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# Introduction to Rational Functions

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**Summary.** In this article we formalize rational functions as pairs of polynomials and define some basic notions including the degree and evaluation of rational functions [8]. The main goal of the article is to provide properties of rational functions necessary to prove a theorem on the stability of networks.

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The notation and terminology used in this paper are introduced in the following articles: [14], [3], [4], [5], [18], [20], [16], [17], [1], [15], [2], [6], [12], [10], [11], [22], [19], [21], [9], [13], [23], and [7].

## 1. Preliminaries

One can prove the following three propositions:

(1) Let $L$ be an add-associative right zeroed right complementable right distributive non empty double loop structure, $a$ be an element of $L$, and $p$, $q$ be finite sequences of elements of $L$. Suppose $\operatorname{len} p = \operatorname{len} q$ and for every element $i$ of $\mathbb{N}$ such that $i \in \operatorname{dom} p$ holds $q_i = a \cdot p_i$. Then $\sum q = a \cdot \sum p$.

(2) Let $L$ be an add-associative right zeroed right complementable right distributive non empty double loop structure, $f$ be a finite sequence of elements of $L$, and $i$, $j$ be elements of $\mathbb{N}$. If $i \in \operatorname{dom} f$ and $j = i - 1$, then $\operatorname{Ins}(f_{\restriction i}, j, f_i) = f$.

(3) Let $L$ be an add-associative right zeroed right complementable associative unital right distributive commutative non empty double loop structure, $f$ be a finite sequence of elements of $L$, and $i$ be an element of $\mathbb{N}$. If $i \in \operatorname{dom} f$, then $\prod f = f_i \cdot \prod(f_{\restriction i})$.



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Let $L$ be an add-associative right zeroed right complementable well unital associative left distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $x$, $y$ be non zero elements of $L$. Note that $\frac{x}{y}$ is non zero.

Let us note that every add-associative right zeroed right complementable right distributive non empty double loop structure which is integral domain-like is also almost left cancelable and every add-associative right zeroed right complementable left distributive non empty double loop structure which is integral domain-like is also almost right cancelable.

Let $x$, $y$ be integers. Note that $\max(x, y)$ is integer and $\min(x, y)$ is integer.

One can prove the following proposition

(4) For all integers $x$, $y$, $z$ holds $\max(x + y, x + z) = x + \max(y, z)$.

## 2. More on Polynomials

Let $L$ be a non empty zero structure and let $p$ be a polynomial of $L$. We say that $p$ is zero if and only if:

(Def. 1) $p = \mathbf{0}. L$.

We say that $p$ is constant if and only if:

(Def. 2) $\deg p \leq 0$.

Let $L$ be a non trivial zero structure. One can verify that there exists a polynomial of $L$ which is non zero.

Let $L$ be a non empty zero structure. One can verify that $\mathbf{0}. L$ is zero and constant.

Let $L$ be a non degenerated multiplicative loop with zero structure. Note that $\mathbf{1}. L$ is non zero.

Let $L$ be a non empty multiplicative loop with zero structure. Note that $\mathbf{1}. L$ is constant.

Let $L$ be a non empty zero structure. One can verify that every polynomial of $L$ which is zero is also constant. Note that every polynomial of $L$ which is non constant is also non zero.

Let $L$ be a non trivial zero structure. One can verify that there exists a polynomial of $L$ which is non constant.

Let $L$ be a well unital non degenerated non empty double loop structure, let $z$ be an element of $L$, and let $k$ be an element of $\mathbb{N}$. Observe that $\mathrm{rpoly}(k, z)$ is non zero.

Let $L$ be an add-associative right zeroed right complementable distributive non degenerated double loop structure. One can check that Polynom-Ring $L$ is non degenerated.

Let $L$ be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure. Observe that Polynom-Ring $L$ is integral domain-like.

Next we state two propositions:

(5) Let $L$ be an add-associative right zeroed right complementable right distributive associative non empty double loop structure, $p$, $q$ be polynomials of $L$, and $a$ be an element of $L$. Then $(a \cdot p) * q = a \cdot (p * q)$.

