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# Commutativeness of Fundamental Groups of Topological Groups

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**Summary.** In this article we prove that fundamental groups based at the unit point of topological groups are commutative [11].

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The notation and terminology used in this paper have been introduced in the following articles: [3], [19], [9], [10], [16], [20], [4], [5], [22], [23], [21], [1], [6], [17], [18], [2], [25], [26], [24], [15], [12], [13], [8], [14], and [7].

Let $A$ be a non empty set, $x$ be an element, and $a$ be an element of $A$. Let us observe that $(A \longmapsto x)(a)$ reduces to $x$.

Let $A$, $B$ be non empty topological spaces, $C$ be a set, and $f$ be a function from $A \times B$ into $C$. Let $b$ be an element of $B$. Let us note that the functor $f(a, b)$ yields an element of $C$. Let $G$ be a multiplicative magma and $g$ be an element of $G$. We say that $g$ is unital if and only if

(Def. 1) $g = \mathbf{1}_G$.

One can check that $\mathbf{1}_G$ is unital.

Let $G$ be a unital multiplicative magma. Let us note that there exists an element of $G$ which is unital.

Let $g$ be an element of $G$ and $h$ be a unital element of $G$. One can check that $g \cdot h$ reduces to $g$. One can check that $h \cdot g$ reduces to $g$.

Let $G$ be a group. One can verify that $(\mathbf{1}_G)^{-1}$ reduces to $\mathbf{1}_G$.

The scheme *TopFuncEx* deals with non empty topological spaces $\mathcal{S}$, $\mathcal{T}$ and a non empty set $\mathcal{X}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and states that




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(Sch. 1) There exists a function $f$ from $\mathcal{S} \times \mathcal{T}$ into $\mathcal{X}$ such that for every point $s$ of $\mathcal{S}$ for every point $t$ of $\mathcal{T}$, $f(s,t) = \mathcal{F}(s,t)$.

The scheme *TopFuncEq* deals with non empty topological spaces $\mathcal{S}$, $\mathcal{T}$ and a non empty set $\mathcal{X}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and states that

(Sch. 2) For every functions $f$, $g$ from $\mathcal{S} \times \mathcal{T}$ into $\mathcal{X}$ such that for every point $s$ of $\mathcal{S}$ and for every point $t$ of $\mathcal{T}$, $f(s,t) = \mathcal{F}(s,t)$ and for every point $s$ of $\mathcal{S}$ and for every point $t$ of $\mathcal{T}$, $g(s,t) = \mathcal{F}(s,t)$ holds $f = g$.

Let $X$ be a non empty set, $T$ be a non empty multiplicative magma, and $f$, $g$ be functions from $X$ into $T$. The functor $f \cdot g$ yielding a function from $X$ into $T$ is defined by

(Def. 2) Let us consider an element $x$ of $X$. Then $\mathit{it}(x) = f(x) \cdot g(x)$.

Now we state the proposition:

(1) Let us consider a non empty set $X$, an associative non empty multiplicative magma $T$, and functions $f$, $g$, $h$ from $X$ into $T$. Then $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.

Let $X$ be a non empty set, $T$ be a commutative non empty multiplicative magma, and $f$, $g$ be functions from $X$ into $T$. Observe that the functor $f \cdot g$ is commutative.

Let $T$ be a non empty topological group structure, $t$ be a point of $T$, and $f$, $g$ be loops of $t$. The functor $f \bullet g$ yielding a function from $\mathbb{I}$ into $T$ is defined by the term

(Def. 3) $f \cdot g$.

In this paper $T$ denotes a continuous unital topological space-like non empty topological group structure, $x$, $y$ denote points of $\mathbb{I}$, $s$, $t$ denote unital points of $T$, $f$, $g$ denote loops of $t$, and $c$ denotes a constant loop of $t$.

Let us consider $T$, $t$, $f$, and $g$. One can check that the functor $f \bullet g$ yields a loop of $t$. Let $T$ be an inverse-continuous semi topological group. Observe that $\cdot_T^{-1}$ is continuous.

Let $T$ be a semi topological group, $t$ be a point of $T$, and $f$ be a loop of $t$. The functor $f^{-1}$ yielding a function from $\mathbb{I}$ into $T$ is defined by the term

(Def. 4) $\cdot_T^{-1} \cdot f$.

Let us consider a semi topological group $T$, a point $t$ of $T$, and a loop $f$ of $t$. Now we state the propositions:

(2) $(f^{-1})(x) = f(x)^{-1}$.

(3) $(f^{-1})(x) \cdot f(x) = \mathbf{1}_T$.

(4) $f(x) \cdot (f^{-1})(x) = \mathbf{1}_T$.

