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F. Riesz Theorem

Keiko Narita
Hirosaki-city
Aomori, Japan

Kazuhisa Nakasho
Akita Prefectural University
Akita, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, we formalize in the Mizar system [1, 4] the F. Riesz theorem. In the first section, we defined Mizar functor `ClstoCmp`, compact topological spaces as closed interval subset of real numbers. Then using the former definition and referring to the article [10] and the article [5], we defined the normed spaces of continuous functions on closed interval subset of real numbers, and defined the normed spaces of bounded functions on closed interval subset of real numbers. We also proved some related properties.

In Sec.2, we proved some lemmas for the proof of F. Riesz theorem. In Sec.3, we proved F. Riesz theorem, about the dual space of the space of continuous functions on closed interval subset of real numbers, finally. We applied Hahn-Banach theorem (36) in [7], to the proof of the last theorem. For the description of theorems of this section, we also referred to the article [8] and the article [6]. These formalizations are based on [2], [3], [9], and [11].

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1. The Normed Space of Continuous Functions on Closed Interval

Now we state the propositions:

(1) Let us consider a real number d. Then $|\operatorname{sgn} d| \leqslant 1$.

(2) Let us consider a real number x. Then $|x| = \operatorname{sgn} x \cdot x$.

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Let A be a non empty, closed interval subset of $\mathbb{R}$. The functor Cls2Cmp(A) yielding a strict, compact, non empty topological space is defined by

(Def. 1) there exist real numbers a, b such that $a \leqslant b$ and $[a,b] = A$ and $it = [a, b]_{\mathrm{T}}$.

Now we state the propositions:

(3) Let us consider a strict, non empty subspace X of $\mathbb{R}^{\mathbf{1}}$, a real map f of X, a partial function g from $\mathbb{R}$ to $\mathbb{R}$, a point x of X, and a real number x_0. Suppose $g = f$ and $x = x_0$. Then for every subset V of $\mathbb{R}$ such that $f(x) \in V$ and V is open there exists a subset W of X such that $x \in W$ and W is open and $f^{\circ}W \subseteq V$ if and only if g is continuous in x_0.

(4) Let us consider a strict, non empty subspace X of $\mathbb{R}^{\mathbf{1}}$, a real map f of X, and a partial function g from $\mathbb{R}$ to $\mathbb{R}$. If $g = f$, then f is continuous iff g is continuous. The theorem is a consequence of (3).

(5) Let us consider a non empty, closed interval subset A of $\mathbb{R}$. Then the carrier of Cls2Cmp(A) = A.

(6) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and a function u. Then u is a point of C(Cls2Cmp(A); $\mathbb{R}$) if and only if $\operatorname{dom} u = A$ and u is a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$. The theorem is a consequence of (5) and (4).

(7) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and a point v of C(Cls2Cmp(A); $\mathbb{R}$). Then $v \in$ BoundedFunctions(the carrier of Cls2Cmp(A)).

2. Preliminaries

Now we state the proposition:

(8) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and real numbers a, b. Suppose $A = [a,b]$. Then there exists a function x from A into BoundedFunctions A such that for every real number s such that $s \in [a,b]$ holds if $s = a$, then $x(s) = [a,b] \longmapsto 0$ and if $s \neq a$, then $x(s) = ([a,s] \longmapsto 1) + \cdot (]s,b] \longmapsto 0)$.

Proof: Define $\mathcal{C}[\text{object}] \equiv \$_1 = a$. Define $\mathcal{F}(\text{object}) = [a,b] \longmapsto 0$. Define $\mathcal{G}(\text{object}) = ([a, \$_1(\in \mathbb{R})] \longmapsto 1) + \cdot (]\$_1(\in \mathbb{R}), b] \longmapsto 0)$. Set $B =$ BoundedFunctions A. For every object s such that $s \in [a,b]$ holds if $\mathcal{C}[s]$, then $\mathcal{F}(s) \in B$ and if $\mathcal{C}[s]$, then $\mathcal{G}(s) \in B$. Consider x being a function from $[a,b]$ into B such that for every object s such that $s \in [a,b]$ holds if $\mathcal{C}[s]$, then $x(s) = \mathcal{F}(s)$ and if $\mathcal{C}[s]$, then $x(s) = \mathcal{G}(s)$. For every real number s such that $s \in [a,b]$ holds if $s = a$, then $x(s) = [a,b] \longmapsto 0$ and if $s \neq a$, then $x(s) = ([a,s] \longmapsto 1) + \cdot (]s,b] \longmapsto 0)$. $\square$

