FORMALIZED MATHEMATICS
Vol. 23, No. 4, Pages 351–369, 2015
DOI: 10.1515/forma-2015-0028

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degruyter.com/view/j/forma

Exponential Objects

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Summary. In the first part of this article we formalize the concepts of terminal and initial object, categorical product [4] and natural transformation within a free-object category [1]. In particular, we show that this definition of natural transformation is equivalent to the standard definition [13]. Then we introduce the exponential object using its universal property and we show the isomorphism between the exponential object of categories and the functor category [12].

MSC: 18A99 18A25 03B35

Keywords: exponential objects; functor category; natural transformation

MML identifier: CAT_8, version: 8.1.04 5.33.1254

The notation and terminology used in this paper have been introduced in the following articles: [2], [5], [15], [16], [17], [10], [6], [7], [11], [18], [19], [3], [8], [21], [22], [14], [20], and [9].

1. Preliminaries

Now we state the propositions:

(1) Let us consider a composable, associative category structure $\mathscr{C}$, and morphisms f_1, f_2, f_3 of $\mathscr{C}$. Suppose $f_1 \triangleright f_2$ and $f_2 \triangleright f_3$. Then $(f_1 \circ f_2) \circ f_3 = f_1 \circ (f_2 \circ f_3)$.

(2) Let us consider a composable, associative category structure $\mathscr{C}$, and morphisms f_1, f_2, f_3, f_4 of $\mathscr{C}$. Suppose $f_1 \triangleright f_2$ and $f_2 \triangleright f_3$ and $f_3 \triangleright f_4$. Then

(i) $((f_1 \circ f_2) \circ f_3) \circ f_4 = (f_1 \circ f_2) \circ (f_3 \circ f_4)$, and

(ii) $((f_1 \circ f_2) \circ f_3) \circ f_4 = (f_1 \circ (f_2 \circ f_3)) \circ f_4$, and

ISSN 1426–2630(Print), 1898-9934(Online)

(iii) $((f_1 \circ f_2) \circ f_3) \circ f_4 = f_1 \circ ((f_2 \circ f_3) \circ f_4)$, and

(iv) $((f_1 \circ f_2) \circ f_3) \circ f_4 = f_1 \circ (f_2 \circ (f_3 \circ f_4))$.

The theorem is a consequence of (1).

(3) Let us consider a composable category structure $\mathscr{C}$, and morphisms f, f_1, f_2 of $\mathscr{C}$. Suppose $f_1 \triangleright f_2$. Then

(i) $f_1 \circ f_2 \triangleright f$ iff $f_2 \triangleright f$, and

(ii) $f \triangleright f_1 \circ f_2$ iff $f \triangleright f_1$.

(4) Let us consider a composable category structure $\mathscr{C}$ with identities, and morphisms f_1, f_2 of $\mathscr{C}$. Suppose $f_1 \triangleright f_2$. Then

(i) if f_1 is identity, then $f_1 \circ f_2 = f_2$, and

(ii) if f_2 is identity, then $f_1 \circ f_2 = f_1$.

PROOF: If f_1 is identity, then $f_1 \circ f_2 = f_2$ by [16, (6), (5), (9)]. □

(5) Let us consider a non empty category structure $\mathscr{C}$ with identities, and a morphism f of $\mathscr{C}$. Then there exist morphisms f_1, f_2 of $\mathscr{C}$ such that

(i) f_1 is identity, and

(ii) f_2 is identity, and

(iii) $f_1 \triangleright f$, and

(iv) $f \triangleright f_2$.

(6) Let us consider a category structure $\mathscr{C}$, objects a, b of $\mathscr{C}$, and a morphism f from a to b. Suppose $\hom(a, b) = \{f\}$. Let us consider a morphism g from a to b. Then $f = g$.

(7) Let us consider a category structure $\mathscr{C}$, objects a, b of $\mathscr{C}$, and a morphism f from a to b. Suppose $\hom(a, b) \neq \emptyset$ and for every morphism g from a to b, $f = g$. Then $\hom(a, b) = \{f\}$.

(8) Let us consider an object x, and a category structure $\mathscr{C}$. Suppose the carrier of $\mathscr{C} = \{x\}$ and the composition of $\mathscr{C} = \{\langle\langle x, x\rangle, x\rangle\}$. Then $\mathscr{C}$ is a non empty category.

PROOF: For every object y, $y \in$ the composition of the discrete category of $\{x\}$ iff $y \in \{\langle\langle x, x\rangle, x\rangle\}$ by [22, (2)], [9, (29)], [15, (24)], (4). □

(9) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$, and a functor $\mathcal{F}$ from $\mathscr{C}_1$ to $\mathscr{C}_2$. If $\mathcal{F}$ is isomorphism, then $\mathcal{F}$ is bijective.

(10) Let us consider composable category structures $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ with identities. Suppose $\mathscr{C}_1 \cong \mathscr{C}_2$ and $\mathscr{C}_2 \cong \mathscr{C}_3$. Then $\mathscr{C}_1 \cong \mathscr{C}_3$.

(11) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$. Suppose $\mathscr{C}_1 \cong \mathscr{C}_2$. Then $\mathscr{C}_1$ is empty if and only if $\mathscr{C}_2$ is empty.

Let $\mathscr{C}_1$ be an empty category structure with identities and $\mathscr{C}_2$ be category structure with identities. Note that every functor from $\mathscr{C}_1$ to $\mathscr{C}_2$ is covariant.

Now we state the propositions:

(12) Let us consider category structures $\mathscr{C}_1$, $\mathscr{C}_2$ with identities, a morphism f of $\mathscr{C}_1$, and a functor $\mathcal{F}$ from $\mathscr{C}_1$ to $\mathscr{C}_2$. Suppose $\mathcal{F}$ is covariant and f is identity. Then $\mathcal{F}(f)$ is identity.

(13) Let us consider category structures $\mathscr{C}_1$, $\mathscr{C}_2$ with identities, morphisms f_1, f_2 of $\mathscr{C}_1$, and a functor $\mathcal{F}$ from $\mathscr{C}_1$ to $\mathscr{C}_2$. Suppose $\mathcal{F}$ is covariant and $f_1 \rhd f_2$. Then

(i) $\mathcal{F}(f_1) \rhd \mathcal{F}(f_2)$, and

(ii) $\mathcal{F}(f_1 \circ f_2) = \mathcal{F}(f_1) \circ \mathcal{F}(f_2)$.

(14) Let us consider an object-category $\mathscr{C}$, a morphism f of $\mathscr{C}$, and a morphism g of alter $\mathscr{C}$. Suppose $f = g$. Then

(i) $\operatorname{dom} g = \mathrm{id}_{\operatorname{dom} f}$, and

(ii) $\operatorname{cod} g = \mathrm{id}_{\operatorname{cod} f}$.

PROOF: Consider d_1 being a morphism of alter $\mathscr{C}$ such that $\operatorname{dom} g = d_1$ and $g \rhd d_1$ and d_1 is identity. Reconsider $d_2 = \mathrm{id}_{\operatorname{dom} f}$ as a morphism of alter $\mathscr{C}$. For every morphism f_1 of alter $\mathscr{C}$ such that $f_1 \rhd d_2$ holds $f_1 \circ d_2 = f_1$ by [15, (40)], [5, (22)]. Consider c_1 being a morphism of alter $\mathscr{C}$ such that $\operatorname{cod} g = c_1$ and $c_1 \rhd g$ and c_1 is identity. Reconsider $c_2 = \mathrm{id}_{\operatorname{cod} f}$ as a morphism of alter $\mathscr{C}$. For every morphism f_1 of alter $\mathscr{C}$ such that $f_1 \rhd c_2$ holds $f_1 \circ c_2 = f_1$ by [15, (40)], [5, (22)]. □

(15) There exists a morphism f of $\mathbf{1}$ such that

(i) f is identity, and

(ii) $\operatorname{Ob} \mathbf{1} = \{f\}$, and

(iii) $\operatorname{Mor} \mathbf{1} = \{f\}$.

PROOF: Consider $\mathscr{C}$ being a strict, a preorder category such that $\operatorname{Ob} \mathscr{C} = 1$ and for every objects o_1, o_2 of $\mathscr{C}$ such that $o_1 \in o_2$ holds $\hom(o_1, o_2) = \{\langle o_1, o_2 \rangle\}$ and $\operatorname{RelOb} \mathscr{C} = \subseteq_1$ and $\operatorname{Mor} \mathscr{C} = 1 \cup \{\langle o_1, o_2 \rangle$, where o_1, o_2 are elements of $1 : o_1 \in o_2\}$. Consider $\mathcal{F}$ being a functor from $\mathscr{C}$ to $\mathbf{1}$, $\mathcal{G}$ being a functor from $\mathbf{1}$ to $\mathscr{C}$ such that $\mathcal{F}$ is covariant and $\mathcal{G}$ is covariant and $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathscr{C}}$ and $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathbf{1}}$. Reconsider $g = 0$ as a morphism of $\mathscr{C}$. Set $f = \mathcal{F}(g)$. Consider x being an object such that $\operatorname{Ob} \mathbf{1} = \{x\}$. For every object x, $x \in \operatorname{Mor} \mathbf{1}$ iff $x \in \{f\}$ by [15, (22)], [6, (18)], [15, (34)], [2, (49)]. □

(16) Let us consider a non empty category $\mathscr{C}$, and morphisms f_1, f_2 of $\mathscr{C}$. If $\mathcal{M}_{\mathrm{f}_1} = \mathcal{M}_{\mathrm{f}_2}$, then $f_1 = f_2$.

