Veblen Hierarchy

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Summary. The Veblen hierarchy is an extension of the construction of epsilon numbers (fixpoints of the exponential map: $\omega^\epsilon = \epsilon$). It is a collection φ_α of the Veblen Functions where $\varphi_0(\beta) = \omega^\beta$ and $\varphi_1(\beta) = \epsilon_\beta$. The sequence of fixpoints of φ_1 function form φ_2, etc. For a limit non empty ordinal λ the function φ_λ is the sequence of common fixpoints of all functions φ_α where $\alpha < \lambda$.

The Mizar formalization of the concept cannot be done directly as the Veblen functions are classes (not (small) sets). It is done with use of universal sets (Tarski classes). Namely, we define the Veblen functions in a given universal set and $\varphi_\alpha(\beta)$ as a value of Veblen function from the smallest universal set including α and β.

The papers [16], [18], [2], [5], [14], [13], [9], [10], [15], [3], [4], [1], [8], [11], [19], [17], [6], [7], and [12] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: α, β, γ, δ denote ordinal numbers, λ denotes a non empty limit ordinal ordinal number, A denotes a non empty ordinal number, e denotes an element of A, X, Y, x, y denote sets, and n denotes a natural number.

Next we state several propositions:

(1) Let φ be a function. Suppose φ is an isomorphism between $\subseteq_X$ and $\subseteq_Y$. Let given x, y. If x, $y \in X$, then $x \subseteq y$ iff $\varphi(x) \subseteq \varphi(y)$.

(2) Let X, Y be ordinal-membered sets and φ be a function. Suppose φ is an isomorphism between $\subseteq_X$ and $\subseteq_Y$. Let given x, y. If x, $y \in X$, then $x \in y$ iff $\varphi(x) \in \varphi(y)$.

(3)   If $\langle x, y \rangle \in {}^{\subseteq}{}_X$, then $x \subseteq y$.

(4)   For all transfinite sequences f_1, f_2 holds $f_1 \subseteq f_1 \frown f_2$.

(5)   For all transfinite sequences f_1, f_2 holds $\mathrm{rng}(f_1 \frown f_2) = \mathrm{rng}\, f_1 \cup \mathrm{rng}\, f_2$.

(6) $\alpha \subseteq \beta$ iff $\varepsilon_\alpha \subseteq \varepsilon_\beta$.

(7) $\alpha \in \beta$ iff $\varepsilon_\alpha \in \varepsilon_\beta$.

Let X be an ordinal-membered set. Note that $\bigcup X$ is ordinal.

Let φ be an ordinal yielding function. Observe that $\mathrm{rng}\, \varphi$ is ordinal-membered.

Let us consider α. Note that id_α is transfinite sequence-like and ordinal yielding.

Let us consider α. Observe that id_α is increasing.

Let us consider α. Note that id_α is continuous.

Let us observe that there exists a sequence of ordinal numbers which is non empty, increasing, and continuous.

Let φ be a transfinite sequence. We say that φ is normal if and only if:

(Def. 1) φ is an increasing continuous sequence of ordinal numbers.

Let φ be a sequence of ordinal numbers. Let us observe that φ is normal if and only if:

(Def. 2) φ is increasing and continuous.

One can verify the following observations:

∗   every transfinite sequence which is normal is also ordinal yielding,

∗   every sequence of ordinal numbers which is normal is also increasing and continuous, and

∗   every sequence of ordinal numbers which is increasing and continuous is also normal.

Let us observe that there exists a transfinite sequence which is non empty and normal.

Next we state the proposition

(8)   For every sequence φ of ordinal numbers such that φ is non-decreasing holds $\varphi{\restriction}\alpha$ is non-decreasing.

Let us consider X. The functor ord-type X yields an ordinal number and is defined by:

(Def. 3)   ord-type $X = \overline{\overline{{}^{\subseteq}{}_{\mathrm{On}\, X}}}$.

Let X be an ordinal-membered set. Then ord-type X can be characterized by the condition:

(Def. 4)   ord-type $X = \overline{\overline{{}^{\subseteq}{}_X}}$.

Let X be an ordinal-membered set. One can verify that ${}^{\subseteq}{}_X$ is well-ordering.

Let E be an empty set. Observe that $\mathrm{On}\, E$ is empty.

Let E be an empty set. One can verify that $\overline{\overline{E}}$ is empty.

