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Partial Correctness of GCD Algorithm

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Summary. In this paper we present a formalization in the Mizar system [2, 1] of the correctness of the subtraction-based version of Euclid's algorithm computing the greatest common divisor of natural numbers. The algorithm is written in terms of simple-named complex-valued nominative data [11, 4].

The validity of the algorithm is presented in terms of semantic Floyd-Hoare triples over such data [7]. Proofs of the correctness are based on an inference system for an extended Floyd-Hoare logic with partial pre- and post-conditions [8, 10, 5, 3].

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From now on v denotes an object, V, A denote sets, and f denotes a binominative function over simple-named complex-valued nominative data of V and A.

Let us consider A. We say that A is complex containing if and only if

(Def. 1) $\mathbb{C} \subseteq A$.

One can verify that there exists a set which is complex containing and every set which is complex containing is also non empty.

The scheme *BinPredToFunEx* deals with sets $\mathcal{X}$, $\mathcal{Y}$ and a binary predicate $\mathcal{P}$ and states that

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(Sch. 1) There exists a function f from $\mathcal{X} \times \mathcal{Y}$ into *Boolean* such that for every objects x, y such that x, $y \in \mathcal{Y}$ holds if $\mathcal{P}[x, y]$, then $f(x, y) = \mathit{true}$ and if not $\mathcal{P}[x, y]$, then $f(x, y) = \mathit{false}$.

The scheme *BinPredToFunUnique* deals with sets $\mathcal{X}$, $\mathcal{Y}$ and a binary predicate $\mathcal{P}$ and states that

(Sch. 2) For every functions f, g from $\mathcal{X} \times \mathcal{Y}$ into *Boolean* such that for every objects x, y such that x, $y \in \mathcal{Y}$ holds if $\mathcal{P}[x, y]$, then $f(x, y) = \mathit{true}$ and if not $\mathcal{P}[x, y]$, then $f(x, y) = \mathit{false}$ and for every objects x, y such that x, $y \in \mathcal{Y}$ holds if $\mathcal{P}[x, y]$, then $g(x, y) = \mathit{true}$ and if not $\mathcal{P}[x, y]$, then $g(x, y) = \mathit{false}$ holds $f = g$.

The scheme *Lambda2Unique* deals with sets $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and a binary functor $\mathcal{F}$ yielding an object and states that

(Sch. 3) For every functions f, g from $\mathcal{X} \times \mathcal{Y}$ into $\mathcal{Z}$ such that for every objects x, y such that x, $y \in \mathcal{Y}$ holds $f(x, y) = \mathcal{F}(x, y)$ and for every objects x, y such that x, $y \in \mathcal{Y}$ holds $g(x, y) = \mathcal{F}(x, y)$ holds $f = g$.

Let us consider V and A. The functor nonatomicsND(V, A) yielding a set is defined by the term

(Def. 2) the set of all d where d is a non-atomic nominative data of V and A.

Now we state the propositions:

(1) Let us consider an object d. Suppose $d \in$ nonatomicsND(V, A). Then d is a non-atomic nominative data of V and A.

(2) $\emptyset \in$ nonatomicsND(V, A).

Let us consider V and A. One can verify that nonatomicsND(V, A) is non empty and functional.

We say that V is without nonatomic nominative data w.r.t. A if and only if

(Def. 3) A misses nonatomicsND(V, A).

Now we state the propositions:

(3) If V is without nonatomic nominative data w.r.t. A, then for every non-atomic nominative data d of V and A, $d \notin A$.

(4) Suppose V is without nonatomic nominative data w.r.t. A and $v \in V$. Let us consider a non-atomic nominative data d_1 of V and A, and a nominative data d_2 with simple names from V and complex values from A. Then $\operatorname{dom}(d_1 \nabla_a^v d_2) = \{v\} \cup \operatorname{dom} d_1$. The theorem is a consequence of (3).

(5) Suppose V is without nonatomic nominative data w.r.t. A. Let us consider a non-atomic nominative data d of V and A. Suppose $v \in V$ and $d \in \operatorname{dom} f$. Then $\operatorname{dom}((\mathrm{Asg}^v(f))(d)) = \operatorname{dom} d \cup \{v\}$. The theorem is a consequence of (3).

In the sequel d denotes a nominative data with simple names from V and complex values from A.