(17) Let us consider a non empty category $\mathscr{C}$, covariant functors $\mathcal{F}_1$, $\mathcal{F}_2$ from **2** to $\mathscr{C}$, and a morphism f of **2**. Suppose f is not identity and $\mathcal{F}_1(f) = \mathcal{F}_2(f)$. Then $\mathcal{F}_1 = \mathcal{F}_2$.

PROOF: Consider f_1 being a morphism of **2** such that f_1 is not identity and $\mathrm{Ob}\,\mathbf{2} = \{\mathrm{dom}\, f_1, \mathrm{cod}\, f_1\}$ and $\mathrm{Mor}\,\mathbf{2} = \{\mathrm{dom}\, f_1, \mathrm{cod}\, f_1, f_1\}$ and $\mathrm{dom}\, f_1$, $\mathrm{cod}\, f_1$, f_1 are mutually different. For every object x such that $x \in \mathrm{dom}\,\mathcal{F}_1$ holds $\mathcal{F}_1(x) = \mathcal{F}_2(x)$ by [15, (22), (32)]. □

(18) There exist morphisms f_1, f_2 of **3** such that

(i) f_1 is not identity, and

(ii) f_2 is not identity, and

(iii) $\mathrm{cod}\, f_1 = \mathrm{dom}\, f_2$, and

(iv) $\mathrm{Ob}\,\mathbf{3} = \{\mathrm{dom}\, f_1, \mathrm{cod}\, f_1, \mathrm{cod}\, f_2\}$, and

(v) $\mathrm{Mor}\,\mathbf{3} = \{\mathrm{dom}\, f_1, \mathrm{cod}\, f_1, \mathrm{cod}\, f_2, f_1, f_2, f_2 \circ f_1\}$, and

(vi) $\mathrm{dom}\, f_1$,$\mathrm{cod}\, f_1$,$\mathrm{cod}\, f_2$,f_1,f_2,$f_2 \circ f_1$ are mutually different.

PROOF: Consider $\mathscr{C}$ being a strict, a preorder category such that $\mathrm{Ob}\,\mathscr{C} = 3$ and for every objects o_1, o_2 of $\mathscr{C}$ such that $o_1 \in o_2$ holds $\mathrm{hom}(o_1, o_2) = \{\langle o_1, o_2 \rangle\}$ and $\mathrm{RelOb}\,\mathscr{C} = {}^{\subseteq}{}_3$ and $\mathrm{Mor}\,\mathscr{C} = 3 \cup \{\langle o_1, o_2 \rangle$, where o_1, o_2 are elements of $3 : o_1 \in o_2\}$. Consider $\mathcal{F}$ being a functor from $\mathscr{C}$ to **3**, $\mathcal{G}$ being a functor from **3** to $\mathscr{C}$ such that $\mathcal{F}$ is covariant and $\mathcal{G}$ is covariant and $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathscr{C}}$ and $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathbf{3}}$. Reconsider $g_1 = \langle 0, 1 \rangle$ as a morphism of $\mathscr{C}$. g_1 is not identity by [15, (22)]. Set $f_1 = \mathcal{F}(g_1)$. Reconsider $g_2 = \langle 1, 2 \rangle$ as a morphism of $\mathscr{C}$. g_2 is not identity by [15, (22)]. Set $f_2 = \mathcal{F}(g_2)$. f_1 is not identity by [6, (18)], [15, (34)]. f_2 is not identity by [6, (18)], [15, (34)]. For every object x, $x \in \mathrm{Ob}\,\mathbf{3}$ iff $x \in \{\mathrm{dom}\, f_1, \mathrm{cod}\, f_1, \mathrm{cod}\, f_2\}$ by [15, (34)], [6, (18)], [15, (22)], [2, (51)]. For every object x, $x \in \mathrm{Mor}\,\mathbf{3}$ iff $x \in \{\mathrm{dom}\, f_1, \mathrm{cod}\, f_1, \mathrm{cod}\, f_2, f_1, f_2, f_2 \circ f_1\}$ by [15, (22)], [6, (18)], [15, (34)], [2, (51), (49), (50)]. $g_2 \circ g_1$ is not identity by [15, (22)]. $f_2 \circ f_1$ is not identity by [6, (18)], [15, (34)]. $\mathcal{F}$ is bijective. □

Let $\mathscr{C}$ be a non empty category and f_1, f_2 be morphisms of $\mathscr{C}$. Assume $f_1 \triangleright f_2$. The functor $\mathcal{C}_{f_1,f_2}$ yielding a covariant functor from **3** to $\mathscr{C}$ is defined by

(Def. 1) for every morphisms g_1, g_2 of **3** such that $g_1 \triangleright g_2$ and g_1 is not identity and g_2 is not identity holds $it(g_1) = f_1$ and $it(g_2) = f_2$.

2. Terminal Objects

Let $\mathscr{C}$ be a category structure and a be an object of $\mathscr{C}$. We say that a is terminal if and only if

(Def. 2) for every object b of $\mathscr{C}$, $\hom(b, a) \neq \emptyset$ and there exists a morphism f from b to a such that for every morphism g from b to a, $f = g$.

Now we state the propositions:

(19) Let us consider a category structure $\mathscr{C}$, and an object b of $\mathscr{C}$. Then b is terminal if and only if for every object a of $\mathscr{C}$, there exists a morphism f from a to b such that $\hom(a, b) = \{f\}$. The theorem is a consequence of (7) and (6).

(20) Let us consider category structure $\mathscr{C}$ with identities, and an object a of $\mathscr{C}$. Suppose a is terminal. Let us consider a morphism h from a to a. Then id-$a = h$.

(21) Let us consider a composable category structure $\mathscr{C}$ with identities, and objects a, b of $\mathscr{C}$. If a is terminal and b is terminal, then a and b are isomorphic. The theorem is a consequence of (20).

(22) Let us consider a category $\mathscr{C}$, and objects a, b of $\mathscr{C}$. If b is terminal and a and b are isomorphic, then a is terminal.

(23) Let us consider a composable category structure $\mathscr{C}$ with identities, objects a, b of $\mathscr{C}$, and a morphism f from a to b. Suppose $\hom(a, b) \neq \emptyset$ and a is terminal. Then f is monomorphic.

Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ has terminal objects if and only if

(Def. 3) there exists an object a of $\mathscr{C}$ such that a is terminal.

Now we state the proposition:

(24) $\mathbf{1}$ has terminal objects.

Proof: Consider f being a morphism of $\mathbf{1}$ such that f is identity and $\mathrm{Ob}\,\mathbf{1} = \{f\}$ and $\mathrm{Mor}\,\mathbf{1} = \{f\}$. For every objects a, b of $\mathbf{1}$, every morphism of $\mathbf{1}$ is a morphism from a to b by [16, (20)]. □

One can verify that there exists a category which has terminal objects.

Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is terminal if and only if

(Def. 4) for every category $\mathscr{B}$, there exists a functor $\mathcal{F}$ from $\mathscr{B}$ to $\mathscr{C}$ such that $\mathcal{F}$ is covariant and for every functor $\mathcal{G}$ from $\mathscr{B}$ to $\mathscr{C}$ such that $\mathcal{G}$ is covariant holds $\mathcal{F} = \mathcal{G}$.

Let us note that $\mathbf{1}$ is non empty and terminal and there exists a category which is strict, non empty, and terminal and there exists a category which is strict and non terminal.

Now we state the propositions:

(25) Let us consider terminal categories $\mathscr{C}$, $\mathscr{D}$. Then $\mathscr{C} \cong \mathscr{D}$.

PROOF: There exists a functor $\mathcal{F}$ from $\mathscr{C}$ to $\mathscr{D}$ and there exists a functor $\mathcal{G}$ from $\mathscr{D}$ to $\mathscr{C}$ such that $\mathcal{F}$ is covariant and $\mathcal{G}$ is covariant and $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathscr{C}}$ and $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathscr{D}}$ by [15, (35)]. □

(26) Let us consider categories $\mathscr{C}$, $\mathscr{D}$. Suppose $\mathscr{C}$ is terminal and $\mathscr{C} \cong \mathscr{D}$. Then $\mathscr{D}$ is terminal.

PROOF: Consider $\mathcal{F}$ being a functor from $\mathscr{C}$ to $\mathscr{D}$, $\mathcal{G}$ being a functor from $\mathscr{D}$ to $\mathscr{C}$ such that $\mathcal{F}$ is covariant and $\mathcal{G}$ is covariant and $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathscr{C}}$ and $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathscr{D}}$. Consider $\mathcal{F}_1$ being a functor from $\mathscr{B}$ to $\mathscr{C}$ such that $\mathcal{F}_1$ is covariant and for every functor $\mathcal{G}$ from $\mathscr{B}$ to $\mathscr{C}$ such that $\mathcal{G}$ is covariant holds $\mathcal{F}_1 = \mathcal{G}$. Set $\mathcal{F}_2 = \mathcal{F} \circ \mathcal{F}_1$. For every functor $\mathcal{G}_1$ from $\mathscr{B}$ to $\mathscr{D}$ such that $\mathcal{G}_1$ is covariant holds $\mathcal{F}_2 = \mathcal{G}_1$ by [15, (35)], [16, (10), (11)]. □

(27) Let us consider a category $\mathscr{C}$. Then $\mathscr{C}$ is non empty and trivial if and only if $\mathscr{C} \cong \mathbf{1}$. The theorem is a consequence of (15), (4), and (26).

(28) Let us consider non empty categories $\mathscr{C}$, $\mathscr{D}$. Suppose $\mathscr{C}$ is trivial and $\mathscr{D}$ is trivial. Then $\mathscr{C} \cong \mathscr{D}$. The theorem is a consequence of (27) and (10).

