FORMALIZED MATHEMATICS
Vol. 26, No. 1, Pages 1–9, 2018
DOI: 10.2478/forma-2018-0001

DE GRUYTER
OPEN
degruyter.com/view/j/forma

Introduction to Stochastic Finance: Random Variables and Arbitrage Theory

Peter Jaeger
Siegmund-Schacky-Str. 18a
80993 Munich, Germany

Summary. Using the Mizar system [1], [5], we start to show, that the Call-Option, the Put-Option and the Straddle (more generally defined as in the literature) are random variables ([4], p. 15), see (Def. 1) and (Def. 2). Next we construct and prove the simple random variables ([2], p. 14) in (Def. 8).

In the third section, we introduce the definition of arbitrage opportunity, see (Def. 12). Next we show, that this definition can be characterized in a different way (Lemma 1.3. in [4], p. 5), see (17). In our formalization for Lemma 1.3 we make the assumption that φ is a sequence of real numbers (there are only finitely many valued of interest, the values of φ in R^d). For the definition of almost sure with probability 1 see p. 6 in [2]. Last we introduce the risk-neutral probability (Definition 1.4, p. 6 in [4]), here see (Def. 16).

We give an example in real world: Suppose you have some assets like bonds (riskless assets). Then we can fix our price for these bonds with x for today and $x \cdot (1 + r)$ for tomorrow, r is the interest rate. So we simply assume, that in every possible market evolution of tomorrow we have a determinated value. Then every probability measure of Ω_{fut1} is a risk-neutral measure, see (21). This example shows the existence of some risk-neutral measure. If you find more than one of them, you can determine – with an additional conidition to the probability measures – whether a market model is arbitrage free or not (see Theorem 1.6. in [4], p. 6.)

A short graph for (21):

Suppose we have a portfolio with many (in this example infinitely many) assets. For asset d we have the price $\pi(d)$ for today, and the price $\pi(d) \cdot (1 + r)$ for tomorrow with some interest rate $r > 0$.

Let G be a sequence of random variables on Ω_{fut1}, Borel sets. So you have many functions $f_k : \{1, 2, 3, 4\} \to R$ with $G(k) = f_k$ and f_k is a random variable of Ω_{fut1}, Borel sets. For every f_k we have $f_k(w) = \pi(k) \cdot (1+r)$ for $w \in \{1, 2, 3, 4\}$.

ISSN 1426–2630(Print), 1898-9934(Online)

Today	*Tomorrow*
only one scenario	$\begin{cases} w_{21} = \{1,2\}, \\ w_{22} = \{3,4\}, \end{cases}$
for all $d \in \mathbb{N}$ holds $\pi(d)$	$\begin{cases} f_d(w) = G(d)(w) = \pi(d) \cdot (1+r), \\ w \in w_{21} \text{ or } w \in w_{22}, \\ r > 0 \text{ is the interest rate.} \end{cases}$

Here, every probability measure of Ω_{fut1} is a risk-neutral measure.

MSC: 28A05 03B35

Keywords: random variable; arbitrage theory; risk-neutral measure

MML identifier: FINANCE6, version: 8.1.07 5.47.1318

1. Put-Option, Call-Option and Straddle are Random Variables

From now on Ω denotes a non empty set and F denotes a σ-field of subsets of Ω.

Now we state the propositions:

(1) $]0, +\infty[$ is an element of the Borel sets.

(2) Let us consider a random variable R of F and the Borel sets, an element K of $\mathbb{R}$, and a function g from Ω into $\mathbb{R}$. Suppose $g = \chi_{(R-(\Omega \longmapsto K))^{-1}([0,+\infty[),\Omega}$. Then Call-Option$(R, K) = g \cdot (R - (\Omega \longmapsto K))$.

(3) Let us consider a random variable R of F and the Borel sets, and a real number K. Then $(\Omega \longmapsto K) - R$ is a random variable of F and the Borel sets.

(4) Let us consider an element A of F. Then $\chi_{A,\Omega}$ is a random variable of F and the Borel sets.