Next we state four propositions:

(9)   ord-type $\emptyset = 0$.

(10)   ord-type$\{\alpha\} = 1$.

(11)   If $\alpha \neq \beta$, then ord-type$\{\alpha, \beta\} = 2$.

(12)   ord-type $\alpha = \alpha$.

Let us consider X. The functor numbering X yields a sequence of ordinal numbers and is defined as follows:

(Def. 5)   numbering X = the canonical isomorphism between $\subseteq_{\text{ord-type} X}$ and $\subseteq_{\text{On} X}$.

Next we state four propositions:

(13)   dom numbering X = ord-type X and rng numbering X = On X.

(14)   For every ordinal-membered set X holds rng numbering $X = X$.

(15)   Card dom numbering X = Card On X.

(16)   numbering X is an isomorphism between $\subseteq_{\text{ord-type} X}$ and $\subseteq_{\text{On} X}$.

In the sequel φ denotes a sequence of ordinal numbers.

One can prove the following propositions:

(17)   If φ = numbering X and $x, y \in \text{dom } \varphi$, then $x \subseteq y$ iff $\varphi(x) \subseteq \varphi(y)$.

(18)   If φ = numbering X and $x, y \in \text{dom } \varphi$, then $x \in y$ iff $\varphi(x) \in \varphi(y)$.

Let us consider X. Note that numbering X is increasing.

Let X, Y be ordinal-membered sets. One can check that $X \cup Y$ is ordinal-membered.

Let X be an ordinal-membered set and let Y be a set. Observe that $X \setminus Y$ is ordinal-membered.

The following three propositions are true:

(19)   Let X, Y be ordinal-membered sets. Suppose that for all x, y such that $x \in X$ and $y \in Y$ holds $x \in y$. Then (numbering X) $^\frown$ numbering Y is an isomorphism between $\subseteq_{\text{ord-type} X + \text{ord-type} Y}$ and $\subseteq_{X \cup Y}$.

(20)   For all ordinal-membered sets X, Y such that for all x, y such that $x \in X$ and $y \in Y$ holds $x \in y$ holds numbering$(X \cup Y)$ = (numbering X)$^\frown$ numbering Y.

(21)   For all ordinal-membered sets X, Y such that for all x, y such that $x \in X$ and $y \in Y$ holds $x \in y$ holds ord-type$(X \cup Y)$ = ord-type X + ord-type Y.

2. Fixpoints of a Normal Function

Next we state the proposition

(27)   For every function φ such that x is a fixpoint of φ holds $x \in \text{rng } \varphi$.

Let φ be a sequence of ordinal numbers. The functor criticals φ yields a sequence of ordinal numbers and is defined by:

(Def. 7)   criticals φ = numbering$\{\alpha \in \text{dom}\,\varphi : \alpha$ is a fixpoint of $\varphi\}$.

We now state three propositions:

(28)   On$\{\alpha \in \text{dom}\,\varphi : \alpha$ is a fixpoint of $\varphi\} = \{\alpha \in \text{dom}\,\varphi : \alpha$ is a fixpoint of $\varphi\}$.

(29)   If $x \in \text{dom}\,\text{criticals}\,\varphi$, then $(\text{criticals}\,\varphi)(x)$ is a fixpoint of φ.

(30)   rng criticals $\varphi = \{\alpha \in \text{dom}\,\varphi : \alpha$ is a fixpoint of $\varphi\}$ and rng criticals $\varphi \subseteq$ rng φ.

Let us consider φ. One can verify that criticals φ is increasing.

We now state several propositions:

(31)   If $x \in \text{dom}\,\text{criticals}\,\varphi$, then $x \subseteq (\text{criticals}\,\varphi)(x)$.

(32)   dom criticals $\varphi \subseteq \text{dom}\,\varphi$.

(33)   If β is a fixpoint of φ, then there exists α such that $\alpha \in \text{dom}\,\text{criticals}\,\varphi$ and $\beta = (\text{criticals}\,\varphi)(\alpha)$.

(34)   If $\alpha \in \text{dom}\,\text{criticals}\,\varphi$ and β is a fixpoint of φ and $(\text{criticals}\,\varphi)(\alpha) \in \beta$, then succ $\alpha \in \text{dom}\,\text{criticals}\,\varphi$.

(35)   If succ $\alpha \in \text{dom}\,\text{criticals}\,\varphi$ and β is a fixpoint of φ and $(\text{criticals}\,\varphi)(\alpha) \in \beta$, then $(\text{criticals}\,\varphi)(\text{succ}\,\alpha) \subseteq \beta$.