(6) Let $L$ be an add-associative right zeroed right complementable right distributive commutative associative non empty double loop structure, $p$, $q$ be polynomials of $L$, and $a$ be an element of $L$. Then $p*(a \cdot q) = a \cdot (p*q)$.

Let $L$ be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non trivial double loop structure, let $p$ be a non zero polynomial of $L$, and let $a$ be a non zero element of $L$. Note that $a \cdot p$ is non zero.

Let $L$ be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure and let $p_1$, $p_2$ be non zero polynomials of $L$. Observe that $p_1 * p_2$ is non zero.

One can prove the following proposition

(7) Let $L$ be an add-associative right zeroed right complementable distributive Abelian integral domain-like non trivial double loop structure, $p_1, p_2$ be polynomials of $L$, and $p_3$ be a non zero polynomial of $L$. If $p_1 * p_3 = p_2 * p_3$, then $p_1 = p_2$.

Let $L$ be a non trivial zero structure and let $p$ be a non zero polynomial of $L$. One can check that $\operatorname{degree}(p)$ is natural.

Next we state several propositions:

(8) Let $L$ be an add-associative right zeroed right complementable unital right distributive non empty double loop structure and $p$ be a polynomial of $L$. If $\deg p = 0$, then for every element $x$ of $L$ holds $\operatorname{eval}(p, x) \neq 0_L$.

(9) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible non degenerated double loop structure, $p$ be a polynomial of $L$, and $x$ be an element of $L$. Then $\operatorname{eval}(p, x) = 0_L$ if and only if $\operatorname{rpoly}(1, x) \mid p$.

(10) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible integral domain-like non degenerated double loop structure, $p$, $q$ be polynomials of $L$, and $x$ be an element of $L$. If $\operatorname{rpoly}(1, x) \mid p * q$, then $\operatorname{rpoly}(1, x) \mid p$ or $\operatorname{rpoly}(1, x) \mid q$.

(11) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible non degenerated double loop structure and $f$ be a finite sequence of elements

of Polynom-Ring $L$. Suppose that for every natural number $i$ such that $i \in \operatorname{dom} f$ there exists an element $z$ of $L$ such that $f(i) = \operatorname{rpoly}(1, z)$. Let $p$ be a polynomial of $L$. If $p = \prod f$, then $p \neq \mathbf{0}.\, L$.

(12) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible integral domain-like non degenerated double loop structure and $f$ be a finite sequence of elements of Polynom-Ring $L$. Suppose that for every natural number $i$ such that $i \in \operatorname{dom} f$ there exists an element $z$ of $L$ such that $f(i) = \operatorname{rpoly}(1, z)$. Let $p$ be a polynomial of $L$. Suppose $p = \prod f$. Let $x$ be an element of $L$. Then $\operatorname{eval}(p, x) = 0_L$ if and only if there exists a natural number $i$ such that $i \in \operatorname{dom} f$ and $f(i) = \operatorname{rpoly}(1, x)$.

## 3. Common Roots of Polynomials

Let $L$ be a unital non empty double loop structure, let $p_1$, $p_2$ be polynomials of $L$, and let $x$ be an element of $L$. We say that $x$ is a common root of $p_1$ and $p_2$ if and only if:

(Def. 3) $x$ is a root of $p_1$ and $x$ is a root of $p_2$.

Let $L$ be a unital non empty double loop structure and let $p_1$, $p_2$ be polynomials of $L$. We say that $p_1$ and $p_2$ have a common root if and only if:

(Def. 4) There exists an element of $L$ which is a common root of $p_1$ and $p_2$.

Let $L$ be a unital non empty double loop structure and let $p_1$, $p_2$ be polynomials of $L$. We introduce $p_1$ and $p_2$ have common roots as a synonym of $p_1$ and $p_2$ have a common root. We introduce $p_1$ and $p_2$ have no common roots as an antonym of $p_1$ and $p_2$ have a common root.

