

Homography in \mathbb{RP}^2

Roland Coghetto
Rue de la Brasserie 5
7100 La Louvière, Belgium

Summary. The real projective plane has been formalized in Isabelle/HOL by Timothy Makarios [13] and in Coq by Nicolas Magaud, Julien Narboux and Pascal Schreck [12].

Some definitions on the real projective spaces were introduced early in the Mizar Mathematical Library by Wojciech Leonczuk [9], Krzysztof Prazmowski [10] and by Wojciech Skaba [18].

In this article, we check with the Mizar system [4], some properties on the determinants and the Grassmann-Plücker relation in rank 3 [2], [1], [7], [16], [17].

Then we show that the projective space induced (in the sense defined in [9]) by \mathbb{R}^3 is a projective plane (in the sense defined in [10]).

Finally, in the real projective plane, we define the homography induced by a 3-by-3 invertible matrix and we show that the images of 3 collinear points are themselves collinear.

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1. PRELIMINARIES

From now on a, b, c, d, e, f denote real numbers, k, m denote natural numbers, D denotes a non empty set, V denotes a non trivial real linear space, u, v, w denote elements of V , and p, q, r denote elements of the projective space over V .

Now we state the propositions:

- (1) $\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle, \langle 3, 3 \rangle \in \text{Seg } 3 \times \text{Seg } 3$.

- (2) $\langle 1, 1 \rangle, \langle 2, 1 \rangle, \langle 3, 1 \rangle \in \text{Seg } 3 \times \text{Seg } 1$.
- (3) $\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle \in \text{Seg } 1 \times \text{Seg } 3$.
- (4) $\langle \langle a \rangle, \langle b \rangle, \langle c \rangle \rangle$ is a matrix over \mathbb{R}_F of dimension 3×1 .
- (5) Let us consider a matrix N over \mathbb{R}_F of dimension 3×1 . Suppose $N = \langle \langle a \rangle, \langle b \rangle, \langle c \rangle \rangle$. Then $N_{\square, 1} = \langle a, b, c \rangle$. The theorem is a consequence of (2).
- (6) Let us consider a non empty multiplicative magma K , and elements $a_1, a_2, a_3, b_1, b_2, b_3$ of K . Then $\langle a_1, a_2, a_3 \rangle \bullet \langle b_1, b_2, b_3 \rangle = \langle a_1 \cdot b_1, a_2 \cdot b_2, a_3 \cdot b_3 \rangle$.
- (7) Let us consider a commutative, associative, left unital, Abelian, add-associative, right zeroed, right complementable, non empty double loop structure K , and elements $a_1, a_2, a_3, b_1, b_2, b_3$ of K . Then $\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$. The theorem is a consequence of (6).
- (8) Let us consider a square matrix M over \mathbb{R}_F of dimension 3, and a matrix N over \mathbb{R}_F of dimension 3×1 . Suppose $N = \langle \langle 0 \rangle, \langle 0 \rangle, \langle 0 \rangle \rangle$. Then $M \cdot N = \langle \langle 0 \rangle, \langle 0 \rangle, \langle 0 \rangle \rangle$. The theorem is a consequence of (7), (5), and (2).
- (9) u, v and w are linearly dependent if and only if $u = v$ or $u = w$ or $v = w$ or $\{u, v, w\}$ is linearly dependent.
- (10) p, q and r are collinear if and only if there exists u and there exists v and there exists w such that $p =$ the direction of u and $q =$ the direction of v and $r =$ the direction of w and u is not zero and v is not zero and w is not zero and ($u = v$ or $u = w$ or $v = w$ or $\{u, v, w\}$ is linearly dependent). The theorem is a consequence of (9).
- (11) p, q and r are collinear if and only if there exists u and there exists v and there exists w such that $p =$ the direction of u and $q =$ the direction of v and $r =$ the direction of w and u is not zero and v is not zero and w is not zero and there exists a and there exists b and there exists c such that $a \cdot u + b \cdot v + c \cdot w = 0_V$ and ($a \neq 0$ or $b \neq 0$ or $c \neq 0$).
- (12) Let us consider elements u, v, w of V . Suppose $a \neq 0$ and $a \cdot u + b \cdot v + c \cdot w = 0_V$. Then $u = (\frac{-b}{a}) \cdot v + (\frac{-c}{a}) \cdot w$.
- (13) If $a \neq 0$ and $a \cdot b + c \cdot d + e \cdot f = 0$, then $b = -(\frac{c}{a}) \cdot d - (\frac{e}{a}) \cdot f$.
- (14) Let us consider points u, v, w of \mathcal{E}_T^3 . Suppose there exists a and there exists b and there exists c such that $a \cdot u + b \cdot v + c \cdot w = 0_{\mathcal{E}_T^3}$ and $a \neq 0$. Then $\langle |u, v, w| \rangle = 0$. The theorem is a consequence of (12).
- (15) Let us consider a natural number n . Then $\text{dom } 1_{\mathbb{R}} \text{ matrix}(n) = \text{Seg } n$.
- (16) Let us consider a matrix A over \mathbb{R}_F . Then $(\mathbb{R} \rightarrow \mathbb{R}_F)(\mathbb{R}_F \rightarrow \mathbb{R})A = A$.
- (17) Let us consider matrices A, B over \mathbb{R}_F , and matrices R_1, R_2 over \mathbb{R} . If $A = R_1$ and $B = R_2$, then $A \cdot B = R_1 \cdot R_2$. The theorem is a consequence of (16).

