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Polynomially Bounded Sequences and Polynomial Sequences

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Summary. In this article, we formalize polynomially bounded sequences that plays an important role in computational complexity theory. Class P is a fundamental computational complexity class that contains all polynomial-time decision problems [11], [12]. It takes polynomially bounded amount of computation time to solve polynomial-time decision problems by the deterministic Turing machine. Moreover we formalize polynomial sequences [5].

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The notation and terminology used in this paper have been introduced in the following articles: [26], [18], [16], [17], [6], [22], [10], [7], [8], [24], [14], [1], [2], [3], [13], [20], [27], [28], [21], [25], and [9].

1. Preliminaries

Now we state the proposition:

(1) Let us consider natural numbers m, k. If $1 \leqslant m$, then $1 \leqslant m^k$.

Let us consider natural numbers m, n. Now we state the propositions:

(2) $m \leqslant m^{n+1}$.

(3) If $2 \leqslant m$, then $n + 1 \leqslant m^n$.

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(4) Let us consider a natural number k. Then $2 \cdot k \leqslant 2^k$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv 2 \cdot \$_1 \leqslant 2^{\$_1}$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [20, (25)], [24, (5)], [1, (14)], (2). For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. □

(5) Let us consider natural numbers k, n. If $k \leqslant n$, then $n + k \leqslant 2^n$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv \$_1 + k + k \leqslant 2^{\$_1 + k}$. $2 \cdot k \leqslant 2^k$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [20, (27), (25), (24)]. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. □

(6) Let us consider natural numbers k, m. If $2 \cdot k + 1 \leqslant m$, then $2^k \leqslant 2^m/_m$. The theorem is a consequence of (5).

(7) Let us consider real numbers a, b, c. If $1 < a$ and $0 < b \leqslant c$, then $\log_a b \leqslant \log_a c$.

Let us consider a natural number n and a real number a. Now we state the propositions:

(8) If $1 < a$, then $a^n < a^{n+1}$.

(9) If $1 \leqslant a$, then $a^n \leqslant a^{n+1}$.

(10) There exists a partial function g from $\mathbb{R}$ to $\mathbb{R}$ such that

 (i) $\mathrm{dom}\, g =]0, +\infty[$, and

 (ii) for every real number x such that $x \in]0, +\infty[$ holds $g(x) = \log_2 x$, and

 (iii) g is differentiable on $]0, +\infty[$, and

 (iv) for every real number x such that $x \in]0, +\infty[$ holds g is differentiable in x and $g'(x) = \log_2 e/_x$ and $0 < g'(x)$.

PROOF: Set $g = \log_2 e \cdot$ (the function ln). For every real number d such that $d \in]0, +\infty[$ holds $g(d) = \log_2 d$ by [20, (56)]. For every real number x such that $x \in]0, +\infty[$ holds g is differentiable in x and $g'(x) = \log_2 e/_x$ and $0 < g'(x)$ by [23, (18)], [22, (15)], [20, (57)], [23, (11)]. □

(11) There exists a partial function f from $\mathbb{R}$ to $\mathbb{R}$ such that

 (i) $]e, +\infty[= \mathrm{dom}\, f$, and

 (ii) for every real number x such that $x \in \mathrm{dom}\, f$ holds $f(x) = x/_{\log_2 x}$, and

 (iii) f is differentiable on $]e, +\infty[$, and

 (iv) for every real number x_0 such that $x_0 \in]e, +\infty[$ holds $0 \leqslant f'(x_0)$, and

 (v) f is non-decreasing.

PROOF: Consider g being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} g =]0, +\infty[$ and for every real number x such that $x \in]0, +\infty[$ holds $g(x) = \log_2 x$ and g is differentiable on $]0, +\infty[$ and for every real number x such that $x \in]0, +\infty[$ holds g is differentiable in x and $g'(x) = \log_2 e/_x$ and $0 < g'(x)$. Set $g_0 = g\restriction]e, +\infty[$. For every object x such that $x \in]e, +\infty[$ holds $x \in]0, +\infty[$ by [23, (11)]. Set $f = \mathrm{id}_{\Omega_\mathbb{R}}/g_0$. ${g_0}^{-1}(\{0\}) = \emptyset$ by [23, (11)], [7, (49)], [4, (10)], [20, (52)]. For every real number x such that $x \in \operatorname{dom} f$ holds $f(x) = x/_{\log_2 x}$ by [7, (49)]. For every real number x such that $x \in]e, +\infty[$ holds f is differentiable in x and $f'(x) = \log_2 x - \log_2 e/_{(\log_2 x)^2}$ by [23, (11)], [7, (49)], [4, (10)], [20, (52)]. For every real number x such that $x \in]e, +\infty[$ holds $0 \leqslant f'(x)$ by [20, (57)], [23, (11)]. □