(36)   Suppose φ is normal and $\bigcup X \in \text{dom}\,\varphi$ and X is non empty and for every x such that $x \in X$ there exists y such that $x \subseteq y$ and $y \in X$ and y is a fixpoint of φ. Then $\bigcup X = \varphi(\bigcup X)$.

(37)   If φ is normal and $\bigcup X \in \text{dom}\,\varphi$ and X is non empty and for every x such that $x \in X$ holds x is a fixpoint of φ, then $\bigcup X = \varphi(\bigcup X)$.

(38)   If $\lambda \subseteq \text{dom}\,\text{criticals}\,\varphi$ and α is a fixpoint of φ and for every x such that $x \in \lambda$ holds $(\text{criticals}\,\varphi)(x) \in \alpha$, then $\lambda \in \text{dom}\,\text{criticals}\,\varphi$.

(39)   If φ is normal and $\lambda \in \text{dom}\,\text{criticals}\,\varphi$, then $(\text{criticals}\,\varphi)(\lambda) = \bigcup((\text{criticals}\,\varphi){\upharpoonright}\lambda)$.

Let φ be a normal sequence of ordinal numbers. Observe that criticals φ is continuous.

Next we state the proposition

(40)   For all sequences f_1, f_2 of ordinal numbers such that $f_1 \subseteq f_2$ holds criticals $f_1 \subseteq$ criticals f_2.

3. Fixpoints in a Universal Set

In the sequel U, W are universal classes.

Let us consider U. One can check that there exists a transfinite sequence of ordinals of U which is normal.

Let us consider U, α. An ordinal-sequence from α to U is a function from α into On U.

Let us consider U, α. Note that every ordinal-sequence from α to U is transfinite sequence-like and ordinal yielding.

Let us consider U, α, let φ be an ordinal-sequence from α to U, and let us consider x. Then $\varphi(x)$ is an ordinal of U.

The following two propositions are true:

(41)   If $\alpha \in U$, then for every ordinal-sequence φ from α to U holds $\bigcup \varphi \in U$.

(42)   If $\alpha \in U$, then for every ordinal-sequence φ from α to U holds $\sup \varphi \in U$.

In this article we present several logical schemes. The scheme *CriticalNumber2* deals with a universal class $\mathcal{A}$, an ordinal $\mathcal{B}$ of $\mathcal{A}$, an ordinal-sequence $\mathcal{C}$ from ω to $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding an ordinal number, and states that:

$\mathcal{B} \subseteq \bigcup \mathcal{C}$ and $\mathcal{F}(\bigcup \mathcal{C}) = \bigcup \mathcal{C}$ and for every β such that $\mathcal{B} \subseteq \beta$ and $\beta \in \mathcal{A}$ and $\mathcal{F}(\beta) = \beta$ holds $\bigcup \mathcal{C} \subseteq \beta$

provided the following conditions are met:

- $\omega \in \mathcal{A}$,
- For every α such that $\alpha \in \mathcal{A}$ holds $\mathcal{F}(\alpha) \in \mathcal{A}$,
- For all α, β such that $\alpha \in \beta$ and $\beta \in \mathcal{A}$ holds $\mathcal{F}(\alpha) \in \mathcal{F}(\beta)$,
- Let α be an ordinal of $\mathcal{A}$. Suppose α is non empty and limit ordinal. Let φ_1 be a sequence of ordinal numbers. If $\operatorname{dom} \varphi_1 = \alpha$ and for every β such that $\beta \in \alpha$ holds $\varphi_1(\beta) = \mathcal{F}(\beta)$, then $\mathcal{F}(\alpha)$ is the limit of φ_1,
- $\mathcal{C}(0) = \mathcal{B}$, and
- For every α such that $\alpha \in \omega$ holds $\mathcal{C}(\operatorname{succ} \alpha) = \mathcal{F}(\mathcal{C}(\alpha))$.