(5) $\chi_{\Omega,\Omega}$ is random variable on F and the Borel sets. The theorem is a consequence of (4).

(6) Let us consider random variables f, R of F and the Borel sets, and a real number K. Then $(f - R)^{-1}([0, +\infty[)$ is an element of F. The theorem is a consequence of (1).

Let us consider Ω and F. Let R be a random variable of F and the Borel sets and K be a real number. Let us note that the functor Call-Option(R, K) yields a random variable of F and the Borel sets. The functor Put-Option(R, K) yielding a function from Ω into $\mathbb{R}$ is defined by

(Def. 1) for every element w of Ω, if $((\Omega \longmapsto K) - R)(w) \geqslant 0$, then $it(w) = ((\Omega \longmapsto K) - R)(w)$ and if $((\Omega \longmapsto K) - R)(w) < 0$, then $it(w) = 0$.

Now we state the proposition:

(7) Let us consider a random variable R of F and the Borel sets, a real number K, and a function g from Ω into $\mathbb{R}$. Suppose $g = \chi_{((\Omega \longmapsto K)-R)^{-1}([0,+\infty[),\Omega}$. Then Put-Option$(R, K) = g \cdot ((\Omega \longmapsto K) - R)$.

Let us consider Ω and F. Let R be a random variable of F and the Borel sets and K be a real number. Note that the functor Put-Option(R, K) yields a random variable of F and the Borel sets.

2. Simple Random Variables

Let us consider Ω and F. Let R be a random variable of F and the Borel sets and K be a real number. The functor Straddle(R, K) yielding a random variable of F and the Borel sets is defined by the term

(Def. 2) Put-Option(R, K) + Call-Option(R, K).

Now we state the proposition:

(8) Let us consider a random variable R of F and the Borel sets, a real number K, and an element w of Ω. Then $(\text{Straddle}(R, K))(w) = |(R - (\Omega \longmapsto K))(w)|$.

Let us consider Ω and F. The functors: the set of constants F and the set of χ_F yielding sets are defined by terms

(Def. 3) $\{f$, where f is a function from Ω into $\mathbb{R}$: f is random variable on F and the Borel sets and constant$\}$,

(Def. 4) $\{\chi_{A,\Omega}$, where A is an element of F : $\chi_{A,\Omega}$ is random variable on F and the Borel sets$\}$,

respectively. Let X be a set. We say that X is F-random membered if and only if

(Def. 5) for every object x such that $x \in X$ there exists a function f from Ω into $\mathbb{R}$ such that $f = x$ and f is random variable on F and the Borel sets.

Observe that the set of constants F is non empty and the set of χ_F is non empty and the set of constants F is F-random membered and the set of χ_F is F-random membered and there exists a set which is F-random membered and non empty.

Let D be an F-random membered, non empty set, C_1 be a sequence of D, and n be a natural number. The change type of C_1 and n yielding a random variable of F and the Borel sets is defined by the term

(Def. 6) $C_1(n)$.

Let C_2 be a sequence of D and w be an element of Ω. The change all types of C_2 and w yielding a function from $\mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 7) for every natural number n, $it(n) =$ (the change type of C_2 and n)(w).

Let D_1, D_2 be F-random membered, non empty sets, C_1 be a sequence of D_1, C_2 be a sequence of D_2, and n be a natural number. The simple $\mathcal{RV}$ of C_1, C_2 and n yielding a function from Ω into $\mathbb{R}$ is defined by

(Def. 8) for every element w of Ω, $it(w) = (\sum_{\alpha=0}^{\kappa}(($the change all types of C_2 and $w) \cdot ($the change all types of C_1 and $w))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

Observe that the simple $\mathcal{RV}$ of C_1, C_2 and n yields a random variable of F and the Borel sets.

3. Arbitrage Theory: Definition and Alternative Representation

From now on φ denotes a sequence of real numbers and π denotes a price function.

