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Topological Interpretation of Rough Sets

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Summary. Rough sets, developed by Pawlak, are an important model of incomplete or partially known information. In this article, which is essentially a continuation of [11], we characterize rough sets in terms of topological closure and interior, as the approximations have the properties of the Kuratowski operators. We decided to merge topological spaces with tolerance approximation spaces. As a testbed for our developed approach, we restated the results of Isomichi [13] (formalized in Mizar in [14]) and about fourteen sets of Kuratowski [17] (encoded with the help of Mizar adjectives and clusters' registrations in [1]) in terms of rough approximations. The upper bounds which were 14 and 7 in the original paper of Kuratowski, in our case are six and three, respectively.

It turns out that within the classification given by Isomichi, 1^{st} class subsets are precisely crisp sets, 2^{nd} class subsets are proper rough sets, and there are no 3^{rd} class subsets in topological spaces generated by approximations. Also the important results about these spaces is that they are extremally disconnected [15], hence lattices of their domains are Boolean.

Furthermore, we develop the theory of abstract spaces equipped with maps possessing characteristic properties of rough approximations which enables us to freely use the notions from the theory of rough sets and topological spaces formalized in the Mizar Mathematical Library [10].

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The notation and terminology used in this paper have been introduced in the following articles: [2], [22], [4], [9], [24], [20], [21], [5], [6], [14], [1], [25], [3], [7], [19], [27], [11], [12], [18], [26], [15], [28], [16], and [8].

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1. Preliminaries

Now we state the proposition:

(1) Let us consider a set T and a family F of subsets of T. Then $F = \{B$, where B is a subset of $T : B \in F\}$.

Let f be a function and A be a set. We say that A is f-closed if and only if

(Def. 1) $A = f(A)$.

Let X be a set and F be a family of subsets of X. One can check that F is $\cap$-closed if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let us consider subsets a, b of X. If $a, b \in F$, then $a \cap b \in F$.

We say that F is union-closed if and only if

(Def. 3) Let us consider a family a of subsets of X. If $a \subseteq F$, then $\bigcup a \in F$.

We say that F is topology-like if and only if

(Def. 4) (i) $\emptyset, X \in F$, and

(ii) F is union-closed and $\cap$-closed.

Let us observe that there exists a family of subsets of X which is topology-like.

2. Ordinary Properties of Maps

Let X be a set and f be a function from 2^X into 2^X. We say that f is extensive if and only if

(Def. 5) Let us consider a subset A of X. Then $A \subseteq f(A)$.

We say that f is intensive if and only if

(Def. 6) Let us consider a subset A of X. Then $f(A) \subseteq A$.

We say that f is idempotent if and only if

(Def. 7) Let us consider a subset A of X. Then $f(f(A)) = f(A)$.

We say that f is $\subseteq$-monotone if and only if

(Def. 8) Let us consider subsets A, B of X. If $A \subseteq B$, then $f(A) \subseteq f(B)$.

We say that f preserves $\cup$ if and only if

(Def. 9) Let us consider subsets A, B of X. Then $f(A \cup B) = f(A) \cup f(B)$.

We say that f preserves $\cap$ if and only if

(Def. 10) Let us consider subsets A, B of X. Then $f(A \cap B) = f(A) \cap f(B)$.

Let O be a function from 2^X into 2^X. We say that O is a preclosure if and only if

(Def. 11) O is extensive and preserves $\cup$ and empty set.

We say that O is closure if and only if

(Def. 12) O is extensive and idempotent and preserves $\cup$ and empty set.

We say that O is a preinterior if and only if

(Def. 13) O is intensive and preserves $\cap$ and universe.

We say that O is an interior if and only if

(Def. 14) O is intensive and idempotent and preserves $\cap$ and universe.

Let us observe that every function from 2^X into 2^X which preserves $\cup$ is also $\subseteq$-monotone and every function from 2^X into 2^X which preserves $\cap$ is also $\subseteq$-monotone.

One can verify that id_{2^X} is closure as a function from 2^X into 2^X and id_{2^X} is an interior as a function from 2^X into 2^X.

One can check that there exists a function from 2^X into 2^X which is closure and interior.

Observe that every function from 2^X into 2^X which is closure is also a preclosure.

3. Structural Part

Let T be a 1-sorted structure.

A map of T is a function from $2^{(\text{the carrier of } T)}$ into $2^{(\text{the carrier of } T)}$. We consider 1stOpStrs which extend 1-sorted structures and are systems

$$\langle \text{a carrier}, \text{a FirstOp} \rangle$$

where the carrier is a set, the FirstOp is a function from $2^{(\text{the carrier})}$ into $2^{(\text{the carrier})}$.