Let $T$ be an inverse-continuous semi topological group, $t$ be a unital point of $T$, and $f$ be a loop of $t$. One can check that the functor $f^{-1}$ yields a loop of

$t$. Let $s$, $t$ be points of $\mathbb{I}$. One can check that the functor $s \cdot t$ yields a point of $\mathbb{I}$. The functor $\otimes_{\mathbb{R}^1}$ yielding a function from $\mathbb{R}^{\mathbf{1}} \times \mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$ is defined by

(Def. 5) Let us consider points $x$, $y$ of $\mathbb{R}^{\mathbf{1}}$. Then $it(x, y) = x \cdot y$.

Observe that $\otimes_{\mathbb{R}^1}$ is continuous.

Now we state the proposition:

(5) $(\mathbb{R}^{\mathbf{1}} \times \mathbb{R}^{\mathbf{1}}) \restriction (R^1[0,1] \times R^1[0,1]) = \mathbb{I} \times \mathbb{I}$.

The functor $\otimes_{\mathbb{I}}$ yielding a function from $\mathbb{I} \times \mathbb{I}$ into $\mathbb{I}$ is defined by the term

(Def. 6) $\otimes_{\mathbb{R}^1} \restriction R^1[0,1]$.

Now we state the proposition:

(6) $(\otimes_{\mathbb{I}})(x, y) = x \cdot y$.

One can verify that $\otimes_{\mathbb{I}}$ is continuous.

Now we state the proposition:

(7) Let us consider points $a$, $b$ of $\mathbb{I}$ and a neighbourhood $N$ of $a \cdot b$. Then there exists a neighbourhood $N_1$ of $a$ and there exists a neighbourhood $N_2$ of $b$ such that for every points $x$, $y$ of $\mathbb{I}$ such that $x \in N_1$ and $y \in N_2$ holds $x \cdot y \in N$. The theorem is a consequence of (6).

Let $T$ be a non empty multiplicative magma and $F$, $G$ be functions from $\mathbb{I} \times \mathbb{I}$ into $T$. The functor $F * G$ yielding a function from $\mathbb{I} \times \mathbb{I}$ into $T$ is defined by

(Def. 7) Let us consider points $a$, $b$ of $\mathbb{I}$. Then $it(a, b) = F(a, b) \cdot G(a, b)$.

Now we state the proposition:

(8) Let us consider functions $F$, $G$ from $\mathbb{I} \times \mathbb{I}$ into $T$ and subsets $M$, $N$ of $\mathbb{I} \times \mathbb{I}$. Then $(F * G)^{\circ}(M \cap N) \subseteq F^{\circ}M \cdot G^{\circ}N$.

Let us consider $T$. Let $F$, $G$ be continuous functions from $\mathbb{I} \times \mathbb{I}$ into $T$. Observe that $F * G$ is continuous.

Now we state the propositions:

(9) Let us consider loops $f_1$, $f_2$, $g_1$, $g_2$ of $t$. Suppose

  (i) $f_1$, $f_2$ are homotopic, and

  (ii) $g_1$, $g_2$ are homotopic.

Then $f_1 \bullet g_1$, $f_2 \bullet g_2$ are homotopic.

(10) Let us consider loops $f_1$, $f_2$, $g_1$, $g_2$ of $t$, a homotopy $F$ between $f_1$ and $f_2$, and a homotopy $G$ between $g_1$ and $g_2$. Suppose

  (i) $f_1$, $f_2$ are homotopic, and

  (ii) $g_1$, $g_2$ are homotopic.

Then $F * G$ is a homotopy between $f_1 \bullet g_1$ and $f_2 \bullet g_2$. The theorem is a consequence of (9).

(11) $f + g = (f + c) \bullet (c + g)$.

(12) $f \bullet g$, $(f + c) \bullet (c + g)$ are homotopic. The theorem is a consequence of (9).

Let $T$ be a semi topological group, $t$ be a point of $T$, and $f$, $g$ be loops of $t$. The functor HopfHomotopy$(f, g)$ yielding a function from $\mathbb{I} \times \mathbb{I}$ into $T$ is defined by

(Def. 8) Let us consider points $a$, $b$ of $\mathbb{I}$. Then $it(a, b) = (((f^{-1})(a \cdot b) \cdot f(a)) \cdot g(a)) \cdot f(a \cdot b)$.

Note that HopfHomotopy$(f, g)$ is continuous.

In the sequel $T$ denotes a topological group, $t$ denotes a unital point of $T$, and $f$, $g$ denote loops of $t$.

Now we state the proposition:

(13) $f \bullet g$, $g \bullet f$ are homotopic.

Let us consider $T$, $t$, $f$, and $g$. Let us note that the functor HopfHomotopy$(f, g)$ yields a homotopy between $f \bullet g$ and $g \bullet f$.

Now we are at the position where we can present the Main Theorem of the paper: $\pi_1(T, t)$ is commutative.

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