(6) Let us consider a non-atomic nominative data d_1 of V and A. Suppose $v \in V$ and V is without nonatomic nominative data w.r.t. A. Then $d_1 \nabla_a^v d \in \mathrm{dom}(v \Rightarrow_a)$. The theorem is a consequence of (4).

From now on a, b, c, x, y, z denote elements of V and p, q, r, s denote partial predicates over simple-named complex-valued nominative date of V and A.

Let us consider V, A, d, and a. We say that d is an extended real on a if and only if

(Def. 4) $(a \Rightarrow_a)(d)$ is extended real.

We say that d is a complex on a if and only if

(Def. 5) $(a \Rightarrow_a)(d)$ is complex.

We say that d is a value on a if and only if

(Def. 6) $(a \Rightarrow_a)(d) \in A$.

Now we state the propositions:

(7) If A is complex containing and for every d, d is a complex on a, then for every d, d is a value on a.

(8) If for every d, d is a value on a, then $\mathrm{rng}\, a \Rightarrow_a \subseteq A$.

(9) If for every d, d is a value on a and for every d, d is a value on b, then $\mathrm{rng}\langle a \Rightarrow_a, b \Rightarrow_a\rangle \subseteq A \times A$. The theorem is a consequence of (8).

Let us consider V and A. Let a, b be elements of V and p be a function from $A \times A$ into *Boolean*. The functor lift-binary-pred(p, a, b) yielding a partial predicate over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 7) $p \cdot \langle a \Rightarrow_a, b \Rightarrow_a\rangle$.

Let o_1 be a function from $A \times A$ into A. The functor lift-binary-op(o_1, a, b) yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 8) $o_1 \cdot \langle a \Rightarrow_a, b \Rightarrow_a\rangle$.

The functor Equality(A) yielding a function from $A \times A$ into *Boolean* is defined by

(Def. 9) for every objects a, b such that $a, b \in A$ holds if $a = b$, then $it(a, b) = \mathit{true}$ and if $a \neq b$, then $it(a, b) = \mathit{false}$.

Let us consider V. Let x, y be elements of V. The functor Equality(A, x, y) yielding a partial predicate over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 10) lift-binary-pred(Equality$(A), x, y)$.

Let x, y be objects. We say that x is less than y if and only if

(Def. 11) there exist extended reals x_1, y_1 such that $x_1 = x$ and $y_1 = y$ and $x_1 < y_1$.

Observe that the predicate is irreflexive and asymmetric.

Now we state the proposition:

(10) Let us consider extended reals x, y. If x is not less than y, then y is less than x or $x = y$.

Let us consider A. The functor less(A) yielding a function from $A \times A$ into *Boolean* is defined by

(Def. 12) for every objects x, y such that $x, y \in A$ holds if x is less than y, then $it(x, y) = true$ and if x is not less than y, then $it(x, y) = false$.

Let us consider V. Let x, y be elements of V. The functor less(A, x, y) yielding a partial predicate over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 13) lift-binary-pred(less(A)$, x, y$).

Now we state the propositions:

(11) Suppose for every d, d is a value on a and for every d, d is a value on b. Then $\mathrm{dom}(\mathrm{Equality}(A, a, b)) = \mathrm{dom}(a \Rightarrow_a) \cap \mathrm{dom}(b \Rightarrow_a)$. The theorem is a consequence of (9).

(12) Suppose for every d, d is a value on a and for every d, d is a value on b. Then $\mathrm{dom}(\mathrm{less}(A, a, b)) = \mathrm{dom}(a \Rightarrow_a) \cap \mathrm{dom}(b \Rightarrow_a)$. The theorem is a consequence of (9).

(13) Suppose for every d, d is a value on a and for every d, d is a value on b and for every d, d is an extended real on a and for every d, d is an extended real on b. Then $\neg\,\mathrm{Equality}(A, a, b) = \mathrm{less}(A, a, b) \vee \mathrm{less}(A, b, a)$.

(14) Suppose for every d, d is a value on a and for every d, d is a value on b and d is an extended real on a and d is an extended real on b and $d \in \mathrm{dom}(\neg\,\mathrm{Equality}(A, a, b))$ and $(\neg\,\mathrm{Equality}(A, a, b))(d) = true$. Then

(i) $d \in \mathrm{dom}(\mathrm{less}(A, a, b))$ and $(\mathrm{less}(A, a, b))(d) = true$, or

(ii) $d \in \mathrm{dom}(\mathrm{less}(A, b, a))$ and $(\mathrm{less}(A, b, a))(d) = true$.