- (18) Let us consider a natural number n , a square matrix M over \mathbb{R} of dimension n , and a square matrix N over \mathbb{R}_F of dimension n . If $M = N$, then M is invertible iff N is invertible. The theorem is a consequence of (17).

From now on o, p, q, r, s, t denote points of \mathcal{E}_T^3 and M denotes a square matrix over \mathbb{R}_F of dimension 3.

Let us consider real numbers $p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3$. Now we state the propositions:

- (19) $\langle\langle p_1, p_2, p_3 \rangle, \langle q_1, q_2, q_3 \rangle, \langle r_1, r_2, r_3 \rangle\rangle$ is a square matrix over \mathbb{R}_F of dimension 3.
- (20) Suppose $M = \langle\langle p_1, q_1, r_1 \rangle, \langle p_2, q_2, r_2 \rangle, \langle p_3, q_3, r_3 \rangle\rangle$. Then
- (i) $M_{1,1} = p_1$, and
 - (ii) $M_{1,2} = q_1$, and
 - (iii) $M_{1,3} = r_1$, and
 - (iv) $M_{2,1} = p_2$, and
 - (v) $M_{2,2} = q_2$, and
 - (vi) $M_{2,3} = r_2$, and
 - (vii) $M_{3,1} = p_3$, and
 - (viii) $M_{3,2} = q_3$, and
 - (ix) $M_{3,3} = r_3$.

The theorem is a consequence of (1).

- (21) Suppose $M = \langle p, q, r \rangle$. Then

- (i) $M_{1,1} = (p)_1$, and
- (ii) $M_{1,2} = (p)_2$, and
- (iii) $M_{1,3} = (p)_3$, and
- (iv) $M_{2,1} = (q)_1$, and
- (v) $M_{2,2} = (q)_2$, and
- (vi) $M_{2,3} = (q)_3$, and
- (vii) $M_{3,1} = (r)_1$, and
- (viii) $M_{3,2} = (r)_2$, and
- (ix) $M_{3,3} = (r)_3$.

The theorem is a consequence of (1).

- (22) Let us consider real numbers $p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3$. Suppose $M = \langle\langle p_1, q_1, r_1 \rangle, \langle p_2, q_2, r_2 \rangle, \langle p_3, q_3, r_3 \rangle\rangle$. Then $M^T = \langle\langle p_1, p_2, p_3 \rangle, \langle q_1, q_2, q_3 \rangle, \langle r_1, r_2, r_3 \rangle\rangle$. The theorem is a consequence of (1) and (20).

- (23) Suppose $M = \langle p, q, r \rangle$. Then $M^T = \langle \langle (p)_1, (q)_1, (r)_1 \rangle, \langle (p)_2, (q)_2, (r)_2 \rangle, \langle (p)_3, (q)_3, (r)_3 \rangle \rangle$. The theorem is a consequence of (1) and (21).
- (24) $\text{lines}(M) = \{\text{Line}(M, 1), \text{Line}(M, 2), \text{Line}(M, 3)\}$.
 PROOF: $\text{lines}(M) \subseteq \{\text{Line}(M, 1), \text{Line}(M, 2), \text{Line}(M, 3)\}$ by [14, (103)], [19, (1)]. $\{\text{Line}(M, 1), \text{Line}(M, 2), \text{Line}(M, 3)\} \subseteq \text{lines}(M)$ by [3, (1)], [14, (103)]. \square
- (25) Suppose $M = \langle \langle (p)_1, (p)_2, (p)_3 \rangle, \langle (q)_1, (q)_2, (q)_3 \rangle, \langle (r)_1, (r)_2, (r)_3 \rangle \rangle$. Then
 (i) $\text{Line}(M, 1) = p$, and
 (ii) $\text{Line}(M, 2) = q$, and
 (iii) $\text{Line}(M, 3) = r$.
- (26) Let us consider an object x . Then $x \in \text{lines}(M^T)$ if and only if there exists a natural number i such that $i \in \text{Seg } 3$ and $x = M_{\square, i}$.