Next we state several propositions:

(13) Let $L$ be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, $p$ be a polynomial of $L$, and $x$ be an element of $L$. If $x$ is a root of $p$, then $x$ is a root of $-p$.

(14) Let $L$ be an Abelian add-associative right zeroed right complementable unital left distributive non empty double loop structure, $p_1$, $p_2$ be polynomials of $L$, and $x$ be an element of $L$. If $x$ is a common root of $p_1$ and $p_2$, then $x$ is a root of $p_1 + p_2$.

(15) Let $L$ be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, $p_1$, $p_2$ be polynomials of $L$, and $x$ be an element of $L$. If $x$ is a common root of $p_1$ and $p_2$, then $x$ is a root of $-(p_1 + p_2)$.

(16) Let $L$ be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, $p$, $q$ be polynomials

of $L$, and $x$ be an element of $L$. If $x$ is a common root of $p$ and $q$, then $x$ is a root of $p+q$.

(17) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible non trivial double loop structure and $p_1$, $p_2$ be polynomials of $L$. If $p_1 \mid p_2$ and $p_1$ has roots, then $p_1$ and $p_2$ have common roots.

Let $L$ be a unital non empty double loop structure and let $p$, $q$ be polynomials of $L$. The common roots of $p$ and $q$ yields a subset of $L$ and is defined by:

(Def. 5) The common roots of $p$ and $q = \{x \in L\colon x$ is a common root of $p$ and $q\}$.

## 4. Normalized Polynomials

Let $L$ be a non empty zero structure and let $p$ be a polynomial of $L$. The leading coefficient of $p$ yields an element of $L$ and is defined by:

(Def. 6) The leading coefficient of $p = p(\operatorname{len} p -' 1)$.

We introduce $\mathrm{LC}\, p$ as a synonym of the leading coefficient of $p$.

Let $L$ be a non trivial double loop structure and let $p$ be a non zero polynomial of $L$. One can check that $\mathrm{LC}\, p$ is non zero.

One can prove the following proposition

(18) Let $L$ be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non empty double loop structure, $p$ be a polynomial of $L$, and $a$ be an element of $L$. Then $\mathrm{LC}(a \cdot p) = a \cdot \mathrm{LC}\, p$.

Let $L$ be a non empty double loop structure and let $p$ be a polynomial of $L$. We say that $p$ is normalized if and only if:

(Def. 7) $\mathrm{LC}\, p = 1_L$.

Let $L$ be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non trivial double loop structure and let $p$ be a non zero polynomial of $L$. One can check that $\frac{1_L}{\mathrm{LC}\, p} \cdot p$ is normalized.

Let $L$ be a field and let $p$ be a non zero polynomial of $L$. One can verify that $\mathrm{NormPolynomial}\, p$ is normalized.

## 5. Rational Functions

Let $L$ be a non trivial multiplicative loop with zero structure. Rational function of $L$ is defined by:

(Def. 8) There exists a polynomial $p_1$ of $L$ and there exists a non zero polynomial $p_2$ of $L$ such that it $= \langle p_1, p_2 \rangle$.

Let $L$ be a non trivial multiplicative loop with zero structure, let $p_1$ be a polynomial of $L$, and let $p_2$ be a non zero polynomial of $L$. Then $\langle p_1, p_2 \rangle$ is a rational function of $L$.

Let $L$ be a non trivial multiplicative loop with zero structure and let $z$ be a rational function of $L$. Then $z_{\mathbf{1}}$ is a polynomial of $L$. Then $z_{\mathbf{2}}$ is a non zero polynomial of $L$.

Let $L$ be a non trivial multiplicative loop with zero structure and let $z$ be a rational function of $L$. We say that $z$ is zero if and only if:

(Def. 9) $z_{\mathbf{1}} = \mathbf{0}.L$.

Let $L$ be a non trivial multiplicative loop with zero structure. One can check that there exists a rational function of $L$ which is non zero.