Note that every category which is non empty and trivial is also terminal and every category which is terminal is also non empty and trivial.

Let $\mathscr{C}$ be a category. The functor $\mathscr{C} \to \mathbf{1}$ yielding a covariant functor from $\mathscr{C}$ to $\mathbf{1}$ is defined by

(Def. 5) not contradiction.

Now we state the proposition:

(29) Let us consider categories $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$, a functor $\mathcal{F}_1$ from $\mathscr{C}$ to $\mathscr{C}_1$, and a functor $\mathcal{F}_2$ from $\mathscr{C}$ to $\mathscr{C}_2$. Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant. Then $\mathscr{C}_1 \to \mathbf{1} \circ \mathcal{F}_1 = \mathscr{C}_2 \to \mathbf{1} \circ \mathcal{F}_2$.

3. Initial Objects

Let $\mathscr{C}$ be a category structure and a be an object of $\mathscr{C}$. We say that a is initial if and only if

(Def. 6) for every object b of $\mathscr{C}$, $\hom(a, b) \neq \emptyset$ and there exists a morphism f from a to b such that for every morphism g from a to b, $f = g$.

Now we state the propositions:

(30) Let us consider a category structure $\mathscr{C}$, and an object b of $\mathscr{C}$. Then b is initial if and only if for every object a of $\mathscr{C}$, there exists a morphism f from b to a such that $\hom(b, a) = \{f\}$. The theorem is a consequence of (7) and (6).

(31) Let us consider category structure $\mathscr{C}$ with identities, and an object a of $\mathscr{C}$. Suppose a is initial. Let us consider a morphism h from a to a. Then id-$a = h$.

(32) Let us consider a composable category structure $\mathscr{C}$ with identities, and objects a, b of $\mathscr{C}$. If a is initial and b is initial, then a and b are isomorphic. The theorem is a consequence of (31).

(33) Let us consider a category $\mathscr{C}$, and objects a, b of $\mathscr{C}$. If b is initial and b and a are isomorphic, then a is initial.

(34) Let us consider a composable category structure $\mathscr{C}$ with identities, objects a, b of $\mathscr{C}$, and a morphism f from a to b. Suppose $\hom(a, b) \neq \emptyset$ and b is initial. Then f is epimorphic.

Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ has initial objects if and only if

(Def. 7) there exists an object a of $\mathscr{C}$ such that a is initial.

Now we state the proposition:

(35) $\mathbf{1}$ has initial objects.

PROOF: Consider f being a morphism of $\mathbf{1}$ such that f is identity and $\mathrm{Ob}\,\mathbf{1} = \{f\}$ and $\mathrm{Mor}\,\mathbf{1} = \{f\}$. For every objects a, b of $\mathbf{1}$, every morphism of $\mathbf{1}$ is a morphism from a to b by [16, (20)]. □

Let us note that there exists a category which has initial objects.

Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is initial if and only if

(Def. 8) for every category $\mathscr{C}_1$, there exists a functor $\mathcal{F}$ from $\mathscr{C}$ to $\mathscr{C}_1$ such that $\mathcal{F}$ is covariant and for every functor $\mathcal{F}_1$ from $\mathscr{C}$ to $\mathscr{C}_1$ such that $\mathcal{F}_1$ is covariant holds $\mathcal{F} = \mathcal{F}_1$.

One can verify that $\mathbf{0}$ is empty and initial and there exists a category which is strict, empty, and initial and there exists a category which is strict and non initial.

Now we state the propositions:

(36) Let us consider initial categories $\mathscr{C}$, $\mathscr{D}$. Then $\mathscr{C} \cong \mathscr{D}$.

PROOF: There exists a functor $\mathcal{F}$ from $\mathscr{C}$ to $\mathscr{D}$ and there exists a functor $\mathcal{G}$ from $\mathscr{D}$ to $\mathscr{C}$ such that $\mathcal{F}$ is covariant and $\mathcal{G}$ is covariant and $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathscr{C}}$ and $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathscr{D}}$ by [15, (35)]. □

(37) Let us consider categories $\mathscr{C}$, $\mathscr{D}$. Suppose $\mathscr{C}$ is initial and $\mathscr{C} \cong \mathscr{D}$. Then $\mathscr{D}$ is initial.

PROOF: Consider $\mathcal{F}$ being a functor from $\mathscr{C}$ to $\mathscr{D}$, $\mathcal{G}$ being a functor from $\mathscr{D}$ to $\mathscr{C}$ such that $\mathcal{F}$ is covariant and $\mathcal{G}$ is covariant and $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathscr{C}}$ and $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathscr{D}}$. Consider $\mathcal{F}_1$ being a functor from $\mathscr{C}$ to $\mathscr{B}$ such that $\mathcal{F}_1$ is covariant and for every functor $\mathcal{G}$ from $\mathscr{C}$ to $\mathscr{B}$ such that $\mathcal{G}$ is covariant holds $\mathcal{F}_1 = \mathcal{G}$. Set $\mathcal{F}_2 = \mathcal{F}_1 \circ \mathcal{G}$. For every functor $\mathcal{G}_1$ from $\mathscr{D}$ to $\mathscr{B}$ such that $\mathcal{G}_1$ is covariant holds $\mathcal{F}_2 = \mathcal{G}_1$ by [15, (35)], [16, (10), (11)]. □

Let us note that every category which is empty is also initial.

Let $\mathscr{C}$ be a category. The functor $\mathbf{0} \to \mathscr{C}$ yielding a covariant functor from $\mathbf{0}$ to $\mathscr{C}$ is defined by

(Def. 9) not contradiction.

Now we state the proposition:

(38) Let us consider categories $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$, a functor $\mathcal{F}_1$ from $\mathscr{C}_1$ to $\mathscr{C}$, and a functor $\mathcal{F}_2$ from $\mathscr{C}_2$ to $\mathscr{C}$. Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant. Then $\mathcal{F}_1 \circ \mathbf{0} \to \mathscr{C}_1 = \mathcal{F}_2 \circ \mathbf{0} \to \mathscr{C}_2$.

4. Categorical Products

Let $\mathscr{C}$ be a category, a, b, c be objects of $\mathscr{C}$, and p_1 be a morphism from c to a. Assume $\hom(c, a) \neq \emptyset$. Let p_2 be a morphism from c to b. Assume $\hom(c, b) \neq \emptyset$. We say that $\langle c, p_1, p_2\rangle$ is a product of a and b if and only if

(Def. 10) for every object c_1 of $\mathscr{C}$ and for every morphism q_1 from c_1 to a and for every morphism q_2 from c_1 to b such that $\hom(c_1, a) \neq \emptyset$ and $\hom(c_1, b) \neq \emptyset$ holds $\hom(c_1, c) \neq \emptyset$ and there exists a morphism h from c_1 to c such that $p_1 \cdot h = q_1$ and $p_2 \cdot h = q_2$ and for every morphism h_1 from c_1 to c such that $p_1 \cdot h_1 = q_1$ and $p_2 \cdot h_1 = q_2$ holds $h = h_1$.

Now we state the propositions:

(39) Let us consider a category $\mathscr{C}$, objects c_1, c_2, a, b of $\mathscr{C}$, a morphism p_1 from a to c_1, a morphism p_2 from a to c_2, a morphism q_1 from b to c_1, and a morphism q_2 from b to c_2. Suppose $\hom(a, c_1) \neq \emptyset$ and $\hom(a, c_2) \neq \emptyset$ and $\hom(b, c_1) \neq \emptyset$ and $\hom(b, c_2) \neq \emptyset$ and $\langle a, p_1, p_2\rangle$ is a product of c_1 and c_2 and $\langle b, q_1, q_2\rangle$ is a product of c_1 and c_2. Then a and b are isomorphic.

Proof: There exists a morphism f from a to b and there exists a morphism g from b to a such that $\hom(a, b) \neq \emptyset$ and $\hom(b, a) \neq \emptyset$ and $g \cdot f = \text{id-}a$ and $f \cdot g = \text{id-}b$ by [16, (23), (18)]. □

(40) Let us consider a category $\mathscr{C}$, objects c_1, c_2, d of $\mathscr{C}$, a morphism p_1 from d to c_1, and a morphism p_2 from d to c_2. Suppose $\hom(d, c_1) \neq \emptyset$ and $\hom(d, c_2) \neq \emptyset$ and $\langle d, p_1, p_2\rangle$ is a product of c_1 and c_2. Then $\langle d, p_2, p_1\rangle$ is a product of c_2 and c_1.

Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ has binary products if and only if

(Def. 11) for every objects a, b of $\mathscr{C}$, there exists an object d of $\mathscr{C}$ and there exists a morphism p_1 from d to a and there exists a morphism p_2 from d to b

such that $\hom(d, a) \neq \emptyset$ and $\hom(d, b) \neq \emptyset$ and $\langle d, p_1, p_2 \rangle$ is a product of a and b.

Now we state the proposition:

(41) **1** has binary products.