Let A be a non empty, closed interval subset of $\mathbb{R}$, D be a partition of A, m be a function from A into BoundedFunctions A, and i be a natural number. Assume $i \in \mathrm{Seg}(\mathrm{len}\, D + 1)$. The functor $\mathrm{Dp1}(m, D, i)$ yielding a point of the $\mathbb{R}$-normed algebra of bounded functions on the carrier of $\mathrm{Cls2Cmp}(A)$ is defined by the term

(Def. 2) $\begin{cases} m(\inf A), & \textbf{if } i = 1, \\ m(D(i-1)), & \textbf{otherwise}. \end{cases}$

Let ϱ be a function from A into $\mathbb{R}$. The functor $\mathrm{Dp2}(\varrho, D, i)$ yielding a real number is defined by the term

(Def. 3) $\begin{cases} \varrho(\inf A), & \textbf{if } i = 1, \\ \varrho(D(i-1)), & \textbf{otherwise}. \end{cases}$

Now we state the propositions:

(9) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a partition D of A, a function m from A into BoundedFunctions A, and a function ϱ from A into $\mathbb{R}$. Then there exists a finite sequence s of elements of the $\mathbb{R}$-normed algebra of bounded functions on the carrier of $\mathrm{Cls2Cmp}(A)$ such that

(i) $\mathrm{len}\, s = \mathrm{len}\, D$, and

(ii) for every natural number i such that $i \in \mathrm{dom}\, s$ holds $s(i) = \mathrm{sgn}(\mathrm{Dp2}(\varrho, D, i+1) - \mathrm{Dp2}(\varrho, D, i)) \cdot (\mathrm{Dp1}(m, D, i+1) - \mathrm{Dp1}(m, D, i))$.

Proof: Set $V =$ the $\mathbb{R}$-normed algebra of bounded functions on the carrier of $\mathrm{Cls2Cmp}(A)$. Define $\mathcal{P}[\text{natural number}, \text{set}] \equiv \$_2 = \mathrm{sgn}(\mathrm{Dp2}(\varrho, D, \$_1 + 1) - \mathrm{Dp2}(\varrho, D, \$_1)) \cdot (\mathrm{Dp1}(m, D, \$_1 + 1) - \mathrm{Dp1}(m, D, \$_1))$. Consider s being a finite sequence of elements of V such that $\mathrm{dom}\, s = \mathrm{Seg}\,\mathrm{len}\, D$ and for every natural number i such that $i \in \mathrm{Seg}\,\mathrm{len}\, D$ holds $\mathcal{P}[i, s(i)]$. $\square$

(10) Let us consider a real linear space V, a functional f in V, and a finite sequence s of elements of V. If f is additive, then $f(\sum s) = \sum(f \cdot s)$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every real linear space V for every functional f in V for every finite sequence s of elements of V such that $\mathrm{len}\, s = \$_1$ and f is additive holds $f(\sum s) = \sum(f \cdot s)$. $\mathcal{P}[0]$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(11) Let us consider a non empty set A. Then every element of the $\mathbb{R}$-normed algebra of bounded functions on A is a function from A into $\mathbb{R}$.

(12) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a finite sequence s of elements of the $\mathbb{R}$-normed algebra of bounded functions on the carrier of $\mathrm{Cls2Cmp}(A)$, a finite sequence z of elements of $\mathbb{R}$, a function g from A into $\mathbb{R}$, and an element t of A. Suppose $\mathrm{len}\, s = \mathrm{len}\, z$ and $g = \sum s$

and for every natural number k such that $k \in \operatorname{dom} z$ there exists a function s_1 from A into $\mathbb{R}$ such that $s_1 = s(k)$ and $z(k) = s_1(t)$. Then $g(t) = \sum z$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every non empty, closed interval subset A of $\mathbb{R}$ for every finite sequence s of elements of the $\mathbb{R}$-normed algebra of bounded functions on the carrier of Cls2Cmp(A) for every finite sequence z of elements of $\mathbb{R}$ for every function g from A into $\mathbb{R}$ for every element t of A such that $\operatorname{len} s = \$_1$ and $\operatorname{len} s = \operatorname{len} z$ and $g = \sum s$ and for every natural number k such that $k \in \operatorname{dom} z$ there exists a function s_1 from A into $\mathbb{R}$ such that $s_1 = s(k)$ and $z(k) = s_1(t)$ holds $g(t) = \sum z$. $\mathcal{P}[0]$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

(13) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a partition D of A, and an element t of A. Suppose $\inf A < D(1)$. Then there exists an element i of $\mathbb{N}$ such that

(i) $i \in \operatorname{dom} D$, and

(ii) $t \in \operatorname{divset}(D, i)$, and

(iii) $i = 1$ or $\inf \operatorname{divset}(D, i) < t \leqslant \sup \operatorname{divset}(D, i)$.