(12) Let us consider real numbers x, y. If $e < x \leqslant y$, then $x/_{\log_2 x} \leqslant y/_{\log_2 y}$. The theorem is a consequence of (11).

(13) Let us consider a natural number k. Suppose $e < k$. Then there exists a natural number N such that for every natural number n such that $N \leqslant n$ holds $2^k \leqslant n/_{\log_2 n}$. The theorem is a consequence of (12) and (6).

Let us consider a natural number x. Let us assume that $1 < x$.

(14) There exists a natural number N such that for every natural number n such that $N \leqslant n$ holds $4 < n/_{\log_x n}$.

(15) There exist natural numbers N, c such that for every natural number n such that $N \leqslant n$ holds $n^x \leqslant c \cdot x^n$.

(16) Let us consider a natural number x. Suppose $1 < x$. Then there exist no natural numbers N, c such that for every natural number n such that $N \leqslant n$ holds $2^n \leqslant c \cdot n^x$.

PROOF: Consider N being a natural number such that there exists a natural number c such that for every natural number n such that $N \leqslant n$ holds $2^n \leqslant c \cdot n^x$. $N \neq 0$ by [20, (42), (24)]. Consider c being a natural number such that for every natural number n such that $N \leqslant n$ holds $2^n \leqslant c \cdot n^x$. There exists an element n of $\mathbb{N}$ such that $N \leqslant n$ and $0 < n - (x/_4)$ by [24, (6), (3)]. Consider n being an element of $\mathbb{N}$ such that $N \leqslant n$ and $0 < n - (x/_4)$. $0 < c$ by [20, (34)]. For every natural number k such that $1 \leqslant k$ holds $2^{k \cdot n} \leqslant c \cdot (k \cdot n)^x$. For every natural number k such that $1 \leqslant k$ holds $k \cdot n \leqslant \log_2 c + x \cdot \log_2 k + x \cdot \log_2 n$ by [20, (34)], (7), [20, (55), (52), (53)]. Consider Z being an element of $\mathbb{N}$ such that for every natural number k such that $Z \leqslant k$ holds $4 < k/_{\log_2 k}$. There exists a natural number k such that $Z \leqslant k$ and $\log_2 c + x \cdot \log_2 n/_{n-(x/_4)} \leqslant k$ by [24, (6), (3)]. There exists a natural number k such that $Z \leqslant k$ and $\log_2 c + x \cdot \log_2 n/_{n-(x/_4)} \leqslant k$ and $1 < k$ by [1, (11)]. Consider k being a natural number such that $Z \leqslant k$ and $1 < k$ and $\log_2 c + x \cdot \log_2 n/_{n-(x/_4)} \leqslant k$. □

(17) Let us consider natural numbers a, b. If $a \leqslant b$, then $\{n^a\}_{n\in\mathbb{N}} \in O(\{n^b\}_{n\in\mathbb{N}})$.

(18) Let us consider a natural number x. Suppose $1 < x$. Then there exist no natural numbers N, c such that for every natural number n such that $N \leqslant n$ holds $x^n \leqslant c \cdot n^x$.

PROOF: There exist natural numbers N, c such that for every natural number n such that $N \leqslant n$ holds $2^n \leqslant c \cdot n^x$ by [24, (7)]. □

(19) Let us consider a non negative real number a, and a natural number n. If $1 \leqslant n$, then $0 < \{n^a\}_{n\in\mathbb{N}}(n)$.