Next we state the proposition

(19) Let $L$ be a non trivial multiplicative loop with zero structure and $z$ be a rational function of $L$. Then $z = \langle z_{\mathbf{1}}, z_{\mathbf{2}} \rangle$.

Let $L$ be an add-associative right zeroed right complementable distributive unital non trivial double loop structure and let $z$ be a rational function of $L$. We say that $z$ is irreducible if and only if:

(Def. 10) $z_{\mathbf{1}}$ and $z_{\mathbf{2}}$ have no common roots.

Let $L$ be an add-associative right zeroed right complementable distributive unital non trivial double loop structure and let $z$ be a rational function of $L$. We introduce $z$ is reducible as an antonym of $z$ is irreducible.

Let $L$ be an add-associative right zeroed right complementable distributive unital non trivial double loop structure and let $z$ be a rational function of $L$. We say that $z$ is normalized if and only if:

(Def. 11) $z$ is irreducible and $z_{\mathbf{2}}$ is normalized.

Let $L$ be an add-associative right zeroed right complementable distributive unital non trivial double loop structure. Observe that every rational function of $L$ which is normalized is also irreducible.

Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $z$ be a rational function of $L$. Note that $\mathrm{LC}(z_{\mathbf{2}})$ is non zero.

Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $z$ be a rational function of $L$. The norm rational function of $z$ yields a rational function of $L$ and is defined by:

(Def. 12) The norm rational function of $z = \langle \frac{1_L}{\mathrm{LC}(z_{\mathbf{2}})} \cdot z_{\mathbf{1}}, \frac{1_L}{\mathrm{LC}(z_{\mathbf{2}})} \cdot z_{\mathbf{2}} \rangle$.

Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $z$ be a rational function of $L$. We introduce NormRatF $z$ as a synonym of the norm rational function of $z$.

Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $z$ be a non zero rational function of $L$. Observe that the norm rational function of $z$ is non zero.

Let $L$ be a non degenerated multiplicative loop with zero structure. The functor $0. L$ yields a rational function of $L$ and is defined by:

(Def. 13) $0. L = \langle \mathbf{0}. L, \mathbf{1}. L \rangle$.

The functor $1. L$ yields a rational function of $L$ and is defined as follows:

(Def. 14) $1. L = \langle \mathbf{1}. L, \mathbf{1}. L \rangle$.

Let $L$ be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can check that $0. L$ is normalized.

Let $L$ be a non degenerated multiplicative loop with zero structure. Note that $1. L$ is non zero.

Let $L$ be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can verify that $1. L$ is irreducible.

Let $L$ be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. Observe that there exists a rational function of $L$ which is irreducible and non zero.

Let $L$ be an add-associative right zeroed right complementable distributive Abelian associative well unital non degenerated double loop structure and let $x$ be an element of $L$. One can check that $\langle \mathrm{rpoly}(1, x), \mathrm{rpoly}(1, x) \rangle$ is reducible and non zero as a rational function of $L$.

Let $L$ be an add-associative right zeroed right complementable distributive Abelian associative well unital non degenerated double loop structure. Observe that there exists a rational function of $L$ which is reducible and non zero.

Let $L$ be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can verify that there exists a rational function of $L$ which is normalized.

Let $L$ be a non degenerated multiplicative loop with zero structure. One can verify that $0. L$ is zero.

Let $L$ be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can check that $1. L$ is normalized.

Let $L$ be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure and let $p$, $q$ be rational functions of $L$. The functor $p + q$ yields a rational function of $L$ and is defined by:

(Def. 15) $p+q = \langle p_{\mathbf{1}} * q_{\mathbf{2}} + p_{\mathbf{2}} * q_{\mathbf{1}},\, p_{\mathbf{2}} * q_{\mathbf{2}} \rangle$.

