More on the Continuity of Real Functions¹

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Summary. In this article we demonstrate basic properties of the continuous functions from \mathbb{R} to \mathcal{R}^n which correspond to state space equations in control engineering.

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The terminology and notation used here have been introduced in the following articles: [3], [7], [17], [2], [4], [12], [13], [14], [16], [1], [5], [9], [15], [18], [10], [8], [20], [21], [19], [11], [22], and [6].

For simplicity, we use the following convention: n, i denote elements of \mathbb{N} , X, X_1 denote sets, r, p, s, x_0 , x_1 , x_2 denote real numbers, f, f_1 , f_2 denote partial functions from \mathbb{R} to \mathcal{R}^n , and h denotes a partial function from \mathbb{R} to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$.

Let us consider n, f, x_0 . We say that f is continuous in x_0 if and only if:

(Def. 1) There exists a partial function g from \mathbb{R} to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ such that f = g and g is continuous in x_0 .

We now state four propositions:

- (1) If h = f, then f is continuous in x_0 iff h is continuous in x_0 .
- (2) If $x_0 \in X$ and f is continuous in x_0 , then $f \upharpoonright X$ is continuous in x_0 .

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- (3) f is continuous in x_0 if and only if the following conditions are satisfied:
- (i) $x_0 \in \text{dom } f$, and
- (ii) for every r such that 0 < r there exists s such that 0 < s and for every x_1 such that $x_1 \in \text{dom } f$ and $|x_1 x_0| < s$ holds $|f_{x_1} f_{x_0}| < r$.
- (4) Let r be a real number, z be an element of \mathbb{R}^n , and w be a point of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose z = w. Then $\{y \in \mathbb{R}^n : |y z| < r\} = \{y; y \text{ ranges over points of } \langle \mathcal{E}^n, \| \cdot \| \rangle : \|y w\| < r\}$.

Let n be an element of \mathbb{N} , let Z be a set, and let f be a partial function from Z to \mathbb{R}^n . The functor |f| yielding a partial function from Z to \mathbb{R} is defined by:

(Def. 2) dom |f| = dom f and for every set x such that $x \in \text{dom } |f|$ holds $|f|_x = |f_x|$.

Let n be an element of \mathbb{N} , let Z be a non empty set, and let f be a partial function from Z to \mathbb{R}^n . The functor -f yields a partial function from Z to \mathbb{R}^n and is defined by:

(Def. 3) dom(-f) = dom f and for every set c such that $c \in dom(-f)$ holds $(-f)_c = -f_c$.

One can prove the following propositions:

- (5) Let f_1 , f_2 be partial functions from \mathbb{R} to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ and g_1 , g_2 be partial functions from \mathbb{R} to \mathcal{R}^n . If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 + f_2 = g_1 + g_2$.
- (6) Let f_1 be a partial function from \mathbb{R} to the carrier of $\langle \mathcal{E}^n, || \cdot || \rangle$, g_1 be a partial function from \mathbb{R} to \mathcal{R}^n , and a be a real number. If $f_1 = g_1$, then $a \cdot f_1 = a \cdot g_1$.
- (7) For every partial function f_1 from \mathbb{R} to \mathcal{R}^n holds $(-1) \cdot f_1 = -f_1$.
- (8) Let f_1 be a partial function from \mathbb{R} to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ and g_1 be a partial function from \mathbb{R} to \mathcal{R}^n . If $f_1 = g_1$, then $-f_1 = -g_1$.
- (9) Let f_1 be a partial function from \mathbb{R} to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ and g_1 be a partial function from \mathbb{R} to \mathcal{R}^n . If $f_1 = g_1$, then $\|f_1\| = |g_1|$.
- (10) Let f_1 , f_2 be partial functions from \mathbb{R} to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ and g_1 , g_2 be partial functions from \mathbb{R} to \mathcal{R}^n . If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 f_2 = g_1 g_2$.
- (11) f is continuous in x_0 if and only if the following conditions are satisfied:
 - (i) $x_0 \in \text{dom } f$, and
 - (ii) for every subset N_1 of \mathcal{R}^n such that there exists a real number r such that 0 < r and $\{y \in \mathcal{R}^n : |y f_{x_0}| < r\} = N_1$ there exists a neighbourhood N of x_0 such that for every x_1 such that $x_1 \in \text{dom } f$ and $x_1 \in N$ holds $f_{x_1} \in N_1$.
- (12) f is continuous in x_0 if and only if the following conditions are satisfied:
 - (i) $x_0 \in \text{dom } f$, and