2. GRASSMANN-PLÜCKER RELATION

Now we state the propositions:

- (27) $\langle |p, q, r| \rangle = (p)_1 \cdot (q)_2 \cdot (r)_3 - (p)_3 \cdot (q)_2 \cdot (r)_1 - (p)_1 \cdot (q)_3 \cdot (r)_2 + (p)_2 \cdot (q)_3 \cdot (r)_1 - (p)_2 \cdot (q)_1 \cdot (r)_3 + (p)_3 \cdot (q)_1 \cdot (r)_2$.
- (28) GRASSMANN-PLÜCKER-RELATION IN RANK 3:
 $\langle |p, q, r| \rangle \cdot \langle |p, s, t| \rangle - \langle |p, q, s| \rangle \cdot \langle |p, r, t| \rangle + \langle |p, q, t| \rangle \cdot \langle |p, r, s| \rangle = 0$. The theorem is a consequence of (27).
- (29) $\langle |p, q, r| \rangle = -\langle |p, r, q| \rangle$. The theorem is a consequence of (27).
- (30) $\langle |p, q, r| \rangle = -\langle |q, p, r| \rangle$. The theorem is a consequence of (27).
- (31) $\langle |a \cdot p, q, r| \rangle = a \cdot \langle |p, q, r| \rangle$. The theorem is a consequence of (27).
- (32) $\langle |p, a \cdot q, r| \rangle = a \cdot \langle |p, q, r| \rangle$. The theorem is a consequence of (30) and (31).
- (33) $\langle |p, q, a \cdot r| \rangle = a \cdot \langle |p, q, r| \rangle$. The theorem is a consequence of (29) and (32).
- (34) Suppose $M = \langle \langle (p)_1, (q)_1, (r)_1 \rangle, \langle (p)_2, (q)_2, (r)_2 \rangle, \langle (p)_3, (q)_3, (r)_3 \rangle \rangle$. Then $\langle |p, q, r| \rangle = \text{Det } M$. The theorem is a consequence of (22).
- (35) Suppose $M = \langle \langle (p)_1, (p)_2, (p)_3 \rangle, \langle (q)_1, (q)_2, (q)_3 \rangle, \langle (r)_1, (r)_2, (r)_3 \rangle \rangle$. Then $\langle |p, q, r| \rangle = \text{Det } M$.

Let us consider a square matrix M over \mathbb{R}_F of dimension k . Now we state the propositions:

- (36) $\text{Det } M = 0_{\mathbb{R}_F}$ if and only if $\text{rk}(M) < k$.

- (37) $\text{rk}(M) < k$ if and only if $\text{lines}(M)$ is linearly dependent or M is not without repeated line.
- (38) Let us consider a matrix M over \mathbb{R}_F of dimension $k \times m$. Then $\text{Mx2Tran}(M)$ is a function from $\text{RLSp2RVSp}(\mathcal{E}_T^k)$ into $\text{RLSp2RVSp}(\mathcal{E}_T^m)$.
- (39) Let us consider a square matrix M over \mathbb{R}_F of dimension k . Then $\text{Mx2Tran}(M)$ is a linear transformation from $\text{RLSp2RVSp}(\mathcal{E}_T^k)$ to $\text{RLSp2RVSp}(\mathcal{E}_T^k)$.
 PROOF: Reconsider $M_1 = \text{Mx2Tran}(M)$ as a function from $\text{RLSp2RVSp}(\mathcal{E}_T^k)$ into $\text{RLSp2RVSp}(\mathcal{E}_T^k)$. For every elements x, y of $\text{RLSp2RVSp}(\mathcal{E}_T^k)$, $M_1(x + y) = M_1(x) + M_1(y)$ by [15, (22)]. For every scalar a of \mathbb{R}_F and for every vector x of $\text{RLSp2RVSp}(\mathcal{E}_T^k)$, $M_1(a \cdot x) = a \cdot M_1(x)$ by [15, (23)]. \square
- (40) Suppose $M = \langle\langle(p)_1, (p)_2, (p)_3\rangle, \langle\langle(q)_1, (q)_2, (q)_3\rangle, \langle\langle(r)_1, (r)_2, (r)_3\rangle\rangle$ and $\text{rk}(M) < 3$. Then there exists a and there exists b and there exists c such that $a \cdot p + b \cdot q + c \cdot r = 0_{\mathcal{E}_T^3}$ and ($a \neq 0$ or $b \neq 0$ or $c \neq 0$). The theorem is a consequence of (37), (25), (24), (39), and (7).
- (41) If $a \cdot p + b \cdot q + c \cdot r = 0_{\mathcal{E}_T^3}$ and ($a \neq 0$ or $b \neq 0$ or $c \neq 0$), then $\langle\langle p, q, r \rangle\rangle = 0$. The theorem is a consequence of (14) and (30).
- (42) Suppose $\langle\langle p, q, r \rangle\rangle = 0$. Then there exists a and there exists b and there exists c such that $a \cdot p + b \cdot q + c \cdot r = 0_{\mathcal{E}_T^3}$ and ($a \neq 0$ or $b \neq 0$ or $c \neq 0$). The theorem is a consequence of (19), (35), (36), and (40).
- (43) p, q and r are linearly dependent if and only if $\langle\langle p, q, r \rangle\rangle = 0$. The theorem is a consequence of (41) and (42).