Let us consider Ω and F. Let q be a natural number and G be a sequence of the set of random variables on F and the Borel sets. The change element to functions G and q yielding a real-valued random variable on F is defined by the term

(Def. 9) $G(q)$.

Let us consider φ. Let n be a natural number. The functors: the first $\mathcal{AO}$-set of φ, Ω, F, G and n and the second $\mathcal{AO}$-set of φ, Ω, F, G and n yielding elements of F are defined by terms

(Def. 10) (the $\mathcal{RV}$-portfolio value for future extension of φ, F, G and $n)^{-1}([0, +\infty[)$,

(Def. 11) (the $\mathcal{RV}$-portfolio value for future extension of φ, F, G and $n)^{-1}(]0, +\infty[)$,

respectively. Let P be a probability on F and π be a price function. We say that there exists an $\mathcal{AO}$ w.r.t. P, G, π and n if and only if

(Def. 12) there exists a sequence φ of real numbers such that the buy portfolio extension of φ, π, and $n \leqslant 0$ and P(the first $\mathcal{AO}$-set of φ, Ω, F, G and $n) = 1$ and P(the second $\mathcal{AO}$-set of φ, Ω, F, G and $n) > 0$.

Let r be a real number. The first $\mathcal{RV}$ of r yielding an element of the set of random variables on Ω_{now} and the Borel sets is defined by the term

(Def. 13) $\{1, 2, 3, 4\} \longmapsto r$.

Let π be a price function and d be a natural number. The first $\mathcal{RV}$ of π, r and d yielding an element of the set of random variables on Ω_{fut1} and the Borel sets is defined by the term

(Def. 14) the first $\mathcal{RV}$ of $\pi(d) \cdot (1 + r)$.

Now we state the propositions:

(9) There exists a sequence G of the set of random variables on Ω_{now} and the Borel sets such that

(i) $G(0) = \{1, 2, 3, 4\} \longmapsto 1$, and

(ii) $G(1) = \{1, 2, 3, 4\} \longmapsto 5$, and

(iii) for every natural number k such that $k > 1$ holds $G(k) = \{1, 2, 3, 4\} \longmapsto 0$.

PROOF: Define $\mathcal{U}$(natural number) $= (\$_1 = 0 \rightarrow$ the first $\mathcal{RV}$ of $1, (\$_1 = 1 \rightarrow$ the first $\mathcal{RV}$ of $5,$ the first $\mathcal{RV}$ of $0))$. Consider f being a sequence of the set of random variables on Ω_{now} and the Borel sets such that for every element d of $\mathbb{N}$, $f(d) = \mathcal{U}(d)$. $f(0) = (0 = 0 \rightarrow$ the first $\mathcal{RV}$ of $1, (0 = 1 \rightarrow$ the first $\mathcal{RV}$ of $5,$ the first $\mathcal{RV}$ of $0))$. $f(1) = (1 = 0 \rightarrow$ the first $\mathcal{RV}$ of $1, (1 = 1 \rightarrow$ the first $\mathcal{RV}$ of $5,$ the first $\mathcal{RV}$ of $0))$. For every natural number k such that $k > 1$ holds $f(k) = \{1, 2, 3, 4\} \longmapsto 0$. □

