FORMALIZED MATHEMATICS
Vol. 26, No. 2, Pages 175–181, 2018
DOI: 10.2478/forma-2018-0015

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Basic Diophantine Relations[1]

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Summary. The main purpose of formalization is to prove that two equations $\mathrm{y}_a(z) = y$, $y = x^z$ are Diophantine. These equations are explored in the proof of Matiyasevich's negative solution of Hilbert's tenth problem.

In our previous work [6], we showed that from the diophantine standpoint these equations can be obtained from lists of several basic Diophantine relations as linear equations, finite products, congruences and inequalities. In this formalization, we express these relations in terms of Diophantine set introduced in [7]. We prove that these relations are Diophantine and then we prove several second-order theorems that provide the ability to combine Diophantine relation using conjunctions and alternatives as well as to substitute the right-hand side of a given Diophantine equality as an argument in a given Diophantine relation. Finally, we investigate the possibilities of our approach to prove that the two equations, being the main purpose of this formalization, are Diophantine.

The formalization by means of Mizar system [3], [2] follows Z. Adamowicz, P. Zbierski [1] as well as M. Davis [4].

MSC: 11D45 03B35 68T99

Keywords: Hilbert's 10th problem; Diophantine relations

MML identifier: `HILB10_3`, version: `8.1.08 5.52.1328`

1. Preliminaries

From now on n, m, k denote natural numbers, p, q denote n-element finite 0-sequences of $\mathbb{N}$, i_1, i_2, i_3, i_4, i_5, i_6 denote elements of n, and a, b, c, d, e denote integers.

[1] This work has been financed by the resources of the Polish National Science Centre granted by decision no. DEC-2015/19/D/ST6/01473.

ISSN 1426–2630(Print), 1898-9934(Online)

Let X be a set, p be a $\mathbb{Z}$-valued series of X, $\mathbb{R}_{\mathrm{F}}$, and a be an integer element of $\mathbb{R}_{\mathrm{F}}$. Observe that $a \cdot p$ is $\mathbb{Z}$-valued.

Now we state the propositions:

(1) Let us consider a non empty ordinal number O, an element i of O, an add-associative, right zeroed, right complementable, well unital, distributive, non trivial double loop structure L, and a function x from O into L. Then $\mathrm{eval}(1\_1(i, L), x) = x(i)$.

(2) i_1 is an element of $n + k$.

(3) If $k < m$, then $n + k \in n + m$.

(4) Let us consider an $(n + k)$-element finite 0-sequence p. If $n \neq 0$ and $k \neq 0$, then $(p\restriction n)(i_1) = p(i_1)$.

2. Basic Diophantine Relations

Now we state the propositions:

(5) Let us consider a diophantine subset A of the n-xtuples of $\mathbb{N}$, and k. Suppose $k \leqslant n$. Then $\{p\restriction k : p \in A\}$ is a diophantine subset of the k-xtuples of $\mathbb{N}$.

PROOF: Consider k_1 being a natural number, Q being a $\mathbb{Z}$-valued polynomial of $n + k_1$,$\mathbb{R}_{\mathrm{F}}$ such that for every object s, $s \in A$ iff there exists an n-element finite 0-sequence x of $\mathbb{N}$ and there exists a k_1-element finite 0-sequence y of $\mathbb{N}$ such that $s = x$ and $\mathrm{eval}(Q, {}^{@}(x \frown y)) = 0$.

Set $D = \{p\restriction k$, where p is an n-element finite 0-sequence of $\mathbb{N} : p \in A\}$. $D \subseteq$ the k-xtuples of $\mathbb{N}$. Reconsider $k_2 = n - k$ as a natural number. Reconsider $P = Q$ as a $\mathbb{Z}$-valued polynomial of $k + (k_2 + k_1)$,$\mathbb{R}_{\mathrm{F}}$. For every object s, $s \in D$ iff there exists a k-element finite 0-sequence x of $\mathbb{N}$ and there exists a $(k_2 + k_1)$-element finite 0-sequence y of $\mathbb{N}$ such that $s = x$ and $\mathrm{eval}(P, {}^{@}(x \frown y)) = 0$ by [5, (13)], [8, (54),(17),(27)]. □

(6) Let us consider integers a, b, c, i_1, and i_2. Then $\{p : a \cdot p(i_1) = b \cdot p(i_2) + c\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. The theorem is a consequence of (1).

(7) $\{p : a \cdot p(i_1) > b \cdot p(i_2) + c\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. The theorem is a consequence of (2) and (1).