Let $L$ be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure and let $p$, $q$ be rational functions of $L$. The functor $p * q$ yielding a rational function of $L$ is defined by:

(Def. 16) $p * q = \langle p_{\mathbf{1}} * q_{\mathbf{1}},\, p_{\mathbf{2}} * q_{\mathbf{2}} \rangle$.

One can prove the following proposition

(20) Let $L$ be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non trivial double loop structure, $p$ be a rational function of $L$, and $a$ be a non zero element of $L$. Then $\langle a \cdot p_{\mathbf{1}},\, a \cdot p_{\mathbf{2}} \rangle$ is irreducible if and only if $p$ is irreducible.

## 6. Normalized Rational Functions

We now state the proposition

(21) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative integral domain-like non trivial double loop structure and $z$ be a rational function of $L$. Then there exists a rational function $z_1$ of $L$ and there exists a non zero polynomial $z_2$ of $L$ such that

(i) $z = \langle z_2 * (z_1)_{\mathbf{1}},\, z_2 * (z_1)_{\mathbf{2}} \rangle$,

(ii) $z_1$ is irreducible, and

(iii) there exists a finite sequence $f$ of elements of Polynom-Ring $L$ such that $z_2 = \prod f$ and for every element $i$ of $\mathbb{N}$ such that $i \in \operatorname{dom} f$ there exists an element $x$ of $L$ such that $x$ is a common root of $z_{\mathbf{1}}$ and $z_{\mathbf{2}}$ and $f(i) = \mathrm{rpoly}(1, x)$.

Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $z$ be a rational function of $L$. The functor NF $z$ yielding a rational function of $L$ is defined by:

(Def. 17)(i) There exists a rational function $z_1$ of $L$ and there exists a non zero polynomial $z_2$ of $L$ such that $z = \langle z_2 * (z_1)_{\mathbf{1}},\, z_2 * (z_1)_{\mathbf{2}} \rangle$ and $z_1$ is irreducible and NF $z$ = the norm rational function of $z_1$ and there exists a finite sequence $f$ of elements of Polynom-Ring $L$ such that $z_2 = \prod f$ and for every element $i$ of $\mathbb{N}$ such that $i \in \operatorname{dom} f$ there exists an element $x$ of $L$ such that $x$ is a common root of $z_{\mathbf{1}}$ and $z_{\mathbf{2}}$ and $f(i) = \mathrm{rpoly}(1, x)$ if $z$ is non zero,

(ii) NF $z = 0.\, L$, otherwise.

Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $z$ be a rational function of $L$. Observe that $\mathrm{NF}\, z$ is normalized and irreducible.

Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $z$ be a non zero rational function of $L$. One can verify that $\mathrm{NF}\, z$ is non zero.

One can prove the following propositions:

(22) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure, $z$ be a non zero rational function of $L$, $z_1$ be a rational function of $L$, and $z_2$ be a non zero polynomial of $L$. Suppose that

(i) $z = \langle z_2 * (z_1)_{\mathbf{1}},\, z_2 * (z_1)_{\mathbf{2}} \rangle$,

(ii) $z_1$ is irreducible, and

(iii) there exists a finite sequence $f$ of elements of Polynom-Ring $L$ such that $z_2 = \prod f$ and for every element $i$ of $\mathbb{N}$ such that $i \in \mathrm{dom}\, f$ there exists an element $x$ of $L$ such that $x$ is a common root of $z_{\mathbf{1}}$ and $z_{\mathbf{2}}$ and $f(i) = \mathrm{rpoly}(1, x)$.

Then $\mathrm{NF}\, z =$ the norm rational function of $z_1$.

(23) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure. Then $\mathrm{NF}\, 0.\, L = 0.\, L$.

(24) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure. Then $\mathrm{NF}\, 1.\, L = 1.\, L$.

(25) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and $z$ be an irreducible non zero rational function of $L$. Then $\mathrm{NF}\, z =$ the norm rational function of $z$.