The theorem is a consequence of (10) and (12).

Let x, y be objects. Assume x is a complex number and y is a complex number. The functor $x - y$ yielding a complex number is defined by

(Def. 14) there exist complex numbers x_1, y_1 such that $x_1 = x$ and $y_1 = y$ and $it = x_1 - y_1$.

Let us consider A. Assume A is complex containing. The functor subtraction A yielding a function from $A \times A$ into A is defined by

(Def. 15) for every objects x, y such that $x, y \in A$ holds $it(x, y) = x - y$.

Let us consider V. Let x, y be elements of V. The functor subtraction(A, x, y) yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 16) lift-binary-op(subtraction $A, x, y)$.

Let us consider a and b. The functor gcd-conditional-step(V, A, a, b) yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 17) IF(less(A, b, a), Asg^a(subtraction(A, a, b)), $\mathrm{id}_{\mathrm{PP}}(\mathrm{ND}_{\mathrm{SC}}(V, A))$).

The functor gcd-loop-body(V, A, a, b) yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 18) gcd-conditional-step$(V, A, a, b) \bullet$ gcd-conditional-step(V, A, b, a).

The functor gcd-main-loop(V, A, a, b) yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 19) WH($\neg$ Equality(A, a, b), gcd-loop-body(V, A, a, b)).

Let us consider x and y. The functor gcd-var-init(V, A, a, b, x, y) yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 20) $\mathrm{Asg}^a(x \Rightarrow_a) \bullet \mathrm{Asg}^b(y \Rightarrow_a)$.

The functor gcd-main-part(V, A, a, b, x, y) yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 21) gcd-var-init$(V, A, a, b, x, y) \bullet$ gcd-main-loop(V, A, a, b).

Let us consider z. The functor gcd-program(V, A, a, b, x, y, z) yielding a binominative function over simple-named complex-valued nominative data of V and A is defined by the term

(Def. 22) gcd-main-part$(V, A, a, b, x, y) \bullet \mathrm{Asg}^z(a \Rightarrow_a)$.

From now on x_0, y_0 denote natural numbers.

Let us consider V, A, x, y, x_0, and y_0. Let d be an object. We say that x_0, y_0 and d constitute a valid input for the gcd w.r.t. V, A, x and y if and only if

(Def. 23) there exists a non-atomic nominative data d_1 of V and A such that $d = d_1$ and $x, y \in \mathrm{dom}\, d_1$ and $d_1(x) = x_0$ and $d_1(y) = y_0$.

The functor valid-gcd-input(V, A, x, y, x_0, y_0) yielding a partial predicate over simple-named complex-valued nominative data of V and A is defined by

(Def. 24) $\operatorname{dom} it = \mathrm{ND_{SC}}(V, A)$ and for every object d such that $d \in \operatorname{dom} it$ holds if x_0, y_0 and d constitute a valid input for the gcd w.r.t. V, A, x and y, then $it(d) = \mathit{true}$ and if x_0, y_0 and d do not constitute a valid input for the gcd w.r.t. V, A, x and y, then $it(d) = \mathit{false}$.

One can check that valid-gcd-input(V, A, x, y, x_0, y_0) is total.

Let us consider z. Let d be an object. We say that x_0, y_0 and d constitute a valid output for the gcd w.r.t. V, A and z if and only if

(Def. 25) there exists a non-atomic nominative data d_1 of V and A such that $d = d_1$ and $z \in \operatorname{dom} d_1$ and $d_1(z) = \gcd(x_0, y_0)$.

The functor valid-gcd-output(V, A, z, x_0, y_0) yielding a partial predicate over simple-named complex-valued nominative data of V and A is defined by

(Def. 26) $\operatorname{dom} it = \{d$, where d is a nominative data with simple names from V and complex values from $A : d \in \operatorname{dom}(z \Rightarrow_a)\}$ and for every object d such that $d \in \operatorname{dom} it$ holds if x_0, y_0 and d constitute a valid output for the gcd w.r.t. V, A and z, then $it(d) = \mathit{true}$ and if x_0, y_0 and d do not constitute a valid output for the gcd w.r.t. V, A and z, then $it(d) = \mathit{false}$.