Proof: Set $\mathscr{C} = \mathbf{1}$. Consider f being a morphism of $\mathbf{1}$ such that f is identity and $\mathrm{Ob}\,\mathbf{1} = \{f\}$ and $\mathrm{Mor}\,\mathbf{1} = \{f\}$. For every objects o_1, o_2 of $\mathscr{C}$, every morphism of $\mathscr{C}$ is a morphism from o_1 to o_2 by [16, (20)]. Reconsider $p_1 = f$ as a morphism from a to a. Reconsider $p_2 = f$ as a morphism from a to b. For every object c_1 of $\mathscr{C}$ and for every morphism q_1 from c_1 to a and for every morphism q_2 from c_1 to b such that $\hom(c_1, a) \neq \emptyset$ and $\hom(c_1, b) \neq \emptyset$ holds $\hom(c_1, a) \neq \emptyset$ and there exists a morphism h from c_1 to a such that $p_1 \cdot h = q_1$ and $p_2 \cdot h = q_2$ and for every morphism h_1 from c_1 to a such that $p_1 \cdot h_1 = q_1$ and $p_2 \cdot h_1 = q_2$ holds $h = h_1$. □

Observe that there exists a category which has binary products.

Let $\mathscr{C}$ be a category with binary products and c_1, c_2 be objects of $\mathscr{C}$.

A categorical product of c_1 and c_2 is a triple object and is defined by

(Def. 12) there exists an object d of $\mathscr{C}$ and there exists a morphism p_1 from d to c_1 and there exists a morphism p_2 from d to c_2 such that $it = \langle d, p_1, p_2 \rangle$ and $\hom(d, c_1) \neq \emptyset$ and $\hom(d, c_2) \neq \emptyset$ and $\langle d, p_1, p_2 \rangle$ is a product of c_1 and c_2.

The functor $c_1 \times c_2$ yielding an object of $\mathscr{C}$ is defined by the term

(Def. 13) (the categorical product of c_1 and $c_2)_{\mathbf{1},3}$.

The functor $\pi_1(c_1 \boxtimes c_2)$ yielding a morphism from $c_1 \times c_2$ to c_1 is defined by the term

(Def. 14) (the categorical product of c_1 and $c_2)_{\mathbf{2},3}$.

The functor $\pi_2(c_1 \boxtimes c_2)$ yielding a morphism from $c_1 \times c_2$ to c_2 is defined by the term

(Def. 15) (the categorical product of c_1 and $c_2)_{\mathbf{3},3}$.

Now we state the propositions:

(42) Let us consider a category $\mathscr{C}$ with binary products, and objects a, b of $\mathscr{C}$. Then

(i) $\langle a \times b, \pi_1(a \boxtimes b), \pi_2(a \boxtimes b) \rangle$ is a product of a and b, and

(ii) $\hom(a \times b, a) \neq \emptyset$, and

(iii) $\hom(a \times b, b) \neq \emptyset$.

(43) Let us consider a category $\mathscr{C}$ with binary products, and objects a, b, c of $\mathscr{C}$. Suppose $\hom(c, a) \neq \emptyset$ and $\hom(c, b) \neq \emptyset$. Then $\hom(c, a \times b) \neq \emptyset$. The theorem is a consequence of (42).

(44) Let us consider a category $\mathscr{C}$ with binary products, and objects a, b, c, d of $\mathscr{C}$. Suppose $\hom(a, b) \neq \emptyset$ and $\hom(c, d) \neq \emptyset$. Then $\hom(a \times c, b \times d) \neq \emptyset$. The theorem is a consequence of (42).

Let $\mathscr{C}$ be a category with binary products, a, b, c, d be objects of $\mathscr{C}$, and f be a morphism from a to b. Assume $\hom(a, b) \neq \emptyset$. Let g be a morphism from c to d. Assume $\hom(c, d) \neq \emptyset$. The functor $f \times g$ yielding a morphism from $a \times c$ to $b \times d$ is defined by

(Def. 16) $f \cdot \pi_1(a \boxtimes c) = \pi_1(b \boxtimes d) \cdot \mathit{it}$ and $g \cdot \pi_2(a \boxtimes c) = \pi_2(b \boxtimes d) \cdot \mathit{it}$.

Let $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{D}$ be categories and $\mathcal{P}_1$ be a functor from $\mathscr{D}$ to $\mathscr{C}_1$. Assume $\mathcal{P}_1$ is covariant. Let $\mathcal{P}_2$ be a functor from $\mathscr{D}$ to $\mathscr{C}_2$. Assume $\mathcal{P}_2$ is covariant. We say that $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a product of $\mathscr{C}_1$ and $\mathscr{C}_2$ if and only if

(Def. 17) for every category $\mathscr{D}_1$ and for every functor $\mathcal{G}_1$ from $\mathscr{D}_1$ to $\mathscr{C}_1$ and for every functor $\mathcal{G}_2$ from $\mathscr{D}_1$ to $\mathscr{C}_2$ such that $\mathcal{G}_1$ is covariant and $\mathcal{G}_2$ is covariant there exists a functor $\mathcal{H}$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{H}$ is covariant and $\mathcal{P}_1 \circ \mathcal{H} = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H} = \mathcal{G}_2$ and for every functor $\mathcal{H}_1$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{H}_1$ is covariant and $\mathcal{P}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$.

Now we state the propositions:

(45) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{A}$, $\mathscr{B}$, a functor $\mathcal{P}_1$ from $\mathscr{A}$ to $\mathscr{C}_1$, a functor $\mathcal{P}_2$ from $\mathscr{A}$ to $\mathscr{C}_2$, a functor $\mathcal{Q}_1$ from $\mathscr{B}$ to $\mathscr{C}_1$, and a functor $\mathcal{Q}_2$ from $\mathscr{B}$ to $\mathscr{C}_2$. Suppose $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\mathcal{Q}_1$ is covariant and $\mathcal{Q}_2$ is covariant and $\langle \mathscr{A}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a product of $\mathscr{C}_1$ and $\mathscr{C}_2$ and $\langle \mathscr{B}, \mathcal{Q}_1, \mathcal{Q}_2 \rangle$ is a product of $\mathscr{C}_1$ and $\mathscr{C}_2$. Then $\mathscr{A} \cong \mathscr{B}$.
PROOF: There exists a functor $\mathcal{F}_4$ from $\mathscr{A}$ to $\mathscr{B}$ and there exists a functor $\mathcal{G}_3$ from $\mathscr{B}$ to $\mathscr{A}$ such that $\mathcal{F}_4$ is covariant and $\mathcal{G}_3$ is covariant and $\mathcal{G}_3 \circ \mathcal{F}_4 = \mathrm{id}_{\mathscr{A}}$ and $\mathcal{F}_4 \circ \mathcal{G}_3 = \mathrm{id}_{\mathscr{B}}$ by [16, (10), (11)], [15, (35)]. □

(46) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{D}$, a functor $\mathcal{P}_1$ from $\mathscr{D}$ to $\mathscr{C}_1$, and a functor $\mathcal{P}_2$ from $\mathscr{D}$ to $\mathscr{C}_2$. Suppose $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a product of $\mathscr{C}_1$ and $\mathscr{C}_2$. Then $\langle \mathscr{D}, \mathcal{P}_2, \mathcal{P}_1 \rangle$ is a product of $\mathscr{C}_2$ and $\mathscr{C}_1$.

Let $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$ be categories, $\mathcal{F}_1$ be a functor from $\mathscr{C}_1$ to $\mathscr{C}$, and $\mathcal{F}_2$ be a functor from $\mathscr{C}_2$ to $\mathscr{C}$. We introduce the notation $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ as a synonym of $[\![\mathcal{F}_1, \mathcal{F}_2]\!]$.

Now we state the proposition:

(47) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$. Then $\langle \mathscr{C}_1 \to \mathbf{1} \boxtimes \mathscr{C}_2 \to \mathbf{1}, \pi_1((\mathscr{C}_1 \to \mathbf{1}) \boxtimes (\mathscr{C}_2 \to \mathbf{1})), \pi_2((\mathscr{C}_1 \to \mathbf{1}) \boxtimes (\mathscr{C}_2 \to \mathbf{1})) \rangle$ is a product of $\mathscr{C}_1$ and $\mathscr{C}_2$.
PROOF: Set $\mathcal{F}_1 = \mathscr{C}_1 \to \mathbf{1}$. Set $\mathcal{F}_2 = \mathscr{C}_2 \to \mathbf{1}$. For every category $\mathscr{D}_1$ and for every functor $\mathcal{G}_1$ from $\mathscr{D}_1$ to $\mathscr{C}_1$ and for every functor $\mathcal{G}_2$ from $\mathscr{D}_1$ to $\mathscr{C}_2$ such that $\mathcal{G}_1$ is covariant and $\mathcal{G}_2$ is covariant there exists a functor $\mathcal{H}$ from $\mathscr{D}_1$ to $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ such that $\mathcal{H}$ is covariant and $\pi_1(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \circ \mathcal{H} = \mathcal{G}_1$ and

$\pi_2(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \circ \mathcal{H} = \mathcal{G}_2$ and for every functor $\mathcal{H}_1$ from $\mathscr{D}_1$ to $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ such that $\mathcal{H}_1$ is covariant and $\pi_1(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \circ \mathcal{H}_1 = \mathcal{G}_1$ and $\pi_2(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \circ \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$ by [16, (52)], (29). □

Let $\mathscr{C}_1$, $\mathscr{C}_2$ be categories.

A categorical product of $\mathscr{C}_1$ and $\mathscr{C}_2$ is a triple object and is defined by

(Def. 18) there exists a strict category $\mathscr{D}$ and there exists a functor $\mathcal{P}_1$ from $\mathscr{D}$ to $\mathscr{C}_1$ and there exists a functor $\mathcal{P}_2$ from $\mathscr{D}$ to $\mathscr{C}_2$ such that $it = \langle\mathscr{D}, \mathcal{P}_1, \mathcal{P}_2\rangle$ and $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\langle\mathscr{D}, \mathcal{P}_1, \mathcal{P}_2\rangle$ is a product of $\mathscr{C}_1$ and $\mathscr{C}_2$.