- (ii) for every subset N_1 of \mathbb{R}^n such that there exists a real number r such that 0 < r and $\{y \in \mathbb{R}^n : |y f_{x_0}| < r\} = N_1$ there exists a neighbourhood N of x_0 such that $f^{\circ}N \subseteq N_1$.
- (13) If there exists a neighbourhood N of x_0 such that dom $f \cap N = \{x_0\}$, then f is continuous in x_0 .
- (14) If $x_0 \in \text{dom } f_1 \cap \text{dom } f_2$ and f_1 is continuous in x_0 and f_2 is continuous in x_0 , then $f_1 + f_2$ is continuous in x_0 .
- (15) If $x_0 \in \text{dom } f_1 \cap \text{dom } f_2$ and f_1 is continuous in x_0 and f_2 is continuous in x_0 , then $f_1 f_2$ is continuous in x_0 .
- (16) If f is continuous in x_0 , then $r \cdot f$ is continuous in x_0 .
- (17) If $x_0 \in \text{dom } f$ and f is continuous in x_0 , then |f| is continuous in x_0 .
- (18) If $x_0 \in \text{dom } f$ and f is continuous in x_0 , then -f is continuous in x_0 .
- (19) Let S be a real normed space, z be a point of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, f_1 be a partial function from \mathbb{R} to \mathcal{R}^n , and f_2 be a partial function from the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ to the carrier of S. Suppose $x_0 \in \text{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and $z = (f_1)_{x_0}$ and f_2 is continuous in z. Then $f_2 \cdot f_1$ is continuous in x_0 .
- (20) Let S be a real normed space, f_1 be a partial function from \mathbb{R} to the carrier of S, and f_2 be a partial function from the carrier of S to \mathbb{R} . Suppose $x_0 \in \text{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and f_2 is continuous in $(f_1)_{x_0}$. Then $f_2 \cdot f_1$ is continuous in x_0 .

Let us consider n, let f be a partial function from \mathbb{R}^n to \mathbb{R} , and let x_0 be an element of \mathbb{R}^n . We say that f is continuous in x_0 if and only if the condition (Def. 4) is satisfied.

(Def. 4) There exists a point y_0 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ and there exists a partial function g from the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ to \mathbb{R} such that $x_0 = y_0$ and f = g and g is continuous in y_0 .

One can prove the following two propositions:

- (21) Let f be a partial function from \mathbb{R}^n to \mathbb{R} , h be a partial function from the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ to \mathbb{R} , x_0 be an element of \mathbb{R}^n , and y_0 be a point of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose f = h and $x_0 = y_0$. Then f is continuous in x_0 if and only if h is continuous in y_0 .
- (22) Let f_1 be a partial function from \mathbb{R} to \mathcal{R}^n and f_2 be a partial function from \mathcal{R}^n to \mathbb{R} . Suppose $x_0 \in \text{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and f_2 is continuous in $(f_1)_{x_0}$. Then $f_2 \cdot f_1$ is continuous in x_0 .

Let us consider n, f. We say that f is continuous if and only if:

(Def. 5) For every x_0 such that $x_0 \in \text{dom } f$ holds f is continuous in x_0 . One can prove the following propositions:

- (23) Let g be a partial function from \mathbb{R} to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ and f be a partial function from \mathbb{R} to \mathcal{R}^n . If g = f, then g is continuous iff f is continuous.
- (24) Suppose $X \subseteq \text{dom } f$. Then $f \upharpoonright X$ is continuous if and only if for all x_0 , r such that $x_0 \in X$ and 0 < r there exists s such that 0 < s and for every x_1 such that $x_1 \in X$ and $|x_1 x_0| < s$ holds $|f_{x_1} f_{x_0}| < r$.

Let us consider n. Observe that every partial function from \mathbb{R} to \mathcal{R}^n which is constant is also continuous.

Let us consider n. Observe that there exists a partial function from \mathbb{R} to \mathcal{R}^n which is continuous.

Let us consider n, let f be a continuous partial function from \mathbb{R} to \mathcal{R}^n , and let X be a set. One can verify that $f \upharpoonright X$ is continuous.

One can prove the following proposition

(25) If $f \upharpoonright X$ is continuous and $X_1 \subseteq X$, then $f \upharpoonright X_1$ is continuous.

Let us consider n. Note that every partial function from \mathbb{R} to \mathcal{R}^n which is empty is also continuous.

Let us consider n, f and let X be a trivial set. One can verify that $f \upharpoonright X$ is continuous.

Let us consider n and let f_1 , f_2 be continuous partial functions from \mathbb{R} to \mathbb{R}^n . One can check that $f_1 + f_2$ is continuous.

The following propositions are true:

- (26) If $X \subseteq \text{dom } f_1 \cap \text{dom } f_2$ and $f_1 \upharpoonright X$ is continuous and $f_2 \upharpoonright X$ is continuous, then $(f_1 + f_2) \upharpoonright X$ is continuous and $(f_1 f_2) \upharpoonright X$ is continuous.
- (27) If $X \subseteq \text{dom } f_1$ and $X_1 \subseteq \text{dom } f_2$ and $f_1 \upharpoonright X$ is continuous and $f_2 \upharpoonright X_1$ is continuous, then $(f_1 + f_2) \upharpoonright (X \cap X_1)$ is continuous and $(f_1 f_2) \upharpoonright (X \cap X_1)$ is continuous.