Let us consider a and b. Let d be an object. We say that x_0, y_0 and d constitute a valid invariant for the gcd w.r.t. V, A, a and b if and only if

(Def. 27) there exists a non-atomic nominative data d_1 of V and A such that $d = d_1$ and a, $b \in \operatorname{dom} d_1$ and there exist natural numbers x, y such that $x = d_1(a)$ and $y = d_1(b)$ and $\gcd(x, y) = \gcd(x_0, y_0)$.

The functor gcd-inv(V, A, a, b, x_0, y_0) yielding a partial predicate over simple-named complex-valued nominative data of V and A is defined by

(Def. 28) $\operatorname{dom} it = \mathrm{ND_{SC}}(V, A)$ and for every object d such that $d \in \operatorname{dom} it$ holds if x_0, y_0 and d constitute a valid invariant for the gcd w.r.t. V, A, a and b, then $it(d) = \mathit{true}$ and if x_0, y_0 and d do not constitute a valid invariant for the gcd w.r.t. V, A, a and b, then $it(d) = \mathit{false}$.

Observe that gcd-inv(V, A, a, b, x_0, y_0) is total.

Now we state the propositions:

(15) $\langle \sim \mathrm{S_P}(p, x \Rightarrow_a, a), \mathrm{Asg}^a(x \Rightarrow_a), p\rangle$ is an SFHT of $\mathrm{ND_{SC}}(V, A)$.

(16) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and $a \neq b$ and $a \neq y$.
Then $\langle$valid-gcd-input(V, A, x, y, x_0, y_0), gcd-var-init(V, A, a, b, x, y), gcd-inv $(V, A, a, b, x_0, y_0)\rangle$ is an SFHT of $\mathrm{ND_{SC}}(V, A)$.
PROOF: Set $D_3 = x \Rightarrow_a$. Set $D_4 = y \Rightarrow_a$. Set $p =$ gcd-inv(V, A, a, b, x_0, y_0). Set $Q = \mathrm{S_P}(p, D_4, b)$. Set $P = \mathrm{S_P}(Q, D_3, a)$. Set $G = \mathrm{Asg}^b(D_4)$. Set $I =$ valid-gcd-input(V, A, x, y, x_0, y_0). $\langle \sim Q, G, p\rangle$ is an SFHT of $\mathrm{ND_{SC}}(V, A)$. $I \models P$. $\square$

(17) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and $a \neq b$ and A is complex containing and for every d, d is a complex on a and for every d, d is a complex on b.
Then $\langle \text{less}(A, b, a) \wedge \text{gcd-inv}(V, A, a, b, x_0, y_0), \text{Asg}^a(\text{subtraction}(A, a, b)), \text{gcd-inv}(V, A, a, b, x_0, y_0)\rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$.
Proof: Set $i = \text{gcd-inv}(V, A, a, b, x_0, y_0)$. Set $l = \text{less}(A, b, a)$. Set $D = \text{subtraction}(A, a, b)$. Set $f = \text{Asg}^a(D)$. Set $p = l \wedge i$. For every d such that $d \in \text{dom}\, p$ and $p(d) = \mathit{true}$ and $d \in \text{dom}\, f$ and $f(d) \in \text{dom}\, i$ holds $i(f(d)) = \mathit{true}$. □

(18) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and $a \neq b$ and A is complex containing and for every d, d is a complex on a and for every d, d is a complex on b.
Then $\langle \text{less}(A, a, b) \wedge \text{gcd-inv}(V, A, a, b, x_0, y_0), \text{Asg}^b(\text{subtraction}(A, b, a)), \text{gcd-inv}(V, A, a, b, x_0, y_0)\rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$.
Proof: Set $i = \text{gcd-inv}(V, A, a, b, x_0, y_0)$. Set $l = \text{less}(A, a, b)$. Set $D = \text{subtraction}(A, b, a)$. Set $f = \text{Asg}^b(D)$. Set $p = l \wedge i$. For every d such that $d \in \text{dom}\, p$ and $p(d) = \mathit{true}$ and $d \in \text{dom}\, f$ and $f(d) \in \text{dom}\, i$ holds $i(f(d)) = \mathit{true}$ by [6, (23)], [9, (9),(10)]. □

(19) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and $a \neq b$ and A is complex containing and for every d, d is a complex on a and for every d, d is a complex on b.
Then $\langle \text{gcd-inv}(V, A, a, b, x_0, y_0), \text{gcd-conditional-step}(V, A, a, b), \text{gcd-inv}(V, A, a, b, x_0, y_0)\rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$. The theorem is a consequence of (17).