The functor $\mathscr{C}_1 \times \mathscr{C}_2$ yielding a strict category is defined by the term

(Def. 19) (the categorical product of $\mathscr{C}_1$ and $\mathscr{C}_2)_{\mathbf{1},3}$.

The functor $\pi_1(\mathscr{C}_1 \boxtimes \mathscr{C}_2)$ yielding a functor from $\mathscr{C}_1 \times \mathscr{C}_2$ to $\mathscr{C}_1$ is defined by the term

(Def. 20) (the categorical product of $\mathscr{C}_1$ and $\mathscr{C}_2)_{\mathbf{2},3}$.

The functor $\pi_2(\mathscr{C}_1 \boxtimes \mathscr{C}_2)$ yielding a functor from $\mathscr{C}_1 \times \mathscr{C}_2$ to $\mathscr{C}_2$ is defined by the term

(Def. 21) (the categorical product of $\mathscr{C}_1$ and $\mathscr{C}_2)_{\mathbf{3},3}$.

Now we state the proposition:

(48) Let us consider categories $\mathscr{C}_1, \mathscr{C}_2$. Then $\langle\mathscr{C}_1 \times \mathscr{C}_2, \pi_1(\mathscr{C}_1 \boxtimes \mathscr{C}_2), \pi_2(\mathscr{C}_1 \boxtimes \mathscr{C}_2)\rangle$ is a product of $\mathscr{C}_1$ and $\mathscr{C}_2$.

Let $\mathscr{C}_1$, $\mathscr{C}_2$ be categories. Note that $\pi_1(\mathscr{C}_1 \boxtimes \mathscr{C}_2)$ is covariant and $\pi_2(\mathscr{C}_1 \boxtimes \mathscr{C}_2)$ is covariant.

Now we state the proposition:

(49) Let us consider categories $\mathscr{C}_1, \mathscr{C}_2$. Then $\mathscr{C}_1 \times \mathscr{C}_2$ is not empty if and only if $\mathscr{C}_1$ is not empty and $\mathscr{C}_2$ is not empty. The theorem is a consequence of (48).

Let $\mathscr{C}_1$ be an empty category and $\mathscr{C}_2$ be a category. One can verify that $\mathscr{C}_1 \times \mathscr{C}_2$ is empty.

Let $\mathscr{C}_1$ be a category and $\mathscr{C}_2$ be an empty category. Observe that $\mathscr{C}_1 \times \mathscr{C}_2$ is empty.

Let $\mathscr{C}_1$ be a non empty category and $\mathscr{C}_2$ be a non empty category. One can verify that $\mathscr{C}_1 \times \mathscr{C}_2$ is non empty.

Let $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{D}_1$, $\mathscr{D}_2$ be categories, $\mathcal{F}_1$ be a functor from $\mathscr{C}_1$ to $\mathscr{D}_1$, and $\mathcal{F}_2$ be a functor from $\mathscr{C}_2$ to $\mathscr{D}_2$. Assume $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant. The functor $\mathcal{F}_1 \times \mathcal{F}_2$ yielding a functor from $\mathscr{C}_1 \times \mathscr{C}_2$ to $\mathscr{D}_1 \times \mathscr{D}_2$ is defined by

(Def. 22) it is covariant and $\mathcal{F}_1 \circ \pi_1(\mathscr{C}_1 \boxtimes \mathscr{C}_2) = \pi_1(\mathscr{D}_1 \boxtimes \mathscr{D}_2) \circ it$ and $\mathcal{F}_2 \circ \pi_2(\mathscr{C}_1 \boxtimes \mathscr{C}_2) = \pi_2(\mathscr{D}_1 \boxtimes \mathscr{D}_2) \circ it$.

Now we state the propositions:

(50) Let us consider categories $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{B}_1$, $\mathscr{B}_2$, $\mathscr{C}_1$, $\mathscr{C}_2$, a functor $\mathcal{F}_1$ from $\mathscr{A}_1$ to $\mathscr{B}_1$, a functor $\mathcal{F}_2$ from $\mathscr{A}_2$ to $\mathscr{B}_2$, a functor $\mathcal{G}_1$ from $\mathscr{B}_1$ to $\mathscr{C}_1$, and a functor $\mathcal{G}_2$ from $\mathscr{B}_2$ to $\mathscr{C}_2$. Suppose $\mathcal{F}_1$ is covariant and $\mathcal{G}_1$ is covariant and $\mathcal{F}_2$ is covariant and $\mathcal{G}_2$ is covariant. Then $(\mathcal{G}_1 \times \mathcal{G}_2) \circ (\mathcal{F}_1 \times \mathcal{F}_2) = (\mathcal{G}_1 \circ \mathcal{F}_1) \times (\mathcal{G}_2 \circ \mathcal{F}_2)$.

(51) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$. Then $\mathrm{id}_{\mathscr{C}_1} \times \mathrm{id}_{\mathscr{C}_2} = \mathrm{id}_{\mathscr{C}_1 \times \mathscr{C}_2}$.

Let x, y be objects. We introduce the notation KuratowskiPair(x, y) as a synonym of $\langle x, y \rangle$.

Let $\mathscr{C}_1$, $\mathscr{C}_2$ be categories, f_1 be a morphism of $\mathscr{C}_1$, and f_2 be a morphism of $\mathscr{C}_2$. The functor $\langle f_1, f_2 \rangle$ yielding a morphism of $\mathscr{C}_1 \times \mathscr{C}_2$ is defined by

(Def. 23) (i) $\pi_1(\mathscr{C}_1 \boxtimes \mathscr{C}_2)(it) = f_1$ and $\pi_2(\mathscr{C}_1 \boxtimes \mathscr{C}_2)(it) = f_2$, **if** $\mathscr{C}_1$ is not empty and $\mathscr{C}_2$ is not empty,

(ii) $it =$ the morphism of $\mathscr{C}_1 \times \mathscr{C}_2$, **otherwise**.

Now we state the propositions:

(52) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$, and a morphism f of $\mathscr{C}_1 \times \mathscr{C}_2$. Then there exists a morphism f_1 of $\mathscr{C}_1$ and there exists a morphism f_2 of $\mathscr{C}_2$ such that $f = \langle f_1, f_2 \rangle$.

(53) Let us consider non empty categories $\mathscr{C}_1$, $\mathscr{C}_2$, morphisms f_1, g_1 of $\mathscr{C}_1$, and morphisms f_2, g_2 of $\mathscr{C}_2$. Suppose $\langle f_1, f_2 \rangle = \langle g_1, g_2 \rangle$. Then

(i) $f_1 = g_1$, and

(ii) $f_2 = g_2$.

Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$, morphisms f_1, g_1 of $\mathscr{C}_1$, and morphisms f_2, g_2 of $\mathscr{C}_2$. Now we state the propositions:

(54) $\langle f_1, f_2 \rangle \rhd \langle g_1, g_2 \rangle$ if and only if $f_1 \rhd g_1$ and $f_2 \rhd g_2$.

(55) Suppose $f_1 \rhd g_1$ and $f_2 \rhd g_2$. Then $\langle f_1, f_2 \rangle \circ \langle g_1, g_2 \rangle = \langle f_1 \circ g_1, f_2 \circ g_2 \rangle$. The theorem is a consequence of (54) and (13).

Now we state the propositions:

(56) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$, a morphism f_1 of $\mathscr{C}_1$, a morphism f_2 of $\mathscr{C}_2$, and a morphism f of $\mathscr{C}_1 \times \mathscr{C}_2$. Suppose $f = \langle f_1, f_2 \rangle$ and $\mathscr{C}_1$ is not empty and $\mathscr{C}_2$ is not empty. Then f is identity if and only if f_1 is identity and f_2 is identity. The theorem is a consequence of (52), (54), (55), and (4).

(57) Let us consider non empty categories $\mathscr{C}_1$, $\mathscr{C}_2$, categories $\mathscr{D}_1$, $\mathscr{D}_2$, a functor $\mathcal{F}_1$ from $\mathscr{C}_1$ to $\mathscr{D}_1$, a functor $\mathcal{F}_2$ from $\mathscr{C}_2$ to $\mathscr{D}_2$, a morphism c_1 of $\mathscr{C}_1$, and a morphism c_2 of $\mathscr{C}_2$. Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant. Then $(\mathcal{F}_1 \times \mathcal{F}_2)(\langle c_1, c_2 \rangle) = \langle \mathcal{F}_1(c_1), \mathcal{F}_2(c_2) \rangle$.

5. Natural Transformations

Let $\mathscr{C}_1$, $\mathscr{C}_2$ be categories, $\mathcal{F}_1$, $\mathcal{F}_2$ be functors from $\mathscr{C}_1$ to $\mathscr{C}_2$, and τ be a functor from $\mathscr{C}_1$ to $\mathscr{C}_2$. We say that τ is a natural transformation of $\mathcal{F}_1$ and $\mathcal{F}_2$ if and only if

(Def. 24) for every morphisms f_1, f_2 of $\mathscr{C}_1$ such that $f_1 \triangleright f_2$ holds $\tau(f_1) \triangleright \mathcal{F}_1(f_2)$ and $\mathcal{F}_2(f_1) \triangleright \tau(f_2)$ and $\tau(f_1 \circ f_2) = \tau(f_1) \circ \mathcal{F}_1(f_2)$ and $\tau(f_1 \circ f_2) = \mathcal{F}_2(f_1) \circ \tau(f_2)$.