We consider 2ndOpStrs which extend 1-sorted structures and are systems

$$\langle \text{a carrier}, \text{a SecondOp} \rangle$$

where the carrier is a set, the SecondOp is a function from $2^{(\text{the carrier})}$ into $2^{(\text{the carrier})}$.

We consider TwoOpStructs which extend 1stOpStrs and 2ndOpStrs and are systems

$$\langle \text{a carrier}, \text{a FirstOp}, \text{a SecondOp} \rangle$$

where the carrier is a set, the FirstOp and the SecondOp are functions from $2^{(\text{the carrier})}$ into $2^{(\text{the carrier})}$.

Let X be a 1stOpStr. We say that X has closure if and only if

(Def. 15) The FirstOp of X is closure.

We say that X has preclosure if and only if

(Def. 16) The FirstOp of X is a preclosure.

Let T be a topological space. Let us observe that $\mathrm{ClMap}\, T$ is closure and $\mathrm{IntMap}\, T$ is an interior and there exists a 1stOpStr which is non empty and has closure and every 1stOpStr which has closure has also preclosure.

Let X be a 1stOpStr and A be a subset of X. We say that A is op-closed if and only if

(Def. 17) $A =$ (the FirstOp of X)(A).

We say that X has op-closed subsets if and only if

(Def. 18) There exists a subset A of X such that A is op-closed.

One can check that there exists a 1stOpStr which has op-closed subsets.

Let X be 1stOpStr with op-closed subsets. One can check that there exists a subset of X which is op-closed.

Let X be a 2ndOpStr and A be a subset of X. We say that A is op-open if and only if

(Def. 19) $A =$ (the SecondOp of X)(A).

We say that X has op-open subsets if and only if

(Def. 20) There exists a subset A of X such that A is op-open.

Let us observe that there exists a 2ndOpStr which has op-open subsets.

Let X be 2ndOpStr with op-open subsets. Let us observe that there exists a subset of X which is op-open.

Let X be a 2ndOpStr. We say that X has interior if and only if

(Def. 21) The SecondOp of X is an interior.

We say that X has preinterior if and only if

(Def. 22) The SecondOp of X is a preinterior.

Note that there exists a TwoOpStruct which has closure and interior.

4. Merging with Topologies

We consider 1TopStructs which extend 1stOpStrs and topological structures and are systems

$$\langle \text{a carrier}, \text{a FirstOp}, \text{a topology} \rangle$$

where the carrier is a set, the FirstOp is a function from $2^{(\text{the carrier})}$ into $2^{(\text{the carrier})}$, the topology is a family of subsets of the carrier.

We consider 2TopStructs which extend 2ndOpStrs and topological structures and are systems

$$\langle \text{a carrier}, \text{a SecondOp}, \text{a topology} \rangle$$

where the carrier is a set, the SecondOp is a function from $2^{(\text{the carrier})}$ into $2^{(\text{the carrier})}$, the topology is a family of subsets of the carrier.

Let us observe that there exists a 1TopStruct which is non empty and strict and there exists a 2TopStruct which is non empty and strict.

Let T be a 1TopStruct. We say that T has properly defined topology if and only if

(Def. 23) Let us consider an object x. Then $x \in$ the topology of T if and only if there exists a subset S of T such that $S^{c} = x$ and S is op-closed.

Let T be a 2TopStruct. We say that T has properly defined Topology if and only if

(Def. 24) Let us consider an object x. Then $x \in$ the topology of T if and only if there exists a subset S of T such that $S = x$ and S is op-open.

One can verify that there exists a 1TopStruct which has closure and properly defined topology and there exists a 2TopStruct which has interior and properly defined Topology.

(2) Let us consider 1TopStruct T with properly defined topology and a subset A of T. Then A is op-closed if and only if A is closed. Proof: If A is op-closed, then A is closed by [28, (3)]. If A is closed, then A is op-closed by [28, (3)]. □

Observe that every 1TopStruct with properly defined topology which has preclosure is also topological space-like.

(3) Let us consider 2TopStruct T with properly defined Topology and a subset A of T. Then A is op-open if and only if A is open.

Note that every 2TopStruct with properly defined Topology which has preinterior is also topological space-like.

