma_{2143}(\sigma_{1324}(x)) = \sigma_{3142}(x)$, and

(vii) $\sigma_{4321}(\sigma_{1324}(x)) = \sigma_{4231}(x)$, and

(viii) $\sigma_{3412}(\sigma_{1423}(x)) = \sigma_{2314}(x)$, and

(ix) $\sigma_{2143}(\sigma_{1423}(x)) = \sigma_{4132}(x)$, and

(x) $\sigma_{4321}(\sigma_{1423}(x)) = \sigma_{3241}(x)$, and

(xi) $\sigma_{1243}(\sigma_{1423}(x)) = \sigma_{1432}(x)$, and

(xii) $\sigma_{4321}(\sigma_{1432}(x)) = \sigma_{2341}(x)$, and

(xiii) $\sigma_{3412}(\sigma_{1432}(x)) = \sigma_{3214}(x)$, and

(xiv) $\sigma_{2143}(\sigma_{1432}(x)) = \sigma_{4123}(x)$, and

(xv) $\sigma_{4321}(\sigma_{3124}(x)) = \sigma_{4213}(x)$, and

(xvi) $\sigma_{3412}(\sigma_{3124}(x)) = \sigma_{2431}(x)$, and

(xvii) $\sigma_{2143}(\sigma_{3124}(x)) = \sigma_{1342}(x)$, and

(xviii) $\sigma_{4312}(\sigma_{3124}(x)) = \sigma_{4231}(x)$, and

(xix) $\sigma_{4321}(\sigma_{3124}(x)) = \sigma_{4213}(x)$.

In the sequel $x$ denotes a 4-tuple of the carrier of $V$ and $P'$, $Q'$, $R'$, $S'$ denote elements of $V$.

Let $V$ be a real linear space and $P$, $Q$, $R$, $S$ be elements of $V$. The functor $\mathrm{CrossRatio}(P, Q, R, S)$ yielding a real number is defined by the term

(Def. 29) $\frac{\mathrm{AffineRatio}(R,P,Q)}{\mathrm{AffineRatio}(S,P,Q)}$.

Now we state the propositions:

(52) If $P$, $Q$, $R$, and $S$ are collinear and $R \neq Q$ and $S \neq Q$ and $S \neq P$, then $R = P$ iff $\mathrm{CrossRatio}(P, Q, R, S) = 0$. The theorem is a consequence of (13).

(53) If $P \neq R$ and $P \neq S$, then $\mathrm{CrossRatio}(P, P, R, S) = 1$. The theorem is a consequence of (11) and (14).

(54) If $P$, $Q$, $R$, and $S$ are collinear and $R \neq Q$ and $S \neq Q$ and $R \neq S$ and $\mathrm{CrossRatio}(P, Q, R, S) = 1$, then $P = Q$. The theorem is a consequence of (15) and (14).

(55) Suppose $P$, $Q$, $R$, and $S$ are collinear and $P'$, $Q'$, $R'$, and $S'$ are collinear and $S \neq P$ and $S \neq Q$ and $S' \neq P'$ and $S' \neq Q'$. Then $\mathrm{CrossRatio}(P, Q, R, S) = \mathrm{CrossRatio}(P', Q', R', S')$ if and only if $\mathrm{AffineRatio}(R, P, Q) \cdot \mathrm{AffineRatio}(S', P', Q') = \mathrm{AffineRatio}(R', P', Q') \cdot \mathrm{AffineRatio}(S, P, Q)$. The theorem is a consequence of (13).

(56) If $P$, $Q$, $R$, and $S$ are collinear and $P \neq S$ and $R \neq Q$ and $S \neq Q$, then $\mathrm{CrossRatio}(P, Q, R, S) = \mathrm{CrossRatio}(R, S, P, Q)$. The theorem is a consequence of (13).

(57) Let us consider a real linear space $V$, and elements $P$, $Q$, $R$, $S$ of $V$. Suppose $P$, $Q$, $R$, and $S$ are collinear and $P \neq R$ and $P \neq S$ and $R \neq Q$ and $S \neq Q$. Then $\mathrm{CrossRatio}(P, Q, R, S) = \mathrm{CrossRatio}(Q, P, S, R)$. The theorem is a consequence of (11), (14), and (49).

(58) If $P$, $Q$, $R$, and $S$ are collinear and $P \neq R$ and $P \neq S$ and $R \neq Q$ and $S \neq Q$, then $\mathrm{CrossRatio}(P, Q, R, S) = \mathrm{CrossRatio}(S, R, Q, P)$. The theorem is a consequence of (57) and (56).

(59) $\mathrm{CrossRatio}(P, Q, S, R) = \frac{1}{\mathrm{CrossRatio}(P, Q, R, S)}$.

(60) If $P$, $Q$, $R$, and $S$ are collinear and $P \neq R$ and $P \neq S$ and $R \neq Q$ and $S \neq Q$, then $\mathrm{CrossRatio}(Q, P, R, S) = \frac{1}{\mathrm{CrossRatio}(P, Q, R, S)}$. The theorem is a consequence of (57).