The scheme *UnionDiophantine* deals with a natural number n and a unary predicate $\mathcal{P}$, $\mathcal{Q}$ and states that

(Sch. 1) $\{p$, where p is an n-element finite 0-sequence of $\mathbb{N} : \mathcal{P}[p]$ or $\mathcal{Q}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$

provided

- $\{p$, where p is an n-element finite 0-sequence of $\mathbb{N} : \mathcal{P}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$ and
- $\{p$, where p is an n-element finite 0-sequence of $\mathbb{N} : \mathcal{Q}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.

The scheme *Eq* deals with a natural number n and a unary predicate $\mathcal{P}$, $\mathcal{Q}$ and states that

(Sch. 2) $\{p$, where p is an n-element finite 0-sequence of $\mathbb{N} : \mathcal{P}[p]\} = \{q$, where q is an n-element finite 0-sequence of $\mathbb{N} : \mathcal{Q}[q]\}$

provided

- for every n-element finite 0-sequence p of $\mathbb{N}$, $\mathcal{P}[p]$ iff $\mathcal{Q}[p]$.

Now we state the propositions:

(8) $\{p : a \cdot p(i_1) \geqslant b \cdot p(i_2) + c\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. PROOF: Define $\mathcal{P}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) > b \cdot \$_1(i_2) + c$. Define $\mathcal{Q}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) = b \cdot \$_1(i_2) + c$. Define $\mathcal{R}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}[\$_1]$ or $\mathcal{Q}[\$_1]$. Define $\mathcal{S}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) \geqslant b \cdot \$_1(i_2) + c$. $\{p : \mathcal{P}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. $\{p : \mathcal{Q}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. $\{p : \mathcal{P}[p]$ or $\mathcal{Q}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. $\{p : \mathcal{R}[p]\} = \{q : \mathcal{S}[q]\}$. □

(9) $\{p : a \cdot p(i_1) = b \cdot p(i_2) \cdot p(i_3)\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. The theorem is a consequence of (1).

(10) $\{p$: there exists a natural number z such that $a \cdot p(i_1) = b \cdot p(i_2) + z \cdot c \cdot p(i_3)\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. The theorem is a consequence of (2) and (1).

The scheme *IntersectionDiophantine* deals with a natural number n and a unary predicate $\mathcal{P}$, $\mathcal{Q}$ and states that

(Sch. 3) $\{p$, where p is an n-element finite 0-sequence of $\mathbb{N} : \mathcal{P}[p]$ and $\mathcal{Q}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$

provided

- $\{p$, where p is an n-element finite 0-sequence of $\mathbb{N} : \mathcal{P}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$ and
- $\{p$, where p is an n-element finite 0-sequence of $\mathbb{N} : \mathcal{Q}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.

The scheme *Substitution* deals with a 6-ary predicate $\mathcal{P}$ and a ternary functor $\mathcal{F}$ yielding a natural object and states that

(Sch. 4) For every i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}[p(i_1), p(i_2), \mathcal{F}(p(i_3), p(i_4), p(i_5)), p(i_3), p(i_4), p(i_5)]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$

provided

- for every i_1, i_2, i_3, i_4, i_5, and i_6, $\{p : \mathcal{P}[p(i_1), p(i_2), p(i_3), p(i_4), p(i_5), p(i_6)]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$ and
- for every i_1, i_2, i_3, and i_4, $\{p : \mathcal{F}(p(i_1), p(i_2), p(i_3)) = p(i_4)\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.

The scheme *SubstitutionInt* deals with a ternary predicate $\mathcal{P}$ and a ternary functor $\mathcal{F}$ yielding an integer and states that

(Sch. 5) For every i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}[p(i_1), p(i_2), \mathcal{F}(p(i_3), p(i_4), p(i_5))]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$

provided

- for every i_1, i_2, i_3, and a, $\{p : \mathcal{P}[p(i_1), p(i_2), a \cdot p(i_3)]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$ and
- for every i_1, i_2, i_3, i_4, and a, $\{p : \mathcal{F}(p(i_1), p(i_2), p(i_3)) = a \cdot p(i_4)\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.

Now we state the propositions:

(11) $\{p : a \cdot p(i_1) = b \cdot p(i_2) + c \cdot p(i_3) + d\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. The theorem is a consequence of (1).

(12) $\{p : p(i_1) = a \cdot p(i_2)\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. The theorem is a consequence of (6).