2. Polynomially Bounded Sequences

Let p be a sequence of real numbers. We say that p is polynomially bounded if and only if

(Def. 1) there exists a natural number k such that $p \in O(\{n^k\}_{n\in\mathbb{N}})$.

Now we state the propositions:

(20) Let us consider a sequence f of real numbers. Suppose f is not polynomially bounded. Let us consider a natural number k. Then $f \notin O(\{n^k\}_{n\in\mathbb{N}})$.

(21) Let us consider a sequence f of real numbers. Suppose for every natural number k, $f \notin O(\{n^k\}_{n\in\mathbb{N}})$. Then f is not polynomially bounded.

(22) Let us consider a positive real number a. Then $\{a^{1\cdot n+0}\}_{n\in\mathbb{N}}$ is positive.

Let us consider a real number a. Now we state the propositions:

(23) If $1 \leqslant a$, then $\{a^{1\cdot n+0}\}_{n\in\mathbb{N}}$ is non-decreasing. The theorem is a consequence of (9).

(24) If $1 < a$, then $\{a^{1\cdot n+0}\}_{n\in\mathbb{N}}$ is increasing. The theorem is a consequence of (8).

(25) Let us consider a natural number a. If $1 < a$, then $\{a^{1\cdot n+0}\}_{n\in\mathbb{N}}$ is not polynomially bounded.

PROOF: Consider k being a natural number such that $\{a^{1\cdot n+0}\}_{n\in\mathbb{N}} \in O(\{n^k\}_{n\in\mathbb{N}})$. Reconsider $f = \{n^k\}_{n\in\mathbb{N}}$ as an eventually positive sequence of real numbers. Reconsider $t = \{a^{1\cdot n+0}\}_{n\in\mathbb{N}}$ as an eventually non-negative sequence of real numbers. $t \in O(f)$ and for every element n of $\mathbb{N}$ such that $1 \leqslant n$ holds $0 < f(n)$. Consider c being a real number such that $c > 0$ and for every element n of $\mathbb{N}$ such that $n \geqslant 1$ holds $(\{a^{1\cdot n+0}\}_{n\in\mathbb{N}})(n) \leqslant c \cdot \{n^k\}_{n\in\mathbb{N}}(n)$. For every natural number n such that $n \geqslant 1$ holds $2^n \leqslant c \cdot n^k$ by [24, (7)]. There exist natural numbers N, b such that for every natural number n such that $N \leqslant n$ holds $2^n \leqslant b \cdot n^k$ by [24, (3)]. □

3. Polynomial Sequences

Now we state the proposition:

(26) Let us consider a finite 0-sequence x of $\mathbb{R}$, and a sequence y of real numbers. Then

(i) $x \cdot y$ is a finite transfinite sequence of elements of $\mathbb{R}$, and

(ii) $\mathrm{dom}(x \cdot y) = \mathrm{dom}\, x$, and

(iii) for every object i such that $i \in \mathrm{dom}\, x$ holds $(x \cdot y)(i) = x(i) \cdot y(i)$.

Let x be a finite 0-sequence of $\mathbb{R}$ and y be a sequence of real numbers. Observe that the functor $x \cdot y$ yields a finite 0-sequence of $\mathbb{R}$. Now we state the proposition:

(27) Let us consider a finite 0-sequence d of $\mathbb{R}$, and natural numbers x, i. Suppose $i \in \mathrm{dom}\, d$. Then $(d \cdot \{x^{1 \cdot n+0}\}_{n \in \mathbb{N}})(i) = d(i) \cdot x^i$. The theorem is a consequence of (26).

Let c be a finite 0-sequence of $\mathbb{R}$. The functor $\mathrm{Seq}_{\mathrm{poly}}(c)$ yielding a sequence of real numbers is defined by

(Def. 2) for every natural number x, $it(x) = \sum(c \cdot \{x^{1 \cdot n+0}\}_{n \in \mathbb{N}})$.