(10) Let us consider a probability P on Ω_{now}, and a sequence G of the set of random variables on Ω_{now} and the Borel sets. Suppose $G(0) = \{1, 2, 3, 4\} \longmapsto 1$ and $G(1) = \{1, 2, 3, 4\} \longmapsto 5$ and for every natural number k such that $k > 1$ holds $G(k) = \{1, 2, 3, 4\} \longmapsto 0$. Then there exists a price function π such that there exists an $\mathcal{AO}$ w.r.t. P, G, π and 1.
PROOF: Set $\Omega = \{1, 2, 3, 4\}$. Set $F = \Omega_{now}$. $P(\Omega) = 1$ and $P(\emptyset) = 0$. Define $\mathcal{U}$(element of $\mathbb{N}$) $= (\$_1 = 0 \rightarrow 1, (\$_1 = 1 \rightarrow 1, 0))(\in \mathbb{R})$. Consider f being a function from $\mathbb{N}$ into $\mathbb{R}$ such that for every element d of $\mathbb{N}$, $f(d) = \mathcal{U}(d)$. f is a price function. Reconsider $\pi = f$ as a price function. Define $\mathcal{U}$(element of $\mathbb{N}$) $= (\$_1 = 0 \rightarrow -1, (\$_1 = 1 \rightarrow 1, 0))(\in \mathbb{R})$. Consider φ being a sequence of real numbers such that for every element k of $\mathbb{N}$, $\varphi(k) = \mathcal{U}(k)$. P(the first $\mathcal{AO}$-set of φ, Ω, F, G and 1) $= 1$ and P(the second $\mathcal{AO}$-set of φ, Ω, F, G and 1) > 0 and the buy portfolio extension of φ, π, and $1 \leqslant 0$ by [7, (9)]. □

(11) Let us consider a natural number n, a real number r, and a sequence G of the set of random variables on F and the Borel sets. Then $\{w$, where w is an element of Ω : the portfolio value for future extension of n, φ, F, G and $w \geqslant 0\}$ = (the $\mathcal{RV}$-portfolio value for future extension of φ, F, G and $n)^{-1}([0, +\infty[)$. The theorem is a consequence of (1).

Let us consider natural numbers d, d_1, a real number r, and a sequence G of the set of random variables on F and the Borel sets.

(12) Suppose $d_1 = d - 1$. Then $\{w$, where w is an element of Ω : the portfolio value for future of d, φ, F, G and $w \geqslant (1 + r) \cdot$ (the buy portfolio of φ, π, and $d)\}$ = ((the $\mathcal{RV}$-portfolio value for future of φ, F, G and $d_1) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ, π, and $d)))^{-1}([0, +\infty[)$.
PROOF: Set $S_1 = \{w$, where w is an element of Ω : the portfolio value for future of d, φ, F, G and $w \geqslant (1 + r) \cdot$ (the buy portfolio of φ, π, and $d)\}$. Set S_2 = ((the $\mathcal{RV}$-portfolio value for future of φ, F, G and

$d_1) - (\Omega \longmapsto (1+r) \cdot$ (the buy portfolio of φ, π, and d)))$^{-1}([0,+\infty[)$. For every object x, $x \in S_1$ iff $x \in S_2$. □

(13) ((The $\mathcal{RV}$-portfolio value for future of φ, F, G and d_1) $-$ ($\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ, π, and d)))$^{-1}([0,+\infty[)$ is an event of F.

(14) Let us consider a natural number d, a real number r, and a sequence G of the set of random variables on F and the Borel sets. Then $\{w$, where w is an element of Ω : the portfolio value for future extension of d, φ, F, G and $w > 0\}$ = (the $\mathcal{RV}$-portfolio value for future extension of φ, F, G and $d)^{-1}(]0,+\infty[)$. The theorem is a consequence of (1).

Let us consider natural numbers d, d_1, a real number r, and a sequence G of the set of random variables on F and the Borel sets.

(15) Suppose $d_1 = d-1$. Then $\{w$, where w is an element of Ω : the portfolio value for future of d, φ, F, G and $w > (1 + r) \cdot$ (the buy portfolio of φ, π, and d)$\}$ = ((the $\mathcal{RV}$-portfolio value for future of φ, F, G and $d_1) - (\Omega \longmapsto (1+r) \cdot$ (the buy portfolio of φ, π, and d)))$^{-1}(]0,+\infty[)$.
Proof: Set $S_1 = \{w$, where w is an element of Ω : the portfolio value for future of d, φ, F, G and $w > (1 + r) \cdot$ (the buy portfolio of φ, π, and d)$\}$. Set S_2 = ((the $\mathcal{RV}$-portfolio value for future of φ, F, G and $d_1) - (\Omega \longmapsto (1+r) \cdot$ (the buy portfolio of φ, π, and d)))$^{-1}(]0,+\infty[)$. For every object x, $x \in S_1$ iff $x \in S_2$. □

(16) ((The $\mathcal{RV}$-portfolio value for future of φ, F, G and d_1) $-$ ($\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ, π, and d)))$^{-1}(]0,+\infty[)$ is an event of F.