(20) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and $a \neq b$ and A is complex containing and for every d, d is a complex on a and for every d, d is a complex on b.
Then $\langle \text{gcd-inv}(V, A, a, b, x_0, y_0), \text{gcd-conditional-step}(V, A, b, a), \text{gcd-inv}(V, A, a, b, x_0, y_0)\rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$. The theorem is a consequence of (18).

(21) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and $a \neq b$ and A is complex containing and for every d, d is a complex on a and for every d, d is a complex on b. Then $\langle \text{gcd-inv}(V, A, a, b, x_0, y_0), \text{gcd-loop-body}(V, A, a, b), \text{gcd-inv}(V, A, a, b, x_0, y_0)\rangle$ is an SFHT of $\text{ND}_{\text{SC}}(V, A)$. The theorem is a consequence of (19) and (20).

(22) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and $a \neq b$ and A is complex containing and for every d, d is a complex on a and for every d, d is a complex on b.
Then $\langle \sim \text{gcd-inv}(V, A, a, b, x_0, y_0), \text{gcd-loop-body}(V, A, a, b), \text{gcd-inv}$

$(V, A, a, b, x_0, y_0))\rangle$ is an SFHT of $\mathrm{ND_{SC}}(V, A)$. The theorem is a consequence of (20).

(23) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and $a \neq b$ and A is complex containing and for every d, d is a complex on a and for every d, d is a complex on b. Then $\langle$gcd-inv(V, A, a, b, x_0, y_0), gcd-main-loop(V, A, a, b), Equality$(A, a, b) \wedge$ gcd-inv$(V, A, a, b, x_0, y_0)\rangle$ is an SFHT of $\mathrm{ND_{SC}}(V, A)$. The theorem is a consequence of (21) and (22).

(24) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and $a \neq b$ and $a \neq y$ and A is complex containing and for every d, d is a complex on a and for every d, d is a complex on b. Then $\langle$valid-gcd-input(V, A, x, y, x_0, y_0), gcd-main-part(V, A, a, b, x, y), Equality $(A, a, b) \wedge$ gcd-inv$(V, A, a, b, x_0, y_0)\rangle$ is an SFHT of $\mathrm{ND_{SC}}(V, A)$. The theorem is a consequence of (16) and (23).

(25) Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and for every d, d is a value on a and for every d, d is a value on b. Then $\langle$Equality$(A, a, b) \wedge$ gcd-inv(V, A, a, b, x_0, y_0), $\mathrm{Asg}^z(a \Rightarrow_a)$, valid-gcd-output$(V, A, z, x_0, y_0)\rangle$ is an SFHT of $\mathrm{ND_{SC}}(V, A)$.

PROOF: Set $D_1 = a \Rightarrow_a$. Set $q =$ Equality$(A, a, b) \wedge$gcd-inv(V, A, a, b, x_0, y_0). Set $r =$ valid-gcd-output(V, A, z, x_0, y_0). Set $s_3 = \mathrm{S_P}(r, D_1, z)$. $q \models s_3$. □

(26) PARTIAL CORRECTNESS OF GCD ALGORITHM:
Suppose V is not empty and V is without nonatomic nominative data w.r.t. A and $a \neq b$ and $a \neq y$ and A is complex containing and for every d, d is a complex on a and for every d, d is a complex on b. Then $\langle$valid-gcd-input(V, A, x, y, x_0, y_0), gcd-program(V, A, a, b, x, y, z), valid-gcd-output$(V, A, z, x_0, y_0)\rangle$ is an SFHT of $\mathrm{ND_{SC}}(V, A)$. The theorem is a consequence of (7), (24), (25), and (11).

REFERENCES

[1] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, and Karol Pąk. The role of the Mizar Mathematical Library for interactive proof development in Mizar. *Journal of Automated Reasoning*, 61(1):9–32, 2018. doi:10.1007/s10817-017-9440-6.