(13) $\{p : a \cdot p(i_1) = b\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.
PROOF: Set i_2 = the element of n. Define $\mathcal{P}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) = b$. Define $\mathcal{Q}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) = 0 \cdot \$_1(i_2) + b$. $\{p : \mathcal{P}[p]\} = \{q : \mathcal{Q}[q]\}$. □

(14) $\{p : p(i_1) = a\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.
PROOF: Set i_2 = the element of n. Define $\mathcal{P}$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(i_1) = a$. Define $\mathcal{Q}$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(i_1) = 0 \cdot \$_1(i_2) + a$. $\{p : \mathcal{P}[p]\} = \{q : \mathcal{Q}[q]\}$. □

(15) $\{p : p(i_1) = a \cdot p(i_2) + b\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.
PROOF: Define $\mathcal{P}$[finite 0-sequence of $\mathbb{N}$] $\equiv \$_1(i_1) = a \cdot \$_1(i_2) + b$. Define $\mathcal{Q}$[finite 0-sequence of $\mathbb{N}$] $\equiv 1 \cdot \$_1(i_1) = a \cdot \$_1(i_2) + b$. $\{p : \mathcal{P}[p]\} = \{q : \mathcal{Q}[q]\}$. □

(16) $\{p : a \cdot p(i_1) \neq b \cdot p(i_2) + c\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.
PROOF: Define $\mathcal{P}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) > b \cdot \$_1(i_2) + c$. Define $\mathcal{Q}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) + -c < b \cdot \$_1(i_2)$. Define $\mathcal{R}$[finite

0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}[\$_1]$ or $\mathcal{Q}[\$_1]$. Define $\mathcal{S}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) \neq b \cdot \$_1(i_2) + c$. $\{p : \mathcal{P}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. $\{p : \mathcal{Q}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. $\{p : \mathcal{P}[p]$ or $\mathcal{Q}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. $\mathcal{R}[p]$ iff $\mathcal{S}[p]$. $\{p : \mathcal{R}[p]\} = \{q : \mathcal{S}[q]\}$. □

(17) $\{p : a \cdot p(i_1) > b \cdot p(i_2) \cdot p(i_3)\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.

PROOF: Define $\mathcal{P}$[natural number, natural number, integer] $\equiv a \cdot \$_1 > \$_3 + 0$. Define $\mathcal{F}$(natural number, natural number, natural number) $= b \cdot \$_2 \cdot \$_3$. Define $\mathcal{Q}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) > b \cdot \$_1(i_2) \cdot \$_1(i_3) + 0$. Define $\mathcal{R}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) > b \cdot \$_1(i_2) \cdot \$_1(i_3)$.

For every n, i_1, i_2, i_3, and c, $\{p : \mathcal{P}[p(i_1), p(i_2), c \cdot p(i_3)]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. For every n, i_1, i_2, i_3, i_4, and c, $\{p : \mathcal{F}(p(i_1), p(i_2), p(i_3)) = c \cdot p(i_4)\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. For every n, i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}[p(i_1), p(i_2), \mathcal{F}(p(i_3), p(i_4), p(i_5))]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. $\{p : \mathcal{Q}[p]\} = \{q : \mathcal{R}[q]\}$. □

Let us consider a, b, c, i_1, i_2, and i_3. Now we state the propositions:

(18) $\{p : a \cdot p(i_1) < b \cdot p(i_2) + c \cdot p(i_3)\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.

PROOF: Define $\mathcal{P}$[natural number, natural number, integer] $\equiv a \cdot \$_1 + 0 < \$_3$. Define $\mathcal{F}$(natural number, natural number, natural number) $= b \cdot \$_2 + c \cdot \$_3 + 0$. Define $\mathcal{Q}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) + 0 < b \cdot \$_1(i_2) + c \cdot \$_1(i_3) + 0$. Define $\mathcal{R}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) < b \cdot \$_1(i_2) + c \cdot \$_1(i_3)$. For every n, i_1, i_2, i_3, and d, $\{p : \mathcal{P}[p(i_1), p(i_2), d \cdot p(i_3)]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.

For every n, i_1, i_2, i_3, i_4, and d, $\{p : \mathcal{F}(p(i_1), p(i_2), p(i_3)) = d \cdot p(i_4)\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. For every n, i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}[p(i_1), p(i_2), \mathcal{F}(p(i_3), p(i_4), p(i_5))]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. $\{p : \mathcal{Q}[p]\} = \{q : \mathcal{R}[q]\}$. □

(19) $\{p : a \cdot p(i_1) = b \cdot p(i_2) -' c \cdot p(i_3)\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.