(17) Let us consider a price function π, and natural numbers d, d_1. Suppose $d > 0$ and $d_1 = d - 1$. Let us consider a probability P on F, and a real number r. Suppose $r > -1$. Let us consider a sequence G of the set of random variables on F and the Borel sets. Suppose the element of F, the Borel sets, G, and $0 = \Omega \longmapsto 1 + r$. Then there exists an $\mathcal{AO}$ w.r.t. P, G, π and d if and only if there exists a sequence φ_1 of real numbers such that P(((the $\mathcal{RV}$-portfolio value for future of φ_1, F, G and $d_1) - (\Omega \longmapsto (1+r) \cdot$ (the buy portfolio of φ_1, π, and d)))$^{-1}([0,+\infty[)) = 1$ and P(((the $\mathcal{RV}$-portfolio value for future of φ_1, F, G and $d_1) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ_1, π, and d)))$^{-1}(]0,+\infty[)) > 0$.
Proof: If there exists an $\mathcal{AO}$ w.r.t. P, G, π and d, then there exists a sequence φ_1 of real numbers such that P(((the $\mathcal{RV}$-portfolio value for future of φ_1, F, G and $d_1) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ_1, π, and d)))$^{-1}([0,+\infty[)) = 1$ and P(((the $\mathcal{RV}$-portfolio value for future of φ_1, F, G and $d_1) - (\Omega \longmapsto (1 + r) \cdot$ (the buy portfolio of φ_1, π, and d)))$^{-1}(]0,+\infty[)) > 0$. Define $\mathcal{U}$(natural number) = ($\$_1 = 0 \rightarrow$ $-$(the buy portfolio of φ_1, π, and d), $\varphi_1(\$_1))(\in \mathbb{R})$. Consider φ being a se-

quence of real numbers such that for every element n of $\mathbb{N}$, $\varphi(n) = \mathcal{U}(n)$. For every natural number n, if $n = 0$, then $\varphi(n) = -$(the buy portfolio of φ_1, π, and d) and if $n > 0$, then $\varphi(n) = \varphi_1(n)$. The buy portfolio extension of φ, π, and $d = 0$. P(the first $\mathcal{AO}$-set of φ, Ω, F, G and d) $= 1$. P(the second $\mathcal{AO}$-set of φ, Ω, F, G and d) > 0. □

4. Risk-Neutral Probability Measure

Let us consider Ω and F. Let R be a real-valued random variable on F and r be a real number. The r-discounted value of R yielding a real-valued random variable on F is defined by the term

(Def. 15) $R \cdot \frac{1}{1+r}$.

Let π be a price function and G be a sequence of the set of random variables on F and the Borel sets. We say that there exists a risk neutral measure w.r.t. G, π and r if and only if

(Def. 16) there exists a probability P on F such that for every natural number d, $\pi(d) = E_P\{$the r-discounted value of (the change element to functions G and d)$\}$.

From now on P denotes a probability on Ω_{fut1}.

Now we state the propositions:

(18) Let us consider a real number r. Suppose $r > 0$. Let us consider a price function π, and a natural number d. Then there exists a real-valued random variable f on Ω_{fut1} such that

(i) $f = \{1, 2, 3, 4\} \longmapsto \pi(d) \cdot (1 + r)$, and

(ii) f is integrable on P2M(P), and

(iii) f is simple function in Ω_{fut1}.