Let us consider n, let f be a continuous partial function from \mathbb{R} to \mathcal{R}^n , and let us consider r. Observe that $r \cdot f$ is continuous.

The following propositions are true:

- (28) If $X \subseteq \text{dom } f$ and $f \upharpoonright X$ is continuous, then $(r \cdot f) \upharpoonright X$ is continuous.
- (29) If $X \subseteq \text{dom } f$ and $f \upharpoonright X$ is continuous, then $|f| \upharpoonright X$ is continuous and $(-f) \upharpoonright X$ is continuous.
- (30) If f is total and for all x_1 , x_2 holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists x_0 such that f is continuous in x_0 , then $f \mid \mathbb{R}$ is continuous.
- (31) For every subset Y of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ such that dom f is compact and $f \upharpoonright \text{dom } f$ is continuous and Y = rng f holds Y is compact.
- (32) Let Y be a subset of \mathbb{R} and Z be a subset of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $Y \subseteq \text{dom } f$ and $Z = f^{\circ}Y$ and Y is compact and $f \upharpoonright Y$ is continuous. Then Z is compact.

Let us consider n, f. We say that f is Lipschitzian if and only if:

(Def. 6) There exists a partial function g from \mathbb{R} to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ such that g = f and g is Lipschitzian.

The following propositions are true:

- (33) f is Lipschitzian if and only if there exists a real number r such that 0 < r and for all x_1, x_2 such that $x_1, x_2 \in \text{dom } f$ holds $|f_{x_1} f_{x_2}| \le r \cdot |x_1 x_2|$.
- (34) If f = h, then f is Lipschitzian iff h is Lipschitzian.
- (35) $f \upharpoonright X$ is Lipschitzian if and only if there exists a real number r such that 0 < r and for all x_1, x_2 such that $x_1, x_2 \in \text{dom}(f \upharpoonright X)$ holds $|f_{x_1} f_{x_2}| \le r \cdot |x_1 x_2|$.

Let us consider n. Note that every partial function from \mathbb{R} to \mathcal{R}^n which is empty is also Lipschitzian.

Let us consider n. Note that there exists a partial function from \mathbb{R} to \mathcal{R}^n which is empty.

Let us consider n, let f be a Lipschitzian partial function from \mathbb{R} to \mathcal{R}^n , and let X be a set. Note that $f \upharpoonright X$ is Lipschitzian.

We now state the proposition

(36) If $f \upharpoonright X$ is Lipschitzian and $X_1 \subseteq X$, then $f \upharpoonright X_1$ is Lipschitzian.

Let us consider n and let f_1 , f_2 be Lipschitzian partial functions from \mathbb{R} to \mathbb{R}^n . Observe that $f_1 + f_2$ is Lipschitzian and $f_1 - f_2$ is Lipschitzian.

We now state two propositions:

- (37) If $f_1 \upharpoonright X$ is Lipschitzian and $f_2 \upharpoonright X_1$ is Lipschitzian, then $(f_1 + f_2) \upharpoonright (X \cap X_1)$ is Lipschitzian.
- (38) If $f_1 \upharpoonright X$ is Lipschitzian and $f_2 \upharpoonright X_1$ is Lipschitzian, then $(f_1 f_2) \upharpoonright (X \cap X_1)$ is Lipschitzian.

Let us consider n, let f be a Lipschitzian partial function from \mathbb{R} to \mathbb{R}^n , and let us consider p. Observe that $p \cdot f$ is Lipschitzian.

Next we state the proposition

(39) If $f \upharpoonright X$ is Lipschitzian and $X \subseteq \text{dom } f$, then $(p \cdot f) \upharpoonright X$ is Lipschitzian.

Let us consider n and let f be a Lipschitzian partial function from \mathbb{R} to \mathcal{R}^n . Observe that |f| is Lipschitzian.

Next we state the proposition

(40) If $f \upharpoonright X$ is Lipschitzian, then $-f \upharpoonright X$ is Lipschitzian and $|f| \upharpoonright X$ is Lipschitzian and $(-f) \upharpoonright X$ is Lipschitzian.

Let us consider n. One can check that every partial function from \mathbb{R} to \mathcal{R}^n which is constant is also Lipschitzian.

Let us consider n. One can verify that every partial function from \mathbb{R} to \mathcal{R}^n which is Lipschitzian is also continuous.

The following propositions are true:

(41) For all elements r, p of \mathbb{R}^n such that for every x_0 such that $x_0 \in X$ holds $f_{x_0} = x_0 \cdot r + p$ holds $f \upharpoonright X$ is continuous.