(61) If $P$, $Q$, $R$, and $S$ are collinear and $P \neq R$ and $P \neq S$ and $R \neq Q$ and $S \neq Q$, then $\mathrm{CrossRatio}(R, S, Q, P) = \frac{1}{\mathrm{CrossRatio}(P, Q, R, S)}$. The theorem is a consequence of (58).

(62) If $P$, $Q$, $R$, and $S$ are collinear and $P \neq R$ and $P \neq S$ and $R \neq Q$ and $S \neq Q$, then $\mathrm{CrossRatio}(S, R, P, Q) = \frac{1}{\mathrm{CrossRatio}(P, Q, R, S)}$. The theorem is a consequence of (56).

(63) If $P$, $Q$, $R$, and $S$ are collinear and $P$, $Q$, $R$, $S$ are mutually different, then $\mathrm{CrossRatio}(P, R, Q, S) = 1 - \mathrm{CrossRatio}(P, Q, R, S)$. The theorem is a consequence of (17), (20), (14), (13), and (15).

(64) If $P$, $Q$, $R$, and $S$ are collinear and $P$, $Q$, $R$, $S$ are mutually different, then $\mathrm{CrossRatio}(Q, S, P, R) = 1 - \mathrm{CrossRatio}(P, Q, R, S)$. The theorem is

a consequence of (56) and (63).

(65) If $P$, $Q$, $R$, and $S$ are collinear and $P$, $Q$, $R$, $S$ are mutually different, then $\mathrm{CrossRatio}(R,P,S,Q) = 1 - \mathrm{CrossRatio}(P,Q,R,S)$. The theorem is a consequence of (57) and (63).

(66) If $P$, $Q$, $R$, and $S$ are collinear and $P$, $Q$, $R$, $S$ are mutually different, then $\mathrm{CrossRatio}(S,Q,R,P) = 1 - \mathrm{CrossRatio}(P,Q,R,S)$. The theorem is a consequence of (58) and (63).

Let $V$ be a real linear space and $x$ be a 4-tuple of the carrier of $V$. The functor $\mathrm{CrossRatio}(x)$ yielding a real number is defined by

(Def. 30) there exist elements $P$, $Q$, $R$, $S$ of $V$ such that $P = x(1)$ and $Q = x(2)$ and $R = x(3)$ and $S = x(4)$ and $\mathit{it} = \mathrm{CrossRatio}(P,Q,R,S)$.

Now we state the propositions:

(67) If $x = \langle P,Q,R,S\rangle$, then $\mathrm{CrossRatio}(P,Q,R,S) = \mathrm{CrossRatio}(x)$.

(68) Suppose $x = \langle P,Q,R,S\rangle$ and $P$, $Q$, $R$, and $S$ are collinear and $P \neq S$ and $Q \neq R$ and $Q \neq S$. Then $\mathrm{CrossRatio}(x) = \mathrm{CrossRatio}(\sigma_{3412}(x))$. The theorem is a consequence of (56).

(69) Suppose $x = \langle P,Q,R,S\rangle$ and $P$, $Q$, $R$, and $S$ are collinear and $P \neq R$ and $P \neq S$ and $Q \neq R$ and $Q \neq S$. Then

(i) $\mathrm{CrossRatio}(x) = \mathrm{CrossRatio}(\sigma_{2143}(x))$, and

(ii) $\mathrm{CrossRatio}(x) = \mathrm{CrossRatio}(\sigma_{4321}(x))$.

The theorem is a consequence of (57) and (58).

(70) $\mathrm{CrossRatio}(\sigma_{1243}(x)) = \frac{1}{\mathrm{CrossRatio}(x)}$.

(71) Suppose $x = \langle P,Q,R,S\rangle$ and $P$, $Q$, $R$, $S$ are mutually different and $P$, $Q$, $R$, and $S$ are collinear. Then there exists a non unit, non zero real number $r$ such that

(i) $r = \mathrm{CrossRatio}(x)$, and

(ii) $\mathrm{CrossRatio}(\sigma_{1243}(x)) = \mathrm{op1}(r)$.

The theorem is a consequence of (54), (52), and (70).

(72) Suppose $x = \langle P,Q,R,S\rangle$ and $P$, $Q$, $R$, and $S$ are collinear and $P \neq R$ and $P \neq S$ and $Q \neq R$ and $Q \neq S$. Then

(i) $\mathrm{CrossRatio}(\sigma_{1243}(x)) = \frac{1}{\mathrm{CrossRatio}(x)}$, and

(ii) $\mathrm{CrossRatio}(\sigma_{2134}(x)) = \frac{1}{\mathrm{CrossRatio}(x)}$, and

(iii) $\mathrm{CrossRatio}(\sigma_{3421}(x)) = \frac{1}{\mathrm{CrossRatio}(x)}$, and

(iv) $\mathrm{CrossRatio}(\sigma_{4312}(x)) = \frac{1}{\mathrm{CrossRatio}(x)}$.