The scheme *CriticalNumber3* deals with a universal class $\mathcal{A}$, an ordinal $\mathcal{B}$ of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding an ordinal number, and states that:

There exists an ordinal α of $\mathcal{A}$ such that $\mathcal{B} \in \alpha$ and $\mathcal{F}(\alpha) = \alpha$

provided the following requirements are met:

- $\omega \in \mathcal{A}$,
- For every α such that $\alpha \in \mathcal{A}$ holds $\mathcal{F}(\alpha) \in \mathcal{A}$,
- For all α, β such that $\alpha \in \beta$ and $\beta \in \mathcal{A}$ holds $\mathcal{F}(\alpha) \in \mathcal{F}(\beta)$, and
- Let α be an ordinal of $\mathcal{A}$. Suppose α is non empty and limit ordinal. Let φ_1 be a sequence of ordinal numbers. If $\operatorname{dom} \varphi_1 = \alpha$ and for every β such that $\beta \in \alpha$ holds $\varphi_1(\beta) = \mathcal{F}(\beta)$, then $\mathcal{F}(\alpha)$ is the limit of φ_1.

In the sequel F, φ_1 denote normal transfinite sequences of ordinals of W.

One can prove the following propositions:

(43)   If ω, $\beta \in W$, then there exists α such that $\beta \in \alpha$ and α is a fixpoint of φ_1.

(44)   If $\omega \in W$, then criticals F is a transfinite sequence of ordinals of W.

(45)   If φ is normal, then for every α such that $\alpha \in \operatorname{dom} \operatorname{criticals} \varphi$ holds $\varphi(\alpha) \subseteq (\operatorname{criticals} \varphi)(\alpha)$.

4. Sequences of Sequences of Ordinals

Let us consider U and let α, β be ordinals of U. Then α^β is an ordinal of U.

Let us consider U, α. Let us assume that $\alpha \in U$. The functor $U \exp \alpha$ yields a transfinite sequence of ordinals of U and is defined as follows:

(Def. 8)   For every ordinal β of U holds $(U \exp \alpha)(\beta) = \alpha^\beta$.

Let us observe that ω is non trivial.

Let us consider U. Observe that there exists an ordinal of U which is non trivial and finite.

One can verify that there exists an ordinal number which is non trivial and finite.

Let us consider U and let α be a non trivial ordinal of U. Note that $U \exp \alpha$ is normal.

Let ψ be a function. We say that ψ is ordinal-sequence-valued if and only if:

(Def. 9)   If $x \in \operatorname{rng} \psi$, then x is a sequence of ordinal numbers.

Let φ be a sequence of ordinal numbers. Observe that $\langle \varphi \rangle$ is ordinal-sequence-valued.

Let φ be a function. We say that φ has the same dom if and only if:

(Def. 10) $\operatorname{rng} \varphi$ has common domain.

Let φ be a function. Observe that $\{\varphi\}$ has common domain.

Let φ be a function. One can verify that $\langle \varphi \rangle$ has the same dom.

One can verify that there exists a transfinite sequence which is non empty and ordinal-sequence-valued and has the same dom.

Let ψ be an ordinal-sequence-valued function and let us consider x. Observe that $\psi(x)$ is relation-like and function-like.

Let ψ be an ordinal-sequence-valued function and let us consider x. Observe that $\psi(x)$ is transfinite sequence-like and ordinal yielding.

Let ψ be a transfinite sequence and let us consider α. Note that $\psi\!\restriction\!\alpha$ is transfinite sequence-like.

Let ψ be an ordinal-sequence-valued function and let us consider X. One can check that $\psi\!\restriction\!X$ is ordinal-sequence-valued.

Let us consider α, β. Observe that every function from α into β is ordinal yielding and transfinite sequence-like.

Let ψ be an ordinal-sequence-valued transfinite sequence. The functor criticals ψ yields a sequence of ordinal numbers and is defined as follows:

(Def. 11)   criticals $\psi = \operatorname{numbering}\{\alpha \in \operatorname{dom} \psi(0) : \alpha \in \operatorname{dom} \psi(0) \ \wedge \ \bigwedge_\varphi (\varphi \in \operatorname{rng} \psi \ \Rightarrow \ \alpha$ is a fixpoint of $\varphi)\}$.

In the sequel ψ is an ordinal-sequence-valued transfinite sequence.

One can prove the following propositions:

(46)   Let given ψ. Then $\{\alpha \in \operatorname{dom} \psi(0) : \alpha \in \operatorname{dom} \psi(0) \wedge \bigwedge_\varphi (\varphi \in \operatorname{rng} \psi \ \Rightarrow \ \alpha$ is a fixpoint of $\varphi)\}$ is ordinal-membered.