Let us consider a finite 0-sequence d of $\mathbb{R}$ and a natural number k. Now we state the propositions:

(28) Suppose $\mathrm{len}\, d = k+1$. Then there exists a real number a and there exists a finite 0-sequence d_1 of $\mathbb{R}$ and there exists a sequence y of real numbers such that $\mathrm{len}\, d_1 = k$ and $d_1 = d \restriction k$ and $a = d(k)$ and $d = d_1 \frown \langle a \rangle$ and $\mathrm{Seq}_{\mathrm{poly}}(d) = \mathrm{Seq}_{\mathrm{poly}}(d_1) + y$ and for every natural number i, $y(i) = a \cdot i^k$.
PROOF: Consider a being a real number, d_1 being a finite 0-sequence of $\mathbb{R}$ such that $\mathrm{len}\, d_1 = k$ and $d_1 = d \restriction k$ and $a = d(k)$ and $d = d_1 \frown \langle a \rangle$. Define $\mathcal{F}(\text{natural number}) = a \cdot \$_1^k$. Consider y being a sequence of real numbers such that for every natural number x, $y(x) = \mathcal{F}(x)$ from [15, Sch. 1]. For every element x of $\mathbb{N}$, $(\mathrm{Seq}_{\mathrm{poly}}(d))(x) = (\mathrm{Seq}_{\mathrm{poly}}(d_1) + y)(x)$ by (26), [1, (13), (44)], (27). □

(29) If $\mathrm{len}\, d = 1$, then there exists a real number a such that $a = d(0)$ and for every natural number x, $(\mathrm{Seq}_{\mathrm{poly}}(d))(x) = a$. The theorem is a consequence of (26).

(30) If $\mathrm{len}\, d = 1$ and d is non-negative yielding, then $\mathrm{Seq}_{\mathrm{poly}}(d) \in O(\{n^k\}_{n \in \mathbb{N}})$. The theorem is a consequence of (29).

(31) Let us consider a natural number k, a real number a, and a sequence y of real numbers. Suppose $0 \leqslant a$ and for every natural number i, $y(i) = a \cdot i^k$. Then $y \in O(\{n^k\}_{n \in \mathbb{N}})$.

(32) Let us consider natural numbers k, n. If $k \leqslant n$, then $O(\{n^k\}_{n\in\mathbb{N}}) \subseteq O(\{n^n\}_{n\in\mathbb{N}})$.
PROOF: Consider i being a natural number such that $n = k + i$. Define $\mathcal{P}$[natural number] $\equiv O(\{n^k\}_{n\in\mathbb{N}}) \subseteq O(\{n^{(k+\$_1)}\}_{n\in\mathbb{N}})$. For every natural number x such that $\mathcal{P}[x]$ holds $\mathcal{P}[x+1]$. For every natural number x, $\mathcal{P}[x]$ from [1, Sch. 2]. □

(33) Let us consider a natural number k, and a non-negative yielding finite 0-sequence c of $\mathbb{R}$. Suppose $\operatorname{len} c = k + 1$. Then $\mathrm{Seq}_{\mathrm{poly}}(c) \in O(\{n^k\}_{n\in\mathbb{N}})$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every non-negative yielding finite 0-sequence c of $\mathbb{R}$ such that $\operatorname{len} c = \$_1 + 1$ holds $\mathrm{Seq}_{\mathrm{poly}}(c) \in O(\{n^{\$_1}\}_{n\in\mathbb{N}})$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by (28), [7, (47)], [1, (13), (39)]. For every natural number k, $\mathcal{P}[k]$ from [1, Sch. 2]. □

(34) Let us consider a natural number k, and a finite 0-sequence c of $\mathbb{R}$. Then there exists a finite 0-sequence d of $\mathbb{R}$ such that

(i) $\operatorname{len} d = \operatorname{len} c$, and

(ii) for every natural number i such that $i \in \operatorname{dom} d$ holds $d(i) = |c(i)|$.