Now we state the propositions:

(58) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$, functors $\mathcal{F}_1$, $\mathcal{F}_2$ from $\mathscr{C}_1$ to $\mathscr{C}_2$, and a functor τ from $\mathscr{C}_1$ to $\mathscr{C}_2$. Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant. Then τ is a natural transformation of $\mathcal{F}_1$ and $\mathcal{F}_2$ if and only if for every morphisms f, f_1, f_2 of $\mathscr{C}_1$ such that f_1 is identity and f_2 is identity and $f_1 \triangleright f$ and $f \triangleright f_2$ holds $\tau(f_1) \triangleright \mathcal{F}_1(f)$ and $\mathcal{F}_2(f) \triangleright \tau(f_2)$ and $\tau(f) = \tau(f_1) \circ \mathcal{F}_1(f)$ and $\tau(f) = \mathcal{F}_2(f) \circ \tau(f_2)$.

Proof: For every morphisms g_1, g_2 of $\mathscr{C}_1$ such that $g_1 \triangleright g_2$ holds $\tau(g_1) \triangleright \mathcal{F}_1(g_2)$ and $\mathcal{F}_2(g_1) \triangleright \tau(g_2)$ and $\tau(g_1 \circ g_2) = \tau(g_1) \circ \mathcal{F}_1(g_2)$ and $\tau(g_1 \circ g_2) = \mathcal{F}_2(g_1) \circ \tau(g_2)$ by [15, (1)], (5), (3), (13). $\square$

(59) Let us consider non empty categories $\mathscr{C}_1$, $\mathscr{C}_2$, covariant functors $\mathcal{F}_1$, $\mathcal{F}_2$ from $\mathscr{C}_1$ to $\mathscr{C}_2$, and a function τ from $\mathrm{Ob}\,\mathscr{C}_1$ into $\mathrm{Mor}\,\mathscr{C}_2$. Then there exists a functor τ_1 from $\mathscr{C}_1$ to $\mathscr{C}_2$ such that $\tau = \tau_1 \restriction \mathrm{Ob}\,\mathscr{C}_1$ and τ_1 is a natural transformation of $\mathcal{F}_1$ and $\mathcal{F}_2$ if and only if for every object a of $\mathscr{C}_1$, $\tau(a) \in \hom(\mathcal{F}_1(a), \mathcal{F}_2(a))$ and for every objects a_1, a_2 of $\mathscr{C}_1$ and for every morphism f from a_1 to a_2 such that $\hom(a_1, a_2) \neq \emptyset$ holds $\tau(a_2) \circ \mathcal{F}_1(f) = \mathcal{F}_2(f) \circ \tau(a_1)$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every morphism f of $\mathscr{C}_1$ such that $\$_1 = f$ holds $\$_2 = \tau(\operatorname{cod} f) \circ \mathcal{F}_1(f)$. For every object x such that $x \in$ the carrier of $\mathscr{C}_1$ there exists an object y such that $y \in$ the carrier of $\mathscr{C}_2$ and $\mathcal{P}[x, y]$. Consider τ_1 being a function from the carrier of $\mathscr{C}_1$ into the carrier of $\mathscr{C}_2$ such that for every object x such that $x \in$ the carrier of $\mathscr{C}_1$ holds $\mathcal{P}[x, \tau_1(x)]$ from [7, Sch. 1]. For every object x such that $x \in \operatorname{dom} \tau$ holds $\tau(x) = (\tau_1 \restriction \mathrm{Ob}\,\mathscr{C}_1)(x)$ by [15, (22)], [16, (20)], [15, (32)], [16, (5), (6)]. For every morphisms f, f_1, f_2 of $\mathscr{C}_1$ such that f_1 is identity and f_2 is identity and $f_1 \triangleright f$ and $f \triangleright f_2$ holds $\tau_1(f_1) \triangleright \mathcal{F}_1(f)$ and $\mathcal{F}_2(f) \triangleright \tau_1(f_2)$ and $\tau_1(f) = \tau_1(f_1) \circ \mathcal{F}_1(f)$ and $\tau_1(f) = \mathcal{F}_2(f) \circ \tau_1(f_2)$ by [15, (22)], [16, (20), (6)], [15, (32)]. $\square$

(60) Let us consider object-categories $\mathscr{C}$, $\mathscr{D}$, functors $\mathcal{F}_1$, $\mathcal{F}_2$ from $\mathscr{C}$ to $\mathscr{D}$, and functors $\mathcal{G}_1$, $\mathcal{G}_2$, τ from alter $\mathscr{C}$ to alter $\mathscr{D}$. Suppose $\mathcal{F}_1 = \mathcal{G}_1$ and $\mathcal{F}_2 = \mathcal{G}_2$ and τ is a natural transformation of $\mathcal{G}_1$ and $\mathcal{G}_2$. Then $(\mathrm{IdMap}\,\mathscr{C}) \cdot \tau$ is a natural transformation from $\mathcal{F}_1$ to $\mathcal{F}_2$.

Proof: For every object a of $\mathscr{C}$, $\tau(\mathrm{id}_a) \in \hom(\mathcal{F}_1(a), \mathcal{F}_2(a))$ by [15, (41), (24), (42)]. Reconsider $\tau_1 = \tau$ as a function from the carrier' of $\mathscr{C}$ into the carrier' of $\mathscr{D}$. There exists a transformation t from $\mathcal{F}_1$ to $\mathcal{F}_2$ such that $t = (\mathrm{IdMap}\,\mathscr{C}) \cdot \tau_1$ and for every objects a, b of $\mathscr{C}$ such that $\hom(a, b) \neq \emptyset$ for every morphism f from a to b, $t(b) \cdot \mathcal{F}_{1f} = \mathcal{F}_{2f} \cdot t(a)$ by [6, (13)], [5, (1), (15), (21)]. Consider t being a transformation from $\mathcal{F}_1$ to $\mathcal{F}_2$ such that $t = (\mathrm{IdMap}\,\mathscr{C}) \cdot \tau_1$ and for every objects a, b of $\mathscr{C}$ such that $\hom(a, b) \neq \emptyset$ for every morphism f from a to b, $t(b) \cdot \mathcal{F}_{1f} = \mathcal{F}_{2f} \cdot t(a)$. □

Let $\mathscr{C}$, $\mathscr{D}$ be categories and $\mathcal{F}_1$, $\mathcal{F}_2$ be functors from $\mathscr{C}$ to $\mathscr{D}$. We say that $\mathcal{F}_1$ is naturally transformable to $\mathcal{F}_2$ if and only if

(Def. 25) there exists a functor τ from $\mathscr{C}$ to $\mathscr{D}$ such that τ is a natural transformation of $\mathcal{F}_1$ and $\mathcal{F}_2$.

Assume $\mathcal{F}_1$ is naturally transformable to $\mathcal{F}_2$.

A natural transformation from $\mathcal{F}_1$ to $\mathcal{F}_2$ is a functor from $\mathscr{C}$ to $\mathscr{D}$ and is defined by

(Def. 26) *it* is a natural transformation of $\mathcal{F}_1$ and $\mathcal{F}_2$.

Now we state the proposition:

(61) Let us consider categories $\mathscr{C}$, $\mathscr{D}$, and a functor $\mathcal{F}$ from $\mathscr{C}$ to $\mathscr{D}$. Suppose $\mathcal{F}$ is covariant. Then $\mathcal{F}$ is a natural transformation of $\mathcal{F}$ and $\mathcal{F}$. The theorem is a consequence of (58).

Let $\mathscr{C}$, $\mathscr{D}$ be categories and $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$ be functors from $\mathscr{C}$ to $\mathscr{D}$. Assume $\mathcal{F}_1$ is naturally transformable to $\mathcal{F}$ and $\mathcal{F}$ is naturally transformable to $\mathcal{F}_2$ and $\mathcal{F}$ is covariant and $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant. Let τ_1 be a natural transformation from $\mathcal{F}_1$ to $\mathcal{F}$ and τ_2 be a natural transformation from $\mathcal{F}$ to $\mathcal{F}_2$. The functor $\tau_2 \circ \tau_1$ yielding a natural transformation from $\mathcal{F}_1$ to $\mathcal{F}_2$ is defined by

(Def. 27) for every morphisms f, f_1, f_2 of $\mathscr{C}$ such that f_1 is identity and f_2 is identity and $f \triangleright f_1$ and $f_2 \triangleright f$ holds $it(f) = (\tau_2(f_2) \circ \mathcal{F}(f)) \circ \tau_1(f_1)$.

Now we state the proposition:

(62) Let us consider categories $\mathscr{C}$, $\mathscr{D}$, and functors $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$ from $\mathscr{C}$ to $\mathscr{D}$. Suppose $\mathcal{F}_1$ is naturally transformable to $\mathcal{F}$ and $\mathcal{F}$ is naturally transformable to $\mathcal{F}_2$ and covariant and $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant. Then $\mathcal{F}_1$ is naturally transformable to $\mathcal{F}_2$.

Let $\mathscr{C}_1$, $\mathscr{C}_2$ be categories. The functor $\mathrm{Functors}(\mathscr{C}_2, \mathscr{C}_1)$ yielding a strict category is defined by

(Def. 28) the carrier of $it = \{\langle\langle\mathcal{F}_1, \mathcal{F}_2\rangle, \tau\rangle$, where $\mathcal{F}_1, \mathcal{F}_2$ are functors from $\mathscr{C}_1$ to $\mathscr{C}_2, \tau$ is a natural transformation from $\mathcal{F}_1$ to $\mathcal{F}_2$: $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant and $\mathcal{F}_1$ is naturally transformable to $\mathcal{F}_2\}$ and the composition of $it = \{\langle\langle x_2, x_1\rangle, x_3\rangle$, where x_1, x_2, x_3 are elements of the carrier of it : there exist functors $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ from $\mathscr{C}_1$ to $\mathscr{C}_2$ and there exists a natural transformation τ_1 from $\mathcal{F}_1$ to $\mathcal{F}_2$ and there exists a natural transformati-on τ_2 from $\mathcal{F}_2$ to $\mathcal{F}_3$ such that $x_1 = \langle\langle \mathcal{F}_1, \mathcal{F}_2\rangle, \tau_1\rangle$ and $x_2 = \langle\langle \mathcal{F}_2, \mathcal{F}_3\rangle, \tau_2\rangle$ and $x_3 = \langle\langle \mathcal{F}_1, \mathcal{F}_3\rangle, \tau_2 \circ \tau_1\rangle\}$.