3. SOME PROPERTIES ABOUT THE CROSS PRODUCT

Now we state the propositions:

- (44) $|\langle p, p \times q \rangle| = 0$.
- (45) $|\langle p, q \times p \rangle| = 0$.
- (46) (i) $\langle\langle o, p, (o \times p) \times (q \times r) \rangle\rangle = 0$, and
 (ii) $\langle\langle q, r, (o \times p) \times (q \times r) \rangle\rangle = 0$.
 The theorem is a consequence of (44) and (45).
- (47) (i) o, p and $(o \times p) \times (q \times r)$ are linearly dependent, and
 (ii) q, r and $(o \times p) \times (q \times r)$ are linearly dependent.
 The theorem is a consequence of (46) and (43).
- (48) (i) $0_{\mathcal{E}_T^3} \times p = 0_{\mathcal{E}_T^3}$, and
 (ii) $p \times 0_{\mathcal{E}_T^3} = 0_{\mathcal{E}_T^3}$.

- (49) $\langle |p, q, 0_{\mathcal{E}_T^3}| \rangle = 0$. The theorem is a consequence of (48).
- (50) If $p \times q = 0_{\mathcal{E}_T^3}$ and $r = [1, 1, 1]$, then p, q and r are lineary dependent.
 PROOF: Reconsider $r = [1, 1, 1]$ as an element of \mathcal{E}_T^3 . $\langle |p, q, r| \rangle = 0$ by [8, (2)], (27). \square
- (51) If p is not zero and q is not zero and $p \times q = 0_{\mathcal{E}_T^3}$, then p and q are proportional.
- (52) Let us consider non zero points p, q, r, s of \mathcal{E}_T^3 . Suppose $(p \times q) \times (r \times s)$ is zero. Then
- (i) p and q are proportional, or
 - (ii) r and s are proportional, or
 - (iii) $p \times q$ and $r \times s$ are proportional.

The theorem is a consequence of (51).

- (53) $\langle |p, q, p \times q| \rangle = |(q, q)| \cdot |(p, p)| - |(q, p)| \cdot |(p, q)|$.
- (54) $|(p \times q, p \times q)| = |(q, q)| \cdot |(p, p)| - |(q, p)| \cdot |(p, q)|$.
- (55) If p is not zero and $|(p, q)| = 0$ and $|(p, r)| = 0$ and $|(p, s)| = 0$, then $\langle |q, r, s| \rangle = 0$. The theorem is a consequence of (13) and (27).
- (56) $\langle |p, q, p \times q| \rangle = |p \times q|^2$. The theorem is a consequence of (53) and (54).
- (57) The projective space over \mathcal{E}_T^3 is a projective plane defined in terms of collinearity.

PROOF: Set P = the projective space over \mathcal{E}_T^3 . There exist elements u, v, w_1 of \mathcal{E}_T^3 such that for every real numbers a, b, c such that $a \cdot u + b \cdot v + c \cdot w_1 = 0_{\mathcal{E}_T^3}$ holds $a = 0$ and $b = 0$ and $c = 0$ by [6, (22)], [8, (4)], [11, (39)], [8, (2)]. For every elements p, p_1, q, q_1 of P , there exists an element r of P such that p, p_1 and r are collinear and q, q_1 and r are collinear by [9, (26)], (52), [9, (22)], [18, (2)]. \square

4. REAL PROJECTIVE PLANE AND HOMOGRAPHY

Let us consider elements u, v, w, x of \mathcal{E}_T^3 . Now we state the propositions:

- (58) Suppose u is not zero and x is not zero and the direction of u = the direction of x . Then $\langle |u, v, w| \rangle = 0$ if and only if $\langle |x, v, w| \rangle = 0$. The theorem is a consequence of (31).
- (59) Suppose v is not zero and x is not zero and the direction of v = the direction of x . Then $\langle |u, v, w| \rangle = 0$ if and only if $\langle |u, x, w| \rangle = 0$. The theorem is a consequence of (32).

(60) Suppose w is not zero and x is not zero and the direction of $w =$ the direction of x . Then $\langle |u, v, w| \rangle = 0$ if and only if $\langle |u, v, x| \rangle = 0$. The theorem is a consequence of (33).