(4) Let us consider 1TopStruct T with closure properly defined topology and a subset A of T. Then (the FirstOp of T)$(A) = \overline{A}$. Proof: Set $f =$ the FirstOp of T. Consider F being a family of subsets of T such that for every subset C of T, $C \in F$ iff C is closed and $A \subseteq C$ and $\overline{A} = \bigcap F$. $\overline{A} \subseteq f(A)$ by (2), [18, (3)]. Define $\mathcal{P}$[subset of T] $\equiv \$_1 \in F$. Set $G = \{f(B)$, where B is a subset of $T : B \in F\}$. Define $\mathcal{T} = 2^{(\text{the carrier of } T)}$. Define $\mathcal{F}$(element of $\mathcal{T}$) $= f(\$_1)$. Define $\mathcal{G}$(element of $\mathcal{T}$) $= \$_1$. For every element B of $\mathcal{T}$ such that $\mathcal{P}[B]$ holds $\mathcal{F}(B) = \mathcal{G}(B)$. $\{\mathcal{F}(B)$, where B is an element of $\mathcal{T} : \mathcal{P}[B]\} = \{\mathcal{G}(B)$, where B is an element of $\mathcal{T} : \mathcal{P}[B]\}$ from [23, Sch. 6]. $F = G$. For every set Z such that $Z \in G$ holds $f(A) \subseteq Z$. □

5. Introducing Rough Sets

Let R be a tolerance space. Let us note that LAp(R) is a preinterior and UAp(R) is a preclosure.

Let R be an approximation space. Observe that LAp(R) is an interior and UAp(R) is closure.

Let X be a set and f be a function from 2^X into 2^X. The functor GenTop f yielding a family of subsets of X is defined by

(Def. 25) Let us consider an object x. Then $x \in$ *it* if and only if there exists a subset S of X such that $S = x$ and S is f-closed.

Now we state the proposition:

(5) Let us consider a set X and a function f from 2^X into 2^X. If f is a preinterior, then $\mathrm{GenTop}\, f$ is topology-like. Proof: Set $F = \mathrm{GenTop}\, f$. There exists a subset S of X such that $S = X$ and S is f-closed. There exists a subset S of X such that $S = \emptyset$ and S is f-closed. F is $\cap$-closed. For every family a of subsets of X such that $a \subseteq F$ holds $\bigcup a \in F$ by [8, (74), (76)]. $\square$

Let C be a set, I be a binary relation on C, and f be a topology-like family of subsets of C. Observe that $\langle C, I, f \rangle$ is topological space-like and there exists a FR-structure which is topological space-like and non empty and has equivalence relation.

6. On Sequential Closure and Frechet Spaces

Let T be a non empty topological space. The functor $\mathrm{Cl}_{\mathrm{Seq}}\, T$ yielding a map of T is defined by

(Def. 26) Let us consider a subset A of T. Then $it(A) = \mathrm{Cl}_{\mathrm{Seq}}\, A$.

One can verify that $\mathrm{Cl}_{\mathrm{Seq}}\, T$ is a preclosure and there exists a non empty topological space which is Frechet.

Let T be a Frechet non empty topological space. Note that $\mathrm{Cl}_{\mathrm{Seq}}\, T$ is closure.

7. Connections between Closures and Approximations

Let T be a non empty FR-structure. We say that T is Natural if and only if

(Def. 27) Let us consider a subset x of T. Then $x \in$ the topology of T if and only if x is $(\mathrm{LAp}(T))$-closed.

We say that T is naturally generated if and only if

(Def. 28) The topology of $T = \mathrm{GenTop}\,\mathrm{LAp}(T)$.

Now we state the proposition:

(6) Let us consider a non empty FR-structure T. Suppose T is naturally generated. Let us consider a subset A of T. Then A is open if and only if $\mathrm{LAp}(A) = A$.

Let us consider a non empty FR-structure T and a non empty relational structure R.

Let us assume that the relational structure of $T =$ the relational structure of R. Now we state the propositions:

(7) $\mathrm{LAp}(T) = \mathrm{LAp}(R)$.

(8) $\mathrm{UAp}(T) = \mathrm{UAp}(R)$.

One can verify that there exists a non empty FR-structure which is Natural and topological space-like and has equivalence relation and every non empty FR-structure with equivalence relation which is naturally generated is also topological space-like and there exists a non empty FR-structure which is naturally generated and topological space-like and has equivalence relation.

Let T be a naturally generated non empty FR-structure with equivalence relation and A be a subset of T. One can check that $\mathrm{LAp}(A)$ is open.