Let $\mathscr{C}_1$ be a non empty category and $\mathscr{C}_2$ be an empty category. One can verify that Functors$(\mathscr{C}_2, \mathscr{C}_1)$ is empty.

Let $\mathscr{C}_1$ be an empty category and $\mathscr{C}_2$ be a category. Let us observe that Functors$(\mathscr{C}_2, \mathscr{C}_1)$ is non empty and trivial.

Let $\mathscr{C}_1$ be a non empty category and $\mathscr{C}_2$ be a non empty category. Let us note that Functors$(\mathscr{C}_2, \mathscr{C}_1)$ is non empty.

Now we state the proposition:

(63) Let us consider non empty categories $\mathscr{C}_1$, $\mathscr{C}_2$, and morphisms f_1, f_2 of Functors$(\mathscr{C}_2, \mathscr{C}_1)$. Then $f_1 \triangleright f_2$ if and only if there exist covariant functors $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$ from $\mathscr{C}_1$ to $\mathscr{C}_2$ and there exists a natural transformation τ_1 from $\mathcal{F}_1$ to $\mathcal{F}$ and there exists a natural transformation τ_2 from $\mathcal{F}$ to $\mathcal{F}_2$ such that $f_1 = \langle\langle \mathcal{F}, \mathcal{F}_2\rangle, \tau_2\rangle$ and $f_2 = \langle\langle \mathcal{F}_1, \mathcal{F}\rangle, \tau_1\rangle$ and $f_1 \circ f_2 = \langle\langle \mathcal{F}_1, \mathcal{F}_2\rangle, \tau_2 \circ \tau_1\rangle$ and for every morphisms g_1, g_2 of $\mathscr{C}_1$ such that $g_2 \triangleright g_1$ holds $\tau_2(g_2) \triangleright \tau_1(g_1)$ and $(\tau_2 \circ \tau_1)(g_2 \circ g_1) = \tau_2(g_2) \circ \tau_1(g_1)$.

Proof: If $f_1 \triangleright f_2$, then there exist covariant functors $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$ from $\mathscr{C}_1$ to $\mathscr{C}_2$ and there exists a natural transformation τ_1 from $\mathcal{F}_1$ to $\mathcal{F}$ and there exists a natural transformation τ_2 from $\mathcal{F}$ to $\mathcal{F}_2$ such that $f_1 = \langle\langle \mathcal{F}, \mathcal{F}_2\rangle, \tau_2\rangle$ and $f_2 = \langle\langle \mathcal{F}_1, \mathcal{F}\rangle, \tau_1\rangle$ and $f_1 \circ f_2 = \langle\langle \mathcal{F}_1, \mathcal{F}_2\rangle, \tau_2 \circ \tau_1\rangle$ and for every morphisms g_1, g_2 of $\mathscr{C}_1$ such that $g_2 \triangleright g_1$ holds $\tau_2(g_2) \triangleright \tau_1(g_1)$ and $(\tau_2 \circ \tau_1)(g_2 \circ g_1) = \tau_2(g_2) \circ \tau_1(g_1)$ by [6, (1)], (5), (58), [16, (5)]. □

Let us consider non empty categories $\mathscr{C}_1, \mathscr{C}_2$ and a morphism f of Functors$(\mathscr{C}_2, \mathscr{C}_1)$. Now we state the propositions:

(64) f is identity if and only if there exists a covariant functor $\mathcal{F}$ from $\mathscr{C}_1$ to $\mathscr{C}_2$ such that $f = \langle\langle \mathcal{F}, \mathcal{F}\rangle, \mathcal{F}\rangle$.

Proof: Set $\mathscr{C} =$ Functors$(\mathscr{C}_2, \mathscr{C}_1)$. If f is identity, then there exists a covariant functor $\mathcal{F}$ from $\mathscr{C}_1$ to $\mathscr{C}_2$ such that $f = \langle\langle \mathcal{F}, \mathcal{F}\rangle, \mathcal{F}\rangle$ by [15, (24)], (63), (61), (5). Consider $\mathcal{F}$ being a covariant functor from $\mathscr{C}_1$ to $\mathscr{C}_2$ such that $f = \langle\langle \mathcal{F}, \mathcal{F}\rangle, \mathcal{F}\rangle$. For every morphism f_1 of $\mathscr{C}$ such that $f \triangleright f_1$ holds $f \circ f_1 = f_1$ by (63), (5), (4), [7, (12)]. □

(65) There exist covariant functors $\mathcal{F}_1$, $\mathcal{F}_2$ from $\mathscr{C}_1$ to $\mathscr{C}_2$ and there exists a natural transformation τ from $\mathcal{F}_1$ to $\mathcal{F}_2$ such that $f = \langle\langle \mathcal{F}_1, \mathcal{F}_2\rangle, \tau\rangle$ and $\operatorname{dom} f = \langle\langle \mathcal{F}_1, \mathcal{F}_1\rangle, \mathcal{F}_1\rangle$ and $\operatorname{cod} f = \langle\langle \mathcal{F}_2, \mathcal{F}_2\rangle, \mathcal{F}_2\rangle$. The theorem is a consequence of (63) and (64).

6. Exponential Objects

Let $\mathscr{C}$ be a category with binary products, a, b, c be objects of $\mathscr{C}$, and e be a morphism from $c \times a$ to b. Assume $\hom(c \times a, b) \neq \emptyset$. We say that $\langle c, e\rangle$ is an exponent of a and b if and only if

(Def. 29) for every object d of $\mathscr{C}$ and for every morphism f from $d \times a$ to b such that $\hom(d \times a, b) \neq \emptyset$ holds $\hom(d, c) \neq \emptyset$ and there exists a morphism h from d to c such that $f = e \cdot (h \times \text{id-}a)$ and for every morphism h_1 from d to c such that $f = e \cdot (h_1 \times \text{id-}a)$ holds $h = h_1$.

Now we state the propositions:

(66) Let us consider a category $\mathscr{C}$ with binary products, objects a_1, a_2, b_1, b_2, c_1, c_2 of $\mathscr{C}$, a morphism f_1 from a_1 to b_1, a morphism f_2 from a_2 to b_2, a morphism g_1 from b_1 to c_1, and a morphism g_2 from b_2 to c_2. Suppose $\hom(a_1, b_1) \neq \emptyset$ and $\hom(b_1, c_1) \neq \emptyset$ and $\hom(a_2, b_2) \neq \emptyset$ and $\hom(b_2, c_2) \neq \emptyset$. Then $(g_1 \times g_2) \cdot (f_1 \times f_2) = g_1 \cdot f_1 \times (g_2 \cdot f_2)$. The theorem is a consequence of (42) and (44).

(67) Let us consider a category $\mathscr{C}$ with binary products, and objects a, b of $\mathscr{C}$. Then $\text{id-}a \times \text{id-}b = \text{id-}(a \times b)$. The theorem is a consequence of (42).

(68) Let us consider a category $\mathscr{C}$ with binary products, objects a, b, c_1, c_2 of $\mathscr{C}$, a morphism e_1 from $c_1 \times a$ to b, and a morphism e_2 from $c_2 \times a$ to b. Suppose $\hom(c_1 \times a, b) \neq \emptyset$ and $\hom(c_2 \times a, b) \neq \emptyset$ and $\langle c_1, e_1\rangle$ is an exponent of a and b and $\langle c_2, e_2\rangle$ is an exponent of a and b. Then c_1 and c_2 are isomorphic.

Proof: There exists a morphism f from c_1 to c_2 such that f is isomorphism by (44), [16, (23)], (66), [16, (18)]. □

Let $\mathscr{C}$ be a category with binary products. We say that $\mathscr{C}$ has exponential objects if and only if

(Def. 30) for every objects a, b of $\mathscr{C}$, there exists an object c of $\mathscr{C}$ and there exists a morphism e from $c \times a$ to b such that $\hom(c \times a, b) \neq \emptyset$ and $\langle c, e\rangle$ is an exponent of a and b.

One can check that $\mathbf{1}$ has binary products.

Now we state the proposition:

(69) $\mathbf{1}$ has exponential objects.

Proof: Set $\mathscr{C} = \mathbf{1}$. Consider f being a morphism of $\mathbf{1}$ such that f is identity and $\text{Ob}\,\mathbf{1} = \{f\}$ and $\text{Mor}\,\mathbf{1} = \{f\}$. For every objects o_1, o_2 of $\mathscr{C}$, every morphism of $\mathscr{C}$ is a morphism from o_1 to o_2 by [16, (20)]. For every objects a, b of $\mathscr{C}$, there exists an object c of $\mathscr{C}$ and there exists a morphism e from $c \times a$ to b such that $\hom(c \times a, b) \neq \emptyset$ and $\langle c, e\rangle$ is an exponent of a and b. □

Let us observe that there exists a category with binary products which has exponential objects.