(61) (i) $(1_{\mathbb{R}} \text{ matrix}(3))(1) = e_1$, and

(ii) $(1_{\mathbb{R}} \text{ matrix}(3))(2) = e_2$, and

(iii) $(1_{\mathbb{R}} \text{ matrix}(3))(3) = e_3$.

(62) (i) the base finite sequence of 3 and 1 $= e_1$, and

(ii) the base finite sequence of 3 and 2 $= e_2$, and

(iii) the base finite sequence of 3 and 3 $= e_3$.

(63) Let us consider a finite sequence p_2 of elements of D . Suppose $\text{len } p_2 = 3$. Then

(i) $\langle p_2 \rangle_{\square, 1} = \langle p_2(1) \rangle$, and

(ii) $\langle p_2 \rangle_{\square, 2} = \langle p_2(2) \rangle$, and

(iii) $\langle p_2 \rangle_{\square, 3} = \langle p_2(3) \rangle$.

The theorem is a consequence of (3).

(64) (i) $\langle e_1 \rangle_{\square, 1} = \langle 1 \rangle$, and

(ii) $\langle e_1 \rangle_{\square, 2} = \langle 0 \rangle$, and

(iii) $\langle e_1 \rangle_{\square, 3} = \langle 0 \rangle$.

The theorem is a consequence of (63).

(65) (i) $\langle e_2 \rangle_{\square, 1} = \langle 0 \rangle$, and

(ii) $\langle e_2 \rangle_{\square, 2} = \langle 1 \rangle$, and

(iii) $\langle e_2 \rangle_{\square, 3} = \langle 0 \rangle$.

The theorem is a consequence of (63).

(66) (i) $\langle e_3 \rangle_{\square, 1} = \langle 0 \rangle$, and

(ii) $\langle e_3 \rangle_{\square, 2} = \langle 0 \rangle$, and

(iii) $\langle e_3 \rangle_{\square, 3} = \langle 1 \rangle$.

The theorem is a consequence of (63).

(67) (i) $(I_{\mathbb{R}_F}^{3 \times 3})_{\square, 1} = \langle 1, 0, 0 \rangle$, and

(ii) $(I_{\mathbb{R}_F}^{3 \times 3})_{\square, 2} = \langle 0, 1, 0 \rangle$, and

(iii) $(I_{\mathbb{R}_F}^{3 \times 3})_{\square, 3} = \langle 0, 0, 1 \rangle$.

The theorem is a consequence of (1) and (15).

(68) (i) $\text{Line}(I_{\mathbb{R}_F}^{3 \times 3}, 1) = \langle 1, 0, 0 \rangle$, and

(ii) $\text{Line}(I_{\mathbb{R}_F}^{3 \times 3}, 2) = \langle 0, 1, 0 \rangle$, and

(iii) $\text{Line}(I_{\mathbb{R}_F}^{3 \times 3}, 3) = \langle 0, 0, 1 \rangle$.

The theorem is a consequence of (1).

(69) (i) $\langle e_1 \rangle^T = \langle \langle 1 \rangle, \langle 0 \rangle, \langle 0 \rangle \rangle$, and

(ii) $\langle e_2 \rangle^T = \langle \langle 0 \rangle, \langle 1 \rangle, \langle 0 \rangle \rangle$, and

(iii) $\langle e_3 \rangle^T = \langle \langle 0 \rangle, \langle 0 \rangle, \langle 1 \rangle \rangle$.

The theorem is a consequence of (64), (65), and (66).

From now on p_1 denotes a finite sequence of elements of D .

Now we state the propositions:

(70) Let us consider a finite sequence p_1 of elements of D . If $k \in \text{dom } p_1$, then $\langle p_1 \rangle_{1,k} = p_1(k)$.

(71) If $k \in \text{dom } p_1$, then $\langle p_1 \rangle_{\square,k} = \langle p_1(k) \rangle$. The theorem is a consequence of (70).

(72) Let us consider an element p_2 of \mathcal{R}^3 . Suppose $p_1 = p_2$. Then $(\mathbb{R} \rightarrow \mathbb{R}_F) \text{ColVec2Mx}(p_2) = \langle p_1 \rangle^T$. The theorem is a consequence of (71).

In the sequel P denotes a square matrix over \mathbb{R}_F of dimension 3.

Now we state the propositions:

(73) Suppose $P = \langle \langle (p)_1, (p)_2, (p)_3 \rangle, \langle (q)_1, (q)_2, (q)_3 \rangle, \langle (r)_1, (r)_2, (r)_3 \rangle \rangle$. Then

(i) $\text{Line}(P, 1) = p$, and

(ii) $\text{Line}(P, 2) = q$, and

(iii) $\text{Line}(P, 3) = r$.