- (42) For every element x_0 of \mathbb{R}^n such that $1 \leq i \leq n$ holds $\operatorname{proj}(i, n)$ is continuous in x_0 .
- (43) Let n be a non empty element of \mathbb{N} and h be a partial function from \mathbb{R} to \mathbb{R}^n . Then h is continuous in x_0 if and only if the following conditions are satisfied:
 - (i) $x_0 \in \text{dom } h$, and
 - (ii) for every element i of \mathbb{N} such that $i \in \operatorname{Seg} n$ holds $\operatorname{proj}(i, n) \cdot h$ is continuous in x_0 .
- (44) Let n be a non empty element of \mathbb{N} and h be a partial function from \mathbb{R} to \mathbb{R}^n . Then h is continuous if and only if for every element i of \mathbb{N} such that $i \in \text{Seg } n$ holds $\text{proj}(i, n) \cdot h$ is continuous.
- (45) For every point x_0 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ such that $1 \leq i \leq n$ holds $\operatorname{Proj}(i, n)$ is continuous in x_0 .
- (46) Let n be a non empty element of \mathbb{N} and h be a partial function from \mathbb{R} to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Then h is continuous in x_0 if and only if for every element i of \mathbb{N} such that $i \in \operatorname{Seg} n$ holds $\operatorname{Proj}(i, n) \cdot h$ is continuous in x_0 .
- (47) Let n be a non empty element of \mathbb{N} and h be a partial function from \mathbb{R} to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Then h is continuous if and only if for every element i of \mathbb{N} such that $i \in \operatorname{Seg} n$ holds $\operatorname{Proj}(i, n) \cdot h$ is continuous.

References

- [1] Grzegorz Bancerek. The ordinal numbers. Formalized Mathematics, 1(1):91-96, 1990.
- [2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. Formalized Mathematics, 1(1):107–114, 1990.
- [3] Czesław Byliński. The complex numbers. Formalized Mathematics, 1(3):507–513, 1990.
- [4] Czesław Byliński. Functions and their basic properties. Formalized Mathematics, 1(1):55–65, 1990.
- [5] Czesław Byliński. Partial functions. Formalized Mathematics, 1(2):357–367, 1990.
- [6] Czesław Byliński. Some basic properties of sets. Formalized Mathematics, 1(1):47–53,
- [7] Agata Darmochwał. The Euclidean space. Formalized Mathematics, 2(4):599–603, 1991.
- [8] Noboru Endou and Yasunari Shidama. Completeness of the real Euclidean space. Formalized Mathematics, 13(4):577–580, 2005.
- [9] Noboru Endou, Yasunari Shidama, and Keiichi Miyajima. Partial differentiation on normed linear spaces \mathbb{R}^n . Formalized Mathematics, 15(2):65–72, 2007, doi:10.2478/v10037-007-0008-5.
- [10] Krzysztof Hryniewiecki. Basic properties of real numbers. Formalized Mathematics, 1(1):35–40, 1990.
- [11] Artur Korniłowicz. Arithmetic operations on functions from sets into functional sets. Formalized Mathematics, 17(1):43–60, 2009, doi:10.2478/v10037-009-0005-y.
- [12] Keiichi Miyajima and Yasunari Shidama. Riemann integral of functions from \mathbb{R} into \mathcal{R}^n . Formalized Mathematics, 17(2):179–185, 2009, doi: 10.2478/v10037-009-0021-y.
- [13] Takaya Nishiyama, Keiji Ohkubo, and Yasunari Shidama. The continuous functions on normed linear spaces. Formalized Mathematics, 12(3):269–275, 2004.
- [14] Hiroyuki Okazaki, Noboru Endou, and Yasunari Shidama. More on continuous functions on normed linear spaces. Formalized Mathematics, 19(1):45–49, 2011, doi: 10.2478/v10037-011-0008-3.

- [15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. Formalized Mathematics, 1(1):223-230, 1990.
- [16] Jan Popiołek. Real normed space. Formalized Mathematics, 2(1):111–115, 1991.
- [17] Konrad Raczkowski and Paweł Sadowski. Real function continuity. Formalized Mathematics, 1(4):787-791, 1990.
- [18] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. Formalized Mathematics, 1(4):777-780, 1990.
- [19] Zinaida Trybulec. Properties of subsets. Formalized Mathematics, 1(1):67-71, 1990.
- [20] Edmund Woronowicz. Relations and their basic properties. Formalized Mathematics, 1(**1**):73-83, 1990.
- [21] Edmund Woronowicz. Relations defined on sets. Formalized Mathematics, 1(1):181–186,
- 1990. [22] Hiroshi Yamazaki and Yasunari Shidama. Algebra of vector functions. Formalized Mathematics, 3(2):171-175, 1992.

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