PROOF: Define $\mathcal{F}$(natural number) $= |c(\$_1)|(\in \mathbb{R})$. Consider d being a finite 0-sequence of $\mathbb{R}$ such that $\operatorname{len} d = \operatorname{len} c$ and for every natural number j such that $j \in \operatorname{len} c$ holds $d(j) = \mathcal{F}(j)$ from [18, Sch. 1]. □

(35) Let us consider a finite 0-sequence c of $\mathbb{R}$, and a finite 0-sequence d of $\mathbb{R}$. Suppose $\operatorname{len} d = \operatorname{len} c$ and for every natural number i such that $i \in \operatorname{dom} d$ holds $d(i) = |c(i)|$. Let us consider a natural number n. Then $(\mathrm{Seq}_{\mathrm{poly}}(c))(n) \leqslant (\mathrm{Seq}_{\mathrm{poly}}(d))(n)$.
PROOF: $\operatorname{dom}(d \cdot \{x^{1\cdot n+0}\}_{n\in\mathbb{N}}) = \operatorname{dom} d$. For every natural number i such that $i \in \operatorname{dom}(c \cdot \{x^{1\cdot n+0}\}_{n\in\mathbb{N}})$ holds $(c \cdot \{x^{1\cdot n+0}\}_{n\in\mathbb{N}})(i) \leqslant (d \cdot \{x^{1\cdot n+0}\}_{n\in\mathbb{N}})(i)$ by (26), (27), [19, (4)]. □

(36) Let us consider a natural number k, and a finite 0-sequence c of $\mathbb{R}$. Suppose $\operatorname{len} c = k + 1$ and $\mathrm{Seq}_{\mathrm{poly}}(c)$ is eventually nonnegative. Then $\mathrm{Seq}_{\mathrm{poly}}(c) \in O(\{n^k\}_{n\in\mathbb{N}})$.
PROOF: Consider d being a finite 0-sequence of $\mathbb{R}$ such that $\operatorname{len} d = \operatorname{len} c$ and for every natural number i such that $i \in \operatorname{dom} d$ holds $d(i) = |c(i)|$. For every natural number i such that $i \in \operatorname{dom} d$ holds $0 \leqslant d(i)$ by [6, (46)]. For every real number r such that $r \in \operatorname{rng} d$ holds $0 \leqslant r$. $\mathrm{Seq}_{\mathrm{poly}}(d) \in O(\{n^k\}_{n\in\mathbb{N}})$. Consider t being an element of $\mathbb{R}^{\mathbb{N}}$ such that $\mathrm{Seq}_{\mathrm{poly}}(d) = t$ and there exists a real number c and there exists an element N of $\mathbb{N}$ such that $c > 0$ and for every element n of $\mathbb{N}$ such that $n \geqslant N$ holds $t(n) \leqslant c \cdot \{n^k\}_{n\in\mathbb{N}}(n)$ and $t(n) \geqslant 0$. Consider N_1 being a natural number such that for every natural number n such that $N_1 \leqslant n$ holds $0 \leqslant (\mathrm{Seq}_{\mathrm{poly}}(c))(n)$.

Consider a being a real number, N_2 being an element of $\mathbb{N}$ such that $a > 0$ and for every element n of $\mathbb{N}$ such that $n \geqslant N_2$ holds $t(n) \leqslant a \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $t(n) \geqslant 0$. Set $N = N_1 + N_2$. For every element n of $\mathbb{N}$ such that $n \geqslant N$ holds $(\mathrm{Seq_{poly}}(c))(n) \leqslant a \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $(\mathrm{Seq_{poly}}(c))(n) \geqslant 0$ by [1, (11)], (35). □

(37) Let us consider natural numbers k, n. If $0 < n$, then $n \cdot \{n^k\}_{n \in \mathbb{N}}(n) = \{n^{(k+1)}\}_{n \in \mathbb{N}}(n)$.

(38) Let us consider a finite 0-sequence c of $\mathbb{R}$. Suppose $\mathrm{len}\, c = 0$. Let us consider a natural number x. Then $(\mathrm{Seq_{poly}}(c))(x) = 0$.

(39) Let us consider an eventually nonnegative sequence f of real numbers, and a natural number k. Suppose $f \in O(\{n^k\}_{n \in \mathbb{N}})$. Then there exists a natural number N such that for every natural number n such that $N \leqslant n$ holds $f(n) \leqslant \{n^{(k+1)}\}_{n \in \mathbb{N}}(n)$. The theorem is a consequence of (37).

(40) Let us consider a finite 0-sequence c of $\mathbb{R}$. Then there exists a finite 0-sequence a_1 of $\mathbb{R}$ such that

(i) $a_1 = |c|$, and

(ii) for every natural number n, $(\mathrm{Seq_{poly}}(c))(n) \leqslant (\mathrm{Seq_{poly}}(a_1))(n)$.