The theorem is a consequence of (69) and (68).

(73) Suppose $x = \langle P, Q, R, S \rangle$ and $P$, $Q$, $R$, $S$ are mutually different and $P$, $Q$, $R$, and $S$ are collinear. Then

(i) $\mathrm{CrossRatio}(\sigma_{1324}(x)) = 1 - \mathrm{CrossRatio}(x)$, and

(ii) $\mathrm{CrossRatio}(\sigma_{2413}(x)) = 1 - \mathrm{CrossRatio}(x)$, and

(iii) $\mathrm{CrossRatio}(\sigma_{3142}(x)) = 1 - \mathrm{CrossRatio}(x)$, and

(iv) $\mathrm{CrossRatio}(\sigma_{4231}(x)) = 1 - \mathrm{CrossRatio}(x)$.

The theorem is a consequence of (68), (69), and (63).

(74) Suppose $x = \langle P, Q, R, S \rangle$ and $P$, $Q$, $R$, $S$ are mutually different and $P$, $Q$, $R$, and $S$ are collinear. Then

(i) $\mathrm{CrossRatio}(\sigma_{3124}(x)) = \frac{1}{1-\mathrm{CrossRatio}(x)}$, and

(ii) $\mathrm{CrossRatio}(\sigma_{2431}(x)) = \frac{1}{1-\mathrm{CrossRatio}(x)}$, and

(iii) $\mathrm{CrossRatio}(\sigma_{1342}(x)) = \frac{1}{1-\mathrm{CrossRatio}(x)}$, and

(iv) $\mathrm{CrossRatio}(\sigma_{4213}(x)) = \frac{1}{1-\mathrm{CrossRatio}(x)}$.

The theorem is a consequence of (70), (73), (68), and (69).

(75) Suppose $x = \langle P, Q, R, S \rangle$ and $P$, $Q$, $R$, $S$ are mutually different and $P$, $Q$, $R$, and $S$ are collinear. Then

(i) $\mathrm{CrossRatio}(\sigma_{1423}(x)) = \frac{\mathrm{CrossRatio}(x)-1}{\mathrm{CrossRatio}(x)}$, and

(ii) $\mathrm{CrossRatio}(\sigma_{2314}(x)) = \frac{\mathrm{CrossRatio}(x)-1}{\mathrm{CrossRatio}(x)}$, and

(iii) $\mathrm{CrossRatio}(\sigma_{4132}(x)) = \frac{\mathrm{CrossRatio}(x)-1}{\mathrm{CrossRatio}(x)}$, and

(iv) $\mathrm{CrossRatio}(\sigma_{3241}(x)) = \frac{\mathrm{CrossRatio}(x)-1}{\mathrm{CrossRatio}(x)}$.

The theorem is a consequence of (52), (67), (73), (72), (68), and (69).

(76) Suppose $x = \langle P, Q, R, S \rangle$ and $P$, $Q$, $R$, $S$ are mutually different and $P$, $Q$, $R$, and $S$ are collinear. Then

(i) $\mathrm{CrossRatio}(\sigma_{1432}(x)) = \frac{\mathrm{CrossRatio}(x)}{\mathrm{CrossRatio}(x)-1}$, and

(ii) $\mathrm{CrossRatio}(\sigma_{2341}(x)) = \frac{\mathrm{CrossRatio}(x)}{\mathrm{CrossRatio}(x)-1}$, and

(iii) $\mathrm{CrossRatio}(\sigma_{3214}(x)) = \frac{\mathrm{CrossRatio}(x)}{\mathrm{CrossRatio}(x)-1}$, and

(iv) $\mathrm{CrossRatio}(\sigma_{4123}(x)) = \frac{\mathrm{CrossRatio}(x)}{\mathrm{CrossRatio}(x)-1}$.

The theorem is a consequence of (70), (75), (69), and (68).

## 4. Cross-Ratio and the Real Line

Now we state the proposition:

(77) Let us consider elements $x_1$, $x_2$, $x_3$, $x_4$ of $\mathcal{E}^1_{\mathrm{T}}$. Suppose $x_2 \neq x_3$ and $x_3 \neq x_1$ and $x_2 \neq x_4$ and $x_1 \neq x_4$. Then there exist real numbers $a$, $b$, $c$, $d$ such that

(i) $x_1 = \langle a \rangle$, and

(ii) $x_2 = \langle b \rangle$, and

(iii) $x_3 = \langle c \rangle$, and

(iv) $x_4 = \langle d \rangle$, and

(v) $\mathrm{CrossRatio}(\langle x_1, x_2, x_3, x_4 \rangle) = \frac{c-a}{c-b} \cdot \frac{d-b}{d-a}$.

The theorem is a consequence of (43).

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