(74) Suppose $P = \langle \langle (p)_1, (p)_2, (p)_3 \rangle, \langle (q)_1, (q)_2, (q)_3 \rangle, \langle (r)_1, (r)_2, (r)_3 \rangle \rangle$. Then

(i) $P_{\square,1} = \langle (p)_1, (q)_1, (r)_1 \rangle$, and

(ii) $P_{\square,2} = \langle (p)_2, (q)_2, (r)_2 \rangle$, and

(iii) $P_{\square,3} = \langle (p)_3, (q)_3, (r)_3 \rangle$.

(75) $\text{width}\langle p_1 \rangle = \text{len } p_1$.

(76) Suppose $\text{len } p_1 = 3$. Then

(i) $\text{Line}\langle p_1 \rangle^T, 1 = \langle p_1(1) \rangle$, and

(ii) $\text{Line}\langle p_1 \rangle^T, 2 = \langle p_1(2) \rangle$, and

(iii) $\text{Line}\langle p_1 \rangle^T, 3 = \langle p_1(3) \rangle$.

The theorem is a consequence of (75) and (63).

(77) If $\text{len } p_1 = 3$, then $\langle p_1 \rangle^T = \langle \langle p_1(1) \rangle, \langle p_1(2) \rangle, \langle p_1(3) \rangle \rangle$. The theorem is a consequence of (76).

Let us consider D . Let p be a finite sequence of elements of D . Assume $\text{len } p = 3$. The functor $\text{F2M}(p)$ yielding a finite sequence of elements of D^1 is defined by the term

(Def. 1) $\langle\langle p(1)\rangle\rangle, \langle\langle p(2)\rangle\rangle, \langle\langle p(3)\rangle\rangle$.

Let us consider a finite sequence p of elements of \mathbb{R} . Now we state the propositions:

(78) If $\text{len } p = 3$, then $\text{len F2M}(p) = 3$.

(79) If $\text{len } p = 3$, then p is a 3-element finite sequence of elements of \mathbb{R} .

(80) If $p = [0, 0, 0]$, then $\text{F2M}(p) = \langle\langle 0\rangle\rangle, \langle\langle 0\rangle\rangle, \langle\langle 0\rangle\rangle$.

(81) Suppose $\text{len } p_1 = 3$. Then $\langle\langle p_1\rangle_{\square,1}\rangle, \langle\langle p_1\rangle_{\square,2}\rangle, \langle\langle p_1\rangle_{\square,3}\rangle = \text{F2M}(p_1)$. The theorem is a consequence of (63).

Let us consider D . Let p be a finite sequence of elements of D^1 . Assume $\text{len } p = 3$. The functor $\text{M2F}(p)$ yielding a finite sequence of elements of D is defined by the term

(Def. 2) $\langle p(1)(1), p(2)(1), p(3)(1)\rangle$.

Now we state the proposition:

(82) Let us consider a finite sequence p of elements of \mathbb{R}^1 . Suppose $\text{len } p = 3$. Then $\text{M2F}(p)$ is a point of \mathcal{E}_T^3 .

Let p be a finite sequence of elements of \mathbb{R}^1 and a be a real number. Assume $\text{len } p = 3$. The functor $a \cdot p$ yielding a finite sequence of elements of \mathbb{R}^1 is defined by

(Def. 3) there exist real numbers p_1, p_2, p_3 such that $p_1 = p(1)(1)$ and $p_2 = p(2)(1)$ and $p_3 = p(3)(1)$ and $it = \langle\langle a \cdot p_1\rangle\rangle, \langle\langle a \cdot p_2\rangle\rangle, \langle\langle a \cdot p_3\rangle\rangle$.

Let us consider a finite sequence p of elements of \mathbb{R}^1 . Now we state the propositions:

(83) If $\text{len } p = 3$, then $\text{M2F}(a \cdot p) = a \cdot \text{M2F}(p)$.

(84) If $\text{len } p = 3$, then $\langle\langle p(1)(1)\rangle\rangle, \langle\langle p(2)(1)\rangle\rangle, \langle\langle p(3)(1)\rangle\rangle = p$.

(85) If $\text{len } p = 3$, then $\text{F2M}(\text{M2F}(p)) = p$. The theorem is a consequence of (84).

(86) Let us consider a finite sequence p of elements of \mathbb{R} . If $\text{len } p = 3$, then $\text{M2F}(\text{F2M}(p)) = p$.

(87) (i) $\langle e_1\rangle^T = \text{F2M}(e_1)$, and

(ii) $\langle e_2\rangle^T = \text{F2M}(e_2)$, and

(iii) $\langle e_3\rangle^T = \text{F2M}(e_3)$.