PROOF: Reconsider $a_1 = |c|$ as a finite 0-sequence of $\mathbb{R}$. Set $m_1 = c \cdot \{n^{1 \cdot n + 0}\}_{n \in \mathbb{N}}$. Set $m_2 = a_1 \cdot \{n^{1 \cdot n + 0}\}_{n \in \mathbb{N}}$. For every natural number x such that $x \in \mathrm{dom}\, m_1$ holds $m_1(x) \leqslant m_2(x)$ by [19, (4)]. □

(41) Let us consider finite 0-sequences c, a_1 of $\mathbb{R}$. Suppose $a_1 = |c|$. Let us consider a natural number n. Then $|(\mathrm{Seq_{poly}}(c))(n)| \leqslant (\mathrm{Seq_{poly}}(a_1))(n)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every finite 0-sequences c, a_1 of $\mathbb{R}$ such that $\mathrm{len}\, c = \$_1$ and $a_1 = |c|$ for every natural number x, $|(\mathrm{Seq_{poly}}(c))(x)| \leqslant (\mathrm{Seq_{poly}}(a_1))(x)$. $\mathcal{P}[0]$ by (26), [6, (44)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by (28), [7, (47)], [15, (7)], [6, (56), (65)]. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. □

(42) Let us consider a real number a. Suppose $0 < a$. Let us consider a natural number k, and a non-negative yielding finite 0-sequence d of $\mathbb{R}$. Suppose $\mathrm{len}\, d = k$. Then there exists a natural number N such that for every natural number x such that $N \leqslant x$ for every natural number i such that $i \in \mathrm{dom}\, d$ holds $d(i) \cdot x^i \cdot k \leqslant a \cdot x^k$.
PROOF: For every natural number i such that $i \in \mathrm{dom}\, d$ holds $0 \leqslant d(i)$ by [7, (3)]. □

(43) Let us consider a natural number k, a finite 0-sequence d of $\mathbb{R}$, a real number a, and a sequence y of real numbers. Suppose $0 < a$ and $\mathrm{len}\, d = k$ and for every natural number x, $y(x) = a \cdot x^k$. Then there exists a natural number N such that for every natural number x such that $N \leqslant x$ holds

$|(\mathrm{Seq}_{\mathrm{poly}}(d))(x)| \leqslant y(x)$. The theorem is a consequence of (38), (42), (26), (27), and (41).

(44) Let us consider a natural number k, and a finite 0-sequence d of $\mathbb{R}$. Suppose $\mathrm{len}\, d = k+1$ and $0 < d(k)$. Then $\mathrm{Seq}_{\mathrm{poly}}(d)$ is eventually nonnegative.

PROOF: Consider a being a real number, d_1 being a finite 0-sequence of $\mathbb{R}$, y being a sequence of real numbers such that $\mathrm{len}\, d_1 = k$ and $d_1 = d{\restriction}k$ and $a = d(k)$ and $d = d_1 \frown \langle a \rangle$ and $\mathrm{Seq}_{\mathrm{poly}}(d) = \mathrm{Seq}_{\mathrm{poly}}(d_1) + y$ and for every natural number i, $y(i) = a \cdot i^k$. Consider N being a natural number such that for every natural number i such that $N \leqslant i$ holds $|(\mathrm{Seq}_{\mathrm{poly}}(d_1))(i)| \leqslant y(i)$. For every natural number i such that $N \leqslant i$ holds $0 \leqslant (\mathrm{Seq}_{\mathrm{poly}}(d))(i)$ by [19, (4)], [15, (7)]. □

Let us consider a natural number k and a finite 0-sequence c of $\mathbb{R}$.

Let us assume that $\mathrm{len}\, c = k+1$ and $0 < c(k)$. Now we state the propositions:

(45) $\mathrm{Seq}_{\mathrm{poly}}(c) \in O(\{n^k\}_{n \in \mathbb{N}})$.

(46) $\mathrm{Seq}_{\mathrm{poly}}(c)$ is polynomially bounded. The theorem is a consequence of (36) and (44).

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