(47) If $\alpha \in \operatorname{dom} \psi$ and $\beta \in \operatorname{dom} \operatorname{criticals} \psi$, then $(\operatorname{criticals} \psi)(\beta)$ is a fixpoint of $\psi(\alpha)$.

(48) If $x \in \operatorname{dom} \operatorname{criticals} \psi$, then $x \subseteq (\operatorname{criticals} \psi)(x)$.

(49) If $\varphi \in \operatorname{rng} \psi$, then $\operatorname{dom} \operatorname{criticals} \psi \subseteq \operatorname{dom} \varphi$.

(50) If $\operatorname{dom} \psi \neq \emptyset$ and for every γ such that $\gamma \in \operatorname{dom} \psi$ holds β is a fixpoint of $\psi(\gamma)$, then there exists α such that $\alpha \in \operatorname{dom} \operatorname{criticals} \psi$ and $\beta = (\operatorname{criticals} \psi)(\alpha)$.

(51) Suppose $\operatorname{dom} \psi \neq \emptyset$ and $\lambda \subseteq \operatorname{dom} \operatorname{criticals} \psi$ and for every φ such that $\varphi \in \operatorname{rng} \psi$ holds α is a fixpoint of φ and for every x such that $x \in \lambda$ holds $(\operatorname{criticals} \psi)(x) \in \alpha$. Then $\lambda \in \operatorname{dom} \operatorname{criticals} \psi$.

(52) For every ψ such that $\operatorname{dom} \psi \neq \emptyset$ and for every α such that $\alpha \in \operatorname{dom} \psi$ holds $\psi(\alpha)$ is normal holds if $\lambda \in \operatorname{dom} \operatorname{criticals} \psi$, then $(\operatorname{criticals} \psi)(\lambda) = \bigcup((\operatorname{criticals} \psi) {\restriction} \lambda)$.

(53) For every ψ such that $\operatorname{dom} \psi \neq \emptyset$ and for every α such that $\alpha \in \operatorname{dom} \psi$ holds $\psi(\alpha)$ is normal holds $\operatorname{criticals} \psi$ is continuous.

(54) Let given ψ. Suppose $\operatorname{dom} \psi \neq \emptyset$ and for every α such that $\alpha \in \operatorname{dom} \psi$ holds $\psi(\alpha)$ is normal. Let given α, φ. If $\alpha \in \operatorname{dom} \operatorname{criticals} \psi$ and $\varphi \in \operatorname{rng} \psi$, then $\varphi(\alpha) \subseteq (\operatorname{criticals} \psi)(\alpha)$.

(55) Let g_1, g_2 be ordinal-sequence-valued transfinite sequences. If $\operatorname{dom} g_1 = \operatorname{dom} g_2$ and $\operatorname{dom} g_1 \neq \emptyset$ and for every α such that $\alpha \in \operatorname{dom} g_1$ holds $g_1(\alpha) \subseteq g_2(\alpha)$, then $\operatorname{criticals} g_1 \subseteq \operatorname{criticals} g_2$.

Let ψ be an ordinal-sequence-valued transfinite sequence. The functor $\operatorname{lims} \psi$ yielding a sequence of ordinal numbers is defined by:

(Def. 12) $\operatorname{dom} \operatorname{lims} \psi = \operatorname{dom} \psi(0)$ and for every α such that $\alpha \in \operatorname{dom} \operatorname{lims} \psi$ holds $(\operatorname{lims} \psi)(\alpha) = \bigcup\{\psi(\beta)(\alpha); \beta$ ranges over elements of $\operatorname{dom} \psi : \beta \in \operatorname{dom} \psi\}$.

Next we state the proposition

(56) Let ψ be an ordinal-sequence-valued transfinite sequence. Suppose $\operatorname{dom} \psi \neq \emptyset$ and $\operatorname{dom} \psi \in U$ and for every α such that $\alpha \in \operatorname{dom} \psi$ holds $\psi(\alpha)$ is a transfinite sequence of ordinals of U. Then $\operatorname{lims} \psi$ is a transfinite sequence of ordinals of U.

5. Veblen Hierarchy

Let us consider x. The functor OSx yields a sequence of ordinal numbers and is defined by:

(Def. 13) $OSx = \begin{cases} x, & \text{if } x \text{ is a sequence of ordinal numbers}, \\ \text{the sequence of ordinal numbers}, & \text{otherwise}. \end{cases}$

The functor $OSVx$ yielding an ordinal-