The theorem is a consequence of (69).

(88) Let us consider a finite sequence p of elements of D . If $\text{len } p = 3$, then $\langle p\rangle^T = \text{F2M}(p)$. The theorem is a consequence of (77).

(89) $\text{Line}(\langle p_1\rangle, 1) = p_1$.

(90) Let us consider a matrix M over D of dimension 3×1 . Then

- (i) $\text{Line}(M, 1) = \langle M_{1,1} \rangle$, and
- (ii) $\text{Line}(M, 2) = \langle M_{2,1} \rangle$, and
- (iii) $\text{Line}(M, 3) = \langle M_{3,1} \rangle$.

From now on R denotes a ring.

Now we state the propositions:

- (91) Let us consider a square matrix N over R of dimension 3, and a finite sequence p of elements of R . If $\text{len } p = 3$, then $N \cdot \langle p \rangle^T$ is 3,1-size.
- (92) Let us consider a finite sequence p_1 of elements of R , and a square matrix N over R of dimension 3. Suppose $\text{len } p_1 = 3$. Then
 - (i) $\text{Line}(N \cdot \langle p_1 \rangle^T, 1) = \langle (N \cdot \langle p_1 \rangle^T)_{1,1} \rangle$, and
 - (ii) $\text{Line}(N \cdot \langle p_1 \rangle^T, 2) = \langle (N \cdot \langle p_1 \rangle^T)_{2,1} \rangle$, and
 - (iii) $\text{Line}(N \cdot \langle p_1 \rangle^T, 3) = \langle (N \cdot \langle p_1 \rangle^T)_{3,1} \rangle$.

The theorem is a consequence of (91) and (90).

- (93) $(\langle p_1 \rangle^T)_{\square,1} = p_1$. The theorem is a consequence of (89).
- (94) Let us consider finite sequences p_1, q_1, r_1 of elements of \mathbb{R}_F . Suppose $p = p_1$ and $q = q_1$ and $r = r_1$ and $\langle |p, q, r| \rangle \neq 0$. Then there exists a square matrix M over \mathbb{R}_F of dimension 3 such that
 - (i) M is invertible, and
 - (ii) $M \cdot p_1 = \text{F2M}(e_1)$, and
 - (iii) $M \cdot q_1 = \text{F2M}(e_2)$, and
 - (iv) $M \cdot r_1 = \text{F2M}(e_3)$.

PROOF: Reconsider $P = \langle \langle (p)_1, (p)_2, (p)_3 \rangle, \langle (q)_1, (q)_2, (q)_3 \rangle, \langle (r)_1, (r)_2, (r)_3 \rangle \rangle$ as a square matrix over \mathbb{R}_F of dimension 3. $\langle |p, q, r| \rangle = \text{Det } P$. Consider N being a square matrix over \mathbb{R}_F of dimension 3 such that N is inverse of P^T . $N \cdot \langle p_1 \rangle^T$ is a matrix over \mathbb{R}_F of dimension 3×1 and $N \cdot \langle q_1 \rangle^T$ is a matrix over \mathbb{R}_F of dimension 3×1 and $N \cdot \langle r_1 \rangle^T$ is a matrix over \mathbb{R}_F of dimension 3×1 . $N \cdot \langle p_1 \rangle^T = \text{F2M}(e_1)$ by (78), [3, (91), (45), (1)]. $N \cdot \langle q_1 \rangle^T = \text{F2M}(e_2)$ by (78), [3, (91), (45), (1)]. $N \cdot \langle r_1 \rangle^T = \text{F2M}(e_3)$ by (78), [3, (91), (45), (1)]. \square

- (95) Let us consider finite sequences p_1, q_1, r_1 of elements of \mathbb{R}_F , and finite sequences p_2, q_2, r_2 of elements of \mathbb{R}^1 . Suppose $P = \langle \langle (p)_1, (q)_1, (r)_1 \rangle, \langle (p)_2, (q)_2, (r)_2 \rangle, \langle (p)_3, (q)_3, (r)_3 \rangle \rangle$ and $p = p_1$ and $q = q_1$ and $r = r_1$ and $p_2 = M \cdot p_1$ and $q_2 = M \cdot q_1$ and $r_2 = M \cdot r_1$. Then $(M \cdot P)^T = \langle \text{M2F}(p_2), \text{M2F}(q_2), \text{M2F}(r_2) \rangle$.