[2] Adam Grabowski, Artur Korniłowicz, and Adam Naumowicz. Four decades of Mizar. *Journal of Automated Reasoning*, 55(3):191–198, 2015. doi:10.1007/s10817-015-9345-1.

[3] Ievgen Ivanov and Mykola Nikitchenko. On the sequence rule for the Floyd-Hoare logic with partial pre- and post-conditions. In *Proceedings of the 14th International Conference on ICT in Education, Research and Industrial Applications. Integration, Harmonization and Knowledge Transfer. Volume II: Workshops, Kyiv, Ukraine, May 14–17, 2018*, volume 2104 of *CEUR Workshop Proceedings*, pages 716–724, 2018.

[4] Ievgen Ivanov, Mykola Nikitchenko, Andrii Kryvolap, and Artur Korniłowicz. Simple-named complex-valued nominative data – definition and basic operations. *Formalized Mathematics*, 25(**3**):205–216, 2017. doi:10.1515/forma-2017-0020.

[5] Ievgen Ivanov, Artur Korniłowicz, and Mykola Nikitchenko. Implementation of the

composition-nominative approach to program formalization in Mizar. *The Computer Science Journal of Moldova*, 26(1):59–76, 2018.

[6] Ievgen Ivanov, Artur Korniłowicz, and Mykola Nikitchenko. On an algorithmic algebra over simple-named complex-valued nominative data. *Formalized Mathematics*, 26(2):149–158, 2018. doi:10.2478/forma-2018-0012.

[7] Ievgen Ivanov, Artur Korniłowicz, and Mykola Nikitchenko. An inference system of an extension of Floyd-Hoare logic for partial predicates. *Formalized Mathematics*, 26(**2**): 159–164, 2018. doi:10.2478/forma-2018-0013.

[8] Artur Korniłowicz, Andrii Kryvolap, Mykola Nikitchenko, and Ievgen Ivanov. An approach to formalization of an extension of Floyd-Hoare logic. In Vadim Ermolayev, Nick Bassiliades, Hans-Georg Fill, Vitaliy Yakovyna, Heinrich C. Mayr, Vyacheslav Kharchenko, Vladimir Peschanenko, Mariya Shyshkina, Mykola Nikitchenko, and Aleksander Spivakovsky, editors, *Proceedings of the 13th International Conference on ICT in Education, Research and Industrial Applications. Integration, Harmonization and Knowledge Transfer, Kyiv, Ukraine, May 15–18, 2017*, volume 1844 of *CEUR Workshop Proceedings*, pages 504–523. CEUR-WS.org, 2017.

[9] Artur Korniłowicz, Ievgen Ivanov, and Mykola Nikitchenko. Kleene algebra of partial predicates. *Formalized Mathematics*, 26(**1**):11–20, 2018. doi:10.2478/forma-2018-0002.

[10] Andrii Kryvolap, Mykola Nikitchenko, and Wolfgang Schreiner. Extending Floyd-Hoare logic for partial pre- and postconditions. In Vadim Ermolayev, Heinrich C. Mayr, Mykola Nikitchenko, Aleksander Spivakovsky, and Grygoriy Zholtkevych, editors, *Information and Communication Technologies in Education, Research, and Industrial Applications: 9th International Conference, ICTERI 2013, Kherson, Ukraine, June 19–22, 2013, Revised Selected Papers*, pages 355–378. Springer International Publishing, 2013. ISBN 978-3-319-03998-5. doi:10.1007/978-3-319-03998-5_18.

[11] Volodymyr G. Skobelev, Mykola Nikitchenko, and Ievgen Ivanov. On algebraic properties of nominative data and functions. In Vadim Ermolayev, Heinrich C. Mayr, Mykola Nikitchenko, Aleksander Spivakovsky, and Grygoriy Zholtkevych, editors, *Information and Communication Technologies in Education, Research, and Industrial Applications – 10th International Conference, ICTERI 2014, Kherson, Ukraine, June 9–12, 2014, Revised Selected Papers*, volume 469 of *Communications in Computer and Information Science*, pages 117–138. Springer, 2014. ISBN 978-3-319-13205-1. doi:10.1007/978-3-319-13206-8_6.

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