PROOF: Define $\mathcal{P}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) = b \cdot \$_1(i_2) + (-c) \cdot \$_1(i_3) + 0$. Define $\mathcal{Q}$[finite 0-sequence of $\mathbb{N}$] $\equiv b \cdot \$_1(i_2) \geqslant c \cdot \$_1(i_3) + 0$. Define $\mathcal{R}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) = 0 \cdot \$_1(i_2) \cdot \$_1(i_3)$. Define $\mathcal{S}$[finite 0-sequence of $\mathbb{N}$] $\equiv b \cdot \$_1(i_2) + 0 < c \cdot \$_1(i_3)$. Define $\mathcal{U}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{P}[\$_1]$ and $\mathcal{Q}[\$_1]$. $\{p : \mathcal{P}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. $\{p : \mathcal{Q}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. $\{p : \mathcal{P}[p]$ and $\mathcal{Q}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.

Define $\mathcal{W}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{R}[\$_1]$ and $\mathcal{S}[\$_1]$. $\{p : \mathcal{R}[p]\}$ is

a diophantine subset of the n-xtuples of $\mathbb{N}$. $\{p : \mathcal{S}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. $\{p : \mathcal{R}[p]$ and $\mathcal{S}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{V}$[finite 0-sequence of $\mathbb{N}$] $\equiv \mathcal{U}[\$_1]$ or $\mathcal{W}[\$_1]$. Define $\mathcal{T}$[finite 0-sequence of $\mathbb{N}$] $\equiv a \cdot \$_1(i_1) = b \cdot \$_1(i_2) -' c \cdot \$_1(i_3)$. $\{p : \mathcal{U}[p]$ or $\mathcal{W}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. $\mathcal{V}[p]$ iff $\mathcal{T}[p]$. $\{p : \mathcal{V}[p]\} = \{q : \mathcal{T}[q]\}$. □

(20) $\{p : a \cdot p(i_1) = b \cdot p(i_2) -' c\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.
PROOF: Define $\mathcal{P}$[natural number, natural number, integer] $\equiv a \cdot \$_1 = b \cdot \$_2 -' \$_3$. For every n, i_1, i_2, i_3, and d, $\{p : \mathcal{P}[p(i_1), p(i_2), d \cdot p(i_3)]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. Define $\mathcal{F}$(natural number, natural number, natural number) $= c$. For every n, i_1, i_2, i_3, i_4, and d, $\{p : \mathcal{F}(p(i_1), p(i_2), p(i_3)) = d \cdot p(i_4)\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. For every n, i_1, i_2, i_3, i_4, and i_5, $\{p : \mathcal{P}[p(i_1), p(i_2), \mathcal{F}(p(i_3), p(i_4), p(i_5))]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. □

(21) $\{p : a \cdot p(i_1) \equiv b \cdot p(i_2) \pmod{c \cdot p(i_3)}\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$.
PROOF: Define $\mathcal{P}$[finite 0-sequence of $\mathbb{N}$] $\equiv$ there exists a natural number z such that $a \cdot \$_1(i_1) = b \cdot \$_1(i_2) + z \cdot c \cdot \$_1(i_3)$. Define $\mathcal{Q}$[finite 0-sequence of $\mathbb{N}$] $\equiv$ there exists a natural number z such that $b \cdot \$_1(i_2) = a \cdot \$_1(i_1) + z \cdot c \cdot \$_1(i_3)$. $\{p : \mathcal{P}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. $\{p : \mathcal{Q}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. $\{p : \mathcal{P}[p]$ or $\mathcal{Q}[p]\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. Set $P = \{p : a \cdot p(i_1) \equiv b \cdot p(i_2) \pmod{c \cdot p(i_3)}\}$. $P \subseteq \{p : \mathcal{P}[p]$ or $\mathcal{Q}[p]\}$. $\{p : \mathcal{P}[p]$ or $\mathcal{Q}[p]\} \subseteq P$. □

(22) $\{p : \langle a \cdot p(i_1),\ b \cdot p(i_2)\rangle$ is Pell's solution of $(c \cdot p(i_3))^{\mathbf{2}} -' 1\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. The theorem is a consequence of (2), (3), (9), (20), (6), (5), and (4).

3. Main Lemmas

Let us consider i_1, i_2, and i_3. Now we state the propositions:

(23) $\{p : p(i_1) = \mathtt{y}_{p(i_2)}(p(i_3))$ and $p(i_2) > 1\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. The theorem is a consequence of (2), (3), (7), (22), (8), (21), (14), (12), (9), (5), and (4).

(24) $\{p : p(i_2) = p(i_1)^{p(i_3)}\}$ is a diophantine subset of the n-xtuples of $\mathbb{N}$. The theorem is a consequence of (2), (3), (14), (7), (6), (9), (23), (17), (8), (18), (5), and (4).

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Accepted June 29, 2018