Proof: Set $\Omega_2 = \{1, 2, 3, 4\}$. Define $\mathcal{U}$(element of Ω_2) $= \pi(d) \cdot (1+r) (\in \mathbb{R})$. Consider f being a function from Ω_2 into $\mathbb{R}$ such that for every element d of Ω_2, $f(d) = \mathcal{U}(d)$. Set $g = \Omega_2 \longmapsto \pi(d) \cdot (1 + r) (\in \mathbb{R})$. For every object x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$. f is integrable on P2M(P) by [6, (9), (3)], [3, (12)]. □

(19) Let us consider a natural number n, and a real number r. Suppose $r > 0$. Let us consider a price function π, a natural number d, and a real-valued random variable R on Ω_{fut1}. Suppose $R = \{1, 2, 3, 4\} \longmapsto \pi(d) \cdot (1 + r)$ and R is integrable on P2M(P) and R is simple function in Ω_{fut1}. Then $\pi(d) = E_P\{$the r-discounted value of $R\}$.

Proof: Set $F = \Omega_{fut1}$. $\overline{\mathbb{R}}(R) = R$ and R is non-negative. Set $m = \pi(d) \cdot (1 + r)$. for every object x such that $x \in \operatorname{dom} \overline{\mathbb{R}}(R)$ holds $(\overline{\mathbb{R}}(R))(x) = m$ and $\operatorname{dom} \overline{\mathbb{R}}(R) \in F$ and $0 \leqslant m$. □

(20) Let us consider a real number r. Suppose $r > 0$. Let us consider a price function π. Then there exists a sequence G of the set of random variables on Ω_{fut1} and the Borel sets such that for every natural number d, $G(d) = \{1, 2, 3, 4\} \longmapsto \pi(d) \cdot (1 + r)$ and the change element to functions G and d is integrable on P2M(P) and the change element to functions G and d is simple function in Ω_{fut1}.

Proof: Define $\mathcal{U}$(natural number) = the first $\mathcal{RV}$ of π, r and $\$_1$. Consider g being a sequence of the set of random variables on Ω_{fut1} and the Borel sets such that for every element d of $\mathbb{N}$, $g(d) = \mathcal{U}(d)$. There exists a real-valued random variable R on Ω_{fut1} such that $R = \{1, 2, 3, 4\} \longmapsto \pi(d) \cdot (1 + r)(\in \mathbb{R})$ and R is integrable on P2M(P) and R is simple function in Ω_{fut1}. □

(21) Let us consider a real number r. Suppose $r > 0$. Let us consider a price function π, and a sequence G of the set of random variables on Ω_{fut1} and the Borel sets. Suppose for every natural number d, $G(d) = \{1, 2, 3, 4\} \longmapsto \pi(d) \cdot (1 + r)$ and the change element to functions G and d is integrable on P2M(P) and the change element to functions G and d is simple function in Ω_{fut1}. Then

(i) there exists a risk neutral measure w.r.t. G, π and r, and

(ii) for every natural number s, $\pi(s) = E_P\{$the r-discounted value of (the change element to functions G and s)$\}$.

The theorem is a consequence of (19).

References

[1] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[2] Heinz Bauer. *Wahrscheinlichkeitstheorie.* de Gruyter-Verlag, Berlin, New York, 2002.

[3] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. The measurability of extended real valued functions. *Formalized Mathematics*, 9(**3**):525–529, 2001.

[4] Hans Föllmer and Alexander Schied. *Stochastic Finance: An Introduction in Discrete Time*, volume 27 of *Studies in Mathematics.* de Gruyter, Berlin, 2nd edition, 2004.

[5] Adam Grabowski, Artur Korniłowicz, and Adam Naumowicz. Four decades of Mizar. *Journal of Automated Reasoning*, 55(3):191–198, 2015. doi:10.1007/s10817-015-9345-1.

[6] Peter Jaeger. Elementary introduction to stochastic finance in discrete time. *Formalized Mathematics*, 20(**1**):1–5, 2012. doi:10.2478/v10037-012-0001-5.

[7] Peter Jaeger. Modelling real world using stochastic processes and filtration. *Formalized Mathematics*, 24(**1**):1–16, 2016. doi:10.1515/forma-2016-0001.

Received March 27, 2018