PROOF: $P^T = \langle \langle (p)_1, (p)_2, (p)_3 \rangle, \langle (q)_1, (q)_2, (q)_3 \rangle, \langle (r)_1, (r)_2, (r)_3 \rangle \rangle$. width $M = \text{len } \langle p_1 \rangle^T$ and width $M = \text{len } \langle q_1 \rangle^T$ and width $M = \text{len } \langle r_1 \rangle^T$ by (75), [11, (50)]. $\text{len } p_2 = 3$ and $\text{len } q_2 = 3$ and $\text{len } r_2 = 3$. \square

(96) Let us consider finite sequences p_2, q_2, r_2 of elements of \mathbb{R}^1 . Suppose $M = \langle \text{M2F}(p_2), \text{M2F}(q_2), \text{M2F}(r_2) \rangle$ and $\text{Det } M = 0$ and $\text{M2F}(p_2) = p$ and $\text{M2F}(q_2) = q$ and $\text{M2F}(r_2) = r$. Then $\langle |p, q, r| \rangle = 0$. The theorem is a consequence of (35).

(97) Let us consider points p_3, q_3, r_3 of \mathcal{E}_T^3 , finite sequences p_2, q_2, r_2 of elements of \mathbb{R}^1 , and finite sequences p_1, q_1, r_1 of elements of \mathbb{R}_F . Suppose M is invertible and $p = p_1$ and $q = q_1$ and $r = r_1$ and $p_2 = M \cdot p_1$ and $q_2 = M \cdot q_1$ and $r_2 = M \cdot r_1$ and $\text{M2F}(p_2) = p_3$ and $\text{M2F}(q_2) = q_3$ and $\text{M2F}(r_2) = r_3$. Then $\langle |p, q, r| \rangle = 0$ if and only if $\langle |p_3, q_3, r_3| \rangle = 0$. The theorem is a consequence of (19), (23), (95), and (35).

(98) If $0 < m$, then every matrix over \mathbb{R}_F of dimension $m \times 1$ is a finite sequence of elements of \mathbb{R}^1 .

PROOF: Consider s being a finite sequence such that $s \in \text{rng } M$ and $\text{len } s = 1$. Consider n being a natural number such that for every object x such that $x \in \text{rng } M$ there exists a finite sequence s such that $s = x$ and $\text{len } s = n$. Consider s_1 being a finite sequence such that $s_1 = s$ and $\text{len } s_1 = n$. $\text{rng } M \subseteq \mathbb{R}^1$ by [5, (132)]. \square

(99) Let us consider a finite sequence u_1 of elements of \mathbb{R}_F . Suppose $\text{len } u_1 = 3$. Then $\langle u_1 \rangle^T = I_{\mathbb{R}_F}^{3 \times 3} \cdot \langle u_1 \rangle^T$. The theorem is a consequence of (77), (91), (2), (68), (7), and (93).

(100) Let us consider an element u of \mathcal{E}_T^3 , and a finite sequence u_1 of elements of \mathbb{R}_F . Suppose $u = u_1$ and $\langle u_1 \rangle^T = \langle \langle 0 \rangle, \langle 0 \rangle, \langle 0 \rangle \rangle$. Then $u = 0_{\mathcal{E}_T^3}$. The theorem is a consequence of (77).

(101) Let us consider an invertible square matrix N over \mathbb{R}_F of dimension 3, elements u, μ of \mathcal{E}_T^3 , a finite sequence u_1 of elements of \mathbb{R}_F , and a finite sequence u_2 of elements of \mathbb{R}^1 . Suppose u is not zero and $u = u_1$ and $u_2 = N \cdot u_1$ and $\mu = \text{M2F}(u_2)$. Then μ is not zero. The theorem is a consequence of (75), (85), (80), (8), (99), and (100).

Let N be an invertible square matrix over \mathbb{R}_F of dimension 3. The homography of N yielding a function from the projective space over \mathcal{E}_T^3 into the projective space over \mathcal{E}_T^3 is defined by

(Def. 4) for every point x of the projective space over \mathcal{E}_T^3 , there exist elements u, v of \mathcal{E}_T^3 and there exists a finite sequence u_1 of elements of \mathbb{R}_F and there exists a finite sequence p of elements of \mathbb{R}^1 such that $x =$ the direction of u and u is not zero and $u = u_1$ and $p = N \cdot u_1$ and $v = \text{M2F}(p)$ and v is not zero and $it(x) =$ the direction of v .

Now we state the proposition:

(102) Let us consider an invertible square matrix N over \mathbb{R}_F of dimension 3, and points p, q, r of the projective space over \mathcal{E}_T^3 . Then p, q and r are

collinear if and only if (the homography of $N(p)$), (the homography of $N(q)$) and (the homography of $N(r)$) are collinear.

PROOF: If p , q and r are collinear, then (the homography of $N(p)$), (the homography of $N(q)$) and (the homography of $N(r)$) are collinear by [10, (23)], (43), [9, (22), (1)]. If (the homography of $N(p)$), (the homography of $N(q)$) and (the homography of $N(r)$) are collinear, then p , q and r are collinear. \square

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