(14) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a function ϱ from A into $\mathbb{R}$, and a real number B. Suppose $0 < \operatorname{vol}(A)$. Suppose for every partition D of A and for every var-volume K of ϱ and D such that $\inf A < D(1)$ holds $\sum K \leqslant B$. Let us consider a partition D of A, and a var-volume K of ϱ and D. Then $\sum K \leqslant B$.

3. F. Riesz Theorem

Now we state the propositions:

(15) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a function ϱ from A into $\mathbb{R}$, and a point f of DualSp C(Cls2Cmp(A); $\mathbb{R}$). Suppose ϱ is bounded-variation and for every continuous partial function u from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} u = A$ holds $f(u) = \int\limits_{\varrho} u(x)dx$. Then $\|f\| \leqslant \operatorname{TotalVD}(\varrho)$.

PROOF: Set $X = $ C(Cls2Cmp(A); $\mathbb{R}$). For every continuous partial function u from $\mathbb{R}$ to $\mathbb{R}$ such that $u \in$ the carrier of X holds $f(u) = \int\limits_{\varrho} u(x)dx$. For every continuous partial function u from $\mathbb{R}$ to $\mathbb{R}$ and for every point v of X such that $\operatorname{dom} u = A$ and $u = v$ holds $|\int\limits_{\varrho} u(x)dx| \leqslant \|v\| \cdot \operatorname{TotalVD}(\varrho)$.

□

(16) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and a point x of $\operatorname{DualSp} \mathrm{C}(\mathrm{Cls2Cmp}(\mathrm{A}); \mathbb{R})$. Suppose $0 < \operatorname{vol}(A)$. Then there exists a function ϱ from A into $\mathbb{R}$ such that

(i) ϱ is bounded-variation, and

(ii) for every continuous partial function u from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} u = A$ holds $x(u) = \int\limits_{\varrho} u(x)dx$, and

(iii) $\|x\| = \operatorname{TotalVD}(\varrho)$.

Proof: Set $X = \mathrm{C}(\mathrm{Cls2Cmp}(\mathrm{A}); \mathbb{R})$. Set $V =$ the $\mathbb{R}$-normed algebra of bounded functions on the carrier of $\mathrm{Cls2Cmp}(A)$. Set $A_1 =$ the carrier of $\mathrm{Cls2Cmp}(A)$. $A_1 = A$. Reconsider $h = x$ as a Lipschitzian linear functional in X. Consider f being a Lipschitzian linear functional in V, F being a point of $\operatorname{DualSp} V$ such that $f = F$ and $f{\restriction}(\text{the carrier of } X) = h$ and $\|F\| = \|x\|$. Consider a, b being real numbers such that $a \leqslant b$ and $[a, b] = A$ and $\mathrm{Cls2Cmp}(A) = [a, b]_{\mathrm{T}}$. Consider m being a function from A into $\operatorname{BoundedFunctions} A$ such that for every real number s such that $s \in [a, b]$ holds if $s = a$, then $m(s) = [a, b] \longmapsto 0$ and if $s \neq a$, then $m(s) = ([a, s] \longmapsto 1) +\cdot (]s, b] \longmapsto 0)$. The carrier of $V = \operatorname{BoundedFunctions} A$. Reconsider $\varrho = f \cdot m$ as a function from A into $\mathbb{R}$. For every partition D of A and for every var-volume K of ϱ and D such that $a < D(1)$ holds $\sum K \leqslant \|x\|$. For every partition D of A and for every var-volume K of ϱ and D, $\sum K \leqslant \|x\|$. Consider V_1 being a non empty subset of $\mathbb{R}$ such that V_1 is upper bounded and $V_1 = \{r$, where r is a real number : there exists a partition t of A and there exists a var-volume F_0 of ϱ and t such that $r = \sum F_0\}$ and $\operatorname{TotalVD}(\varrho) = \sup V_1$. For every continuous partial function u from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} u = A$ holds $x(u) = \int\limits_{\varrho} u(x)dx$. $\|x\| \leqslant \operatorname{TotalVD}(\varrho)$. $\square$

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