Let us consider a naturally generated non empty FR-structure T with equivalence relation and a subset A of T. Now we state the propositions:

(9) $\mathrm{LAp}(A) = \mathrm{Int}\, A$. PROOF: $\mathrm{Int}\, A \subseteq \mathrm{LAp}(A)$ by [28, (22), (23)], [11, (24)]. $\square$

(10) A is closed if and only if $\mathrm{UAp}(A) = A$. PROOF: If A is closed, then $\mathrm{UAp}(A) = A$ by (6), [11, (28)]. $\square$

Let T be a naturally generated non empty FR-structure with equivalence relation and A be a subset of T. One can check that $\mathrm{UAp}(A)$ is closed.

Let us consider a naturally generated non empty FR-structure T with equivalence relation and a subset A of T. Now we state the propositions:

(11) $\mathrm{UAp}(A) = \overline{A}$. PROOF: $\mathrm{UAp}(A) \subseteq \overline{A}$ by (10), [11, (25)], [19, (15)]. $\square$

(12) $\mathrm{BndAp}(A) = \mathrm{Fr}\, A$. The theorem is a consequence of (11) and (9).

Let T be a naturally generated non empty FR-structure with equivalence relation and A be a subset of T. We identify $\mathrm{LAp}(A)$ with $\mathrm{Int}\, A$. We identify $\mathrm{UAp}(A)$ with $\overline{A}$. We identify $\mathrm{Int}\, A$ with $\mathrm{LAp}(A)$. We identify $\overline{A}$ with $\mathrm{UAp}(A)$. We identify $\mathrm{Fr}\, A$ with $\mathrm{BndAp}(A)$. We identify $\mathrm{BndAp}(A)$ with $\mathrm{Fr}\, A$.

8. Isomichi Results Reuse

Let us consider a naturally generated non empty FR-structure T with equivalence relation and a subset A of T. Now we state the propositions:

(13) A is 1^{st} class if and only if $\mathrm{LAp}(\mathrm{UAp}(A)) \subseteq \mathrm{UAp}(\mathrm{LAp}(A))$.

(14) A is 1^{st} class if and only if $\mathrm{UAp}(A) \subseteq \mathrm{LAp}(A)$.

(15) A is 1^{st} class if and only if A is exact. PROOF: If A is 1^{st} class, then A is exact by [11, (14)], (14), [11, (13), (12)]. $\square$

Let T be a naturally generated non empty FR-structure with equivalence relation. Note that every subset of T which is 1^{st} class is also exact and every subset of T which is exact is also 1^{st} class.

Let us consider a naturally generated non empty FR-structure T with equivalence relation and a subset A of T. Now we state the propositions:

(16) A is 2^{nd} class if and only if $\mathrm{LAp}(A) \subset \mathrm{UAp}(A)$.

(17) A is 2^{nd} class if and only if A is rough. PROOF: $\mathrm{LAp}(A) \neq \mathrm{UAp}(A)$ by [11, (13), (12)]. $\square$

Let T be a naturally generated non empty FR-structure with equivalence relation. Note that every subset of T which is 2^{nd} class is also rough and every subset of T which is rough is also 2^{nd} class.

Now we state the propositions:

(18) Let us consider a naturally generated non empty FR-structure T with equivalence relation and a subset A of T. Then $\overline{\operatorname{Int} A}$ and $\overline{A}$ are $\subseteq$-comparable.

(19) Let us consider a naturally generated non empty FR-structure T with equivalence relation and a subset A of T. Then A is not 3^{rd} class.

Let T be a topological space.

Observe that every naturally generated non empty FR-structure with equivalence relation is without 3rd class subsets and there exists a topological space which is without 3rd class subsets.

Let T be a topological space and A be a 1^{st} class subset of T. One can verify that Border A is empty.

Let T be a naturally generated non empty FR-structure with equivalence relation and A be a subset of T. Note that $\overline{A}$ is open and $\operatorname{Int} A$ is closed and every naturally generated non empty FR-structure with equivalence relation is extremally disconnected.

9. Reexamination of Kuratowski's 14 Sets for Approximation Spaces

Let us consider a naturally generated non empty FR-structure T with equivalence relation and a subset A of T. Now we state the propositions:

(20) $\operatorname{Kurat7Set}(A) = \{A, \overline{A}, \operatorname{Int} A\}$.

(21) $\overline{\overline{\operatorname{Kurat7Set}(A)}} \leqslant 3$. The theorem is a consequence of (20).

(22) $\operatorname{Kurat14Set}(A) = \{A, \operatorname{UAp}(A), (\operatorname{UAp}(A))^{\mathrm{c}}, A^{\mathrm{c}}, (\operatorname{LAp}(A))^{\mathrm{c}}, \operatorname{LAp}(A)\}$.

(23) $\overline{\overline{\operatorname{Kurat14Set}(A)}} \leqslant 6$. The theorem is a consequence of (22).

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