Let $\mathscr{C}$ be a category with exponential objects binary products and a, b be objects of $\mathscr{C}$.

A categorical exponent of a and b is a pair object and is defined by

(Def. 31) there exists an object c of $\mathscr{C}$ and there exists a morphism e from $c \times a$ to b such that $it = \langle c, e \rangle$ and $\hom(c \times a, b) \neq \emptyset$ and $\langle c, e \rangle$ is an exponent of a and b.

The functor b^a yielding an object of $\mathscr{C}$ is defined by the term

(Def. 32) (the categorical exponent of a and $b)_{\mathbf{1}}$.

The functor $\mathrm{eval}(a, b)$ yielding a morphism from $b^a \times a$ to b is defined by the term

(Def. 33) (the categorical exponent of a and $b)_{\mathbf{2}}$.

Now we state the propositions:

(70) Let us consider a category $\mathscr{C}$ with exponential objects binary products, and objects a, b of $\mathscr{C}$. Then

(i) $\hom(b^a \times a, b) \neq \emptyset$, and

(ii) $\langle b^a, \mathrm{eval}(a, b) \rangle$ is an exponent of a and b.

(71) Let us consider a category $\mathscr{C}$ with exponential objects binary products, and objects a, b, c of $\mathscr{C}$. Suppose $\hom(c \times a, b) \neq \emptyset$. Then there exists a function L from $\hom(c \times a, b)$ into $\hom(c, b^a)$ such that

(i) for every morphism f from $c \times a$ to b and for every morphism h from c to b^a such that $h = L(f)$ holds $\mathrm{eval}(a, b) \cdot (h \times \text{id-}a) = f$, and

(ii) L is bijective.

Proof: $\hom(b^a \times a, b) \neq \emptyset$ and $\langle b^a, \mathrm{eval}(a, b) \rangle$ is an exponent of a and b. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every morphism f from $c \times a$ to b such that $f = \$_1$ there exists a morphism h from c to b^a such that $h = \$_2$ and $f = \mathrm{eval}(a, b) \cdot (h \times \text{id-}a)$ and for every morphism h_1 from c to b^a such that $f = \mathrm{eval}(a, b) \cdot (h_1 \times \text{id-}a)$ holds $h = h_1$. For every object x such that $x \in \hom(c \times a, b)$ there exists an object y such that $y \in \hom(c, b^a)$ and $\mathcal{P}[x, y]$. Consider L being a function from $\hom(c \times a, b)$ into $\hom(c, b^a)$ such that for every object x such that $x \in \hom(c \times a, b)$ holds $\mathcal{P}[x, L(x)]$ from [7, Sch. 1]. There exists an object y such that $y \in \hom(c, b^a)$. For every morphism f from $c \times a$ to b and for every morphism h from c to b^a such that $h = L(f)$ holds $\mathrm{eval}(a, b) \cdot (h \times \text{id-}a) = f$. For every objects x_1, x_2 such that x_1, $x_2 \in \hom(c \times a, b)$ and $L(x_1) = L(x_2)$ holds $x_1 = x_2$. For every object y such that $y \in \hom(c, b^a)$ holds $y \in \operatorname{rng} L$ by [6, (3)]. □

Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ be categories and $\mathcal{E}$ be a functor from $\mathscr{C} \times \mathscr{A}$ to $\mathscr{B}$. Assume $\mathcal{E}$ is covariant. We say that $\langle \mathscr{C}, \mathcal{E} \rangle$ is an exponent of $\mathscr{A}$ and $\mathscr{B}$ if and only if

(Def. 34) for every category $\mathscr{D}$ and for every functor $\mathcal{F}$ from $\mathscr{D} \times \mathscr{A}$ to $\mathscr{B}$ such that $\mathcal{F}$ is covariant there exists a functor $\mathcal{H}$ from $\mathscr{D}$ to $\mathscr{C}$ such that $\mathcal{H}$ is covariant and $\mathcal{F} = \mathcal{E} \circ (\mathcal{H} \times \mathrm{id}_{\mathscr{A}})$ and for every functor $\mathcal{H}_1$ from $\mathscr{D}$ to $\mathscr{C}$ such that $\mathcal{H}_1$ is covariant and $\mathcal{F} = \mathcal{E} \circ (\mathcal{H}_1 \times \mathrm{id}_{\mathscr{A}})$ holds $\mathcal{H} = \mathcal{H}_1$.

Let $\mathscr{C}_1$, $\mathscr{C}_2$ be categories.

A categorical exponent of $\mathscr{C}_1$ and $\mathscr{C}_2$ is a pair object and is defined by

(Def. 35) there exists a category $\mathscr{C}$ and there exists a functor $\mathcal{E}$ from $\mathscr{C} \times \mathscr{C}_1$ to $\mathscr{C}_2$ such that $it = \langle \mathscr{C}, \mathcal{E} \rangle$ and $\mathcal{E}$ is covariant and $\langle \mathscr{C}, \mathcal{E} \rangle$ is an exponent of $\mathscr{C}_1$ and $\mathscr{C}_2$.

The functor ${\mathscr{C}_2}^{\mathscr{C}_1}$ yielding a category is defined by the term

(Def. 36) (the categorical exponent of $\mathscr{C}_1$ and $\mathscr{C}_2)_{\mathbf{1}}$.

The functor $\mathrm{eval}(\mathscr{C}_1, \mathscr{C}_2)$ yielding a functor from ${\mathscr{C}_2}^{\mathscr{C}_1} \times \mathscr{C}_1$ to $\mathscr{C}_2$ is defined by the term

(Def. 37) (the categorical exponent of $\mathscr{C}_1$ and $\mathscr{C}_2)_{\mathbf{2}}$.

Now we state the propositions:

(72) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$. Then $\langle {\mathscr{C}_2}^{\mathscr{C}_1}, \mathrm{eval}(\mathscr{C}_1, \mathscr{C}_2) \rangle$ is an exponent of $\mathscr{C}_1$ and $\mathscr{C}_2$.

(73) Let us consider categories $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}_1$, $\mathscr{C}_2$, a functor $\mathcal{E}_1$ from $\mathscr{C}_1 \times \mathscr{A}$ to $\mathscr{B}$, and a functor $\mathcal{E}_2$ from $\mathscr{C}_2 \times \mathscr{A}$ to $\mathscr{B}$. Suppose $\mathcal{E}_1$ is covariant and $\mathcal{E}_2$ is covariant and $\langle \mathscr{C}_1, \mathcal{E}_1 \rangle$ is an exponent of $\mathscr{A}$ and $\mathscr{B}$ and $\langle \mathscr{C}_2, \mathcal{E}_2 \rangle$ is an exponent of $\mathscr{A}$ and $\mathscr{B}$. Then $\mathscr{C}_1 \cong \mathscr{C}_2$.

PROOF: There exists a functor $\mathcal{F}$ from $\mathscr{C}_1$ to $\mathscr{C}_2$ and there exists a functor $\mathcal{G}$ from $\mathscr{C}_2$ to $\mathscr{C}_1$ such that $\mathcal{F}$ is covariant and $\mathcal{G}$ is covariant and $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathscr{C}_1}$ and $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathscr{C}_2}$ by [16, (10)], (50), [16, (11)], [15, (35)]. □

Let $\mathscr{C}_1$, $\mathscr{C}_2$ be categories. Observe that $\mathrm{eval}(\mathscr{C}_1, \mathscr{C}_2)$ is covariant.

Let $\mathscr{C}_1$ be a non empty category and $\mathscr{C}_2$ be an empty category. Let us note that ${\mathscr{C}_2}^{\mathscr{C}_1}$ is empty.

Let $\mathscr{C}_1$ be an empty category and $\mathscr{C}_2$ be a category. Let us observe that ${\mathscr{C}_2}^{\mathscr{C}_1}$ is non empty and trivial.

Let $\mathscr{C}_1$ be a non empty category and $\mathscr{C}_2$ be a non empty category. One can verify that ${\mathscr{C}_2}^{\mathscr{C}_1}$ is non empty.

Now we state the proposition:

(74) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$. Then $\mathrm{Functors}(\mathscr{C}_2, \mathscr{C}_1) \cong {\mathscr{C}_2}^{\mathscr{C}_1}$. The theorem is a consequence of (28), (72), and (73).

References

[1] Jiri Adamek, Horst Herrlich, and George E. Strecker. *Abstract and Concrete Categories: The Joy of Cats*. Dover Publication, New York, 2009.

[2] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[4] Francis Borceaux. *Handbook of Categorical Algebra I. Basic Category Theory*, volume 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994.

[5] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1 (**2**):409–420, 1990.

[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[9] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[10] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[11] Krzysztof Hryniewiecki. Graphs. *Formalized Mathematics*, 2(**3**):365–370, 1991.

[12] F. William Lawvere. Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories. *Reprints in Theory and Applications of Categories*, 5:1–121, 2004.

[13] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer Verlag, New York, Heidelberg, Berlin, 1971.

[14] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[15] Marco Riccardi. Object-free definition of categories. *Formalized Mathematics*, 21(**3**): 193–205, 2013. doi:10.2478/forma-2013-0021.

[16] Marco Riccardi. Categorical pullbacks. *Formalized Mathematics*, 23(**1**):1–14, 2015. doi:10.2478/forma-2015-0001.

[17] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[18] Andrzej Trybulec. Isomorphisms of categories. *Formalized Mathematics*, 2(**5**):629–634, 1991.

[19] Andrzej Trybulec. Natural transformations. Discrete categories. *Formalized Mathematics*, 2(**4**):467–474, 1991.

[20] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[21] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.

[22] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received August 15, 2015