(26) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure, $z$ be a rational function of $L$, and $x$ be an element of $L$. Then $\mathrm{NF}\, \langle \mathrm{rpoly}(1, x) * z_{\mathbf{1}},\, \mathrm{rpoly}(1, x) * z_{\mathbf{2}} \rangle = \mathrm{NF}\, z$.

(27) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and $z$ be a rational function of $L$. Then $\mathrm{NF}\, \mathrm{NF}\, z = \mathrm{NF}\, z$.

(28) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible in-

tegral domain-like non degenerated double loop structure and $z$ be a non zero rational function of $L$. Then $z$ is irreducible if and only if there exists an element $a$ of $L$ such that $a \neq 0_L$ and $\langle a \cdot z_{\mathbf{1}},\, a \cdot z_{\mathbf{2}} \rangle = \mathrm{NF}\, z$.

## 7. Degree of Rational Functions

Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $z$ be a rational function of $L$. The functor degree($z$) yielding an integer is defined as follows:

(Def. 18) $\mathrm{degree}(z) = \max(\mathrm{degree}((\mathrm{NF}\, z)_{\mathbf{1}}), \mathrm{degree}((\mathrm{NF}\, z)_{\mathbf{2}}))$.

Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let $z$ be a rational function of $L$. We introduce $\deg z$ as a synonym of degree($z$).

Next we state two propositions:

(29) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and $z$ be a rational function of $L$. Then $\mathrm{degree}(z) \leq \max(\mathrm{degree}(z_{\mathbf{1}}), \mathrm{degree}(z_{\mathbf{2}}))$.

(30) Let $L$ be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and $z$ be a non zero rational function of $L$. Then $z$ is irreducible if and only if $\mathrm{degree}(z) = \max(\mathrm{degree}(z_{\mathbf{1}}), \mathrm{degree}(z_{\mathbf{2}}))$.

## 8. Evaluation of Rational Functions

Let $L$ be a field, let $z$ be a rational function of $L$, and let $x$ be an element of $L$. The functor eval($z, x$) yielding an element of $L$ is defined by:

(Def. 19) $\mathrm{eval}(z, x) = \frac{\mathrm{eval}(z_{\mathbf{1}}, x)}{\mathrm{eval}(z_{\mathbf{2}}, x)}$.

The following propositions are true:

(31) For every field $L$ and for every element $x$ of $L$ holds $\mathrm{eval}(0.\, L, x) = 0_L$.

(32) For every field $L$ and for every element $x$ of $L$ holds $\mathrm{eval}(1.\, L, x) = 1_L$.

(33) Let $L$ be a field, $p$, $q$ be rational functions of $L$, and $x$ be an element of $L$. If $\mathrm{eval}(p_{\mathbf{2}}, x) \neq 0_L$ and $\mathrm{eval}(q_{\mathbf{2}}, x) \neq 0_L$, then $\mathrm{eval}(p + q, x) = \mathrm{eval}(p, x) + \mathrm{eval}(q, x)$.

(34) Let $L$ be a field, $p$, $q$ be rational functions of $L$, and $x$ be an element of $L$. If $\mathrm{eval}(p_{\mathbf{2}}, x) \neq 0_L$ and $\mathrm{eval}(q_{\mathbf{2}}, x) \neq 0_L$, then $\mathrm{eval}(p * q, x) = \mathrm{eval}(p, x) \cdot \mathrm{eval}(q, x)$.

(35) Let $L$ be a field, $p$ be a rational function of $L$, and $x$ be an element of $L$. If $\mathrm{eval}(p_{\mathbf{2}}, x) \neq 0_L$, then eval(the norm rational function of $p$, $x$) $= \mathrm{eval}(p, x)$.

(36) Let $L$ be a field, $p$ be a rational function of $L$, and $x$ be an element of $L$. If $\mathrm{eval}(p_{\mathbf{2}}, x) \neq 0_L$, then $x$ is a common root of $p_{\mathbf{1}}$ and $p_{\mathbf{2}}$ or $\mathrm{eval}(\mathrm{NF}\, p, x) = \mathrm{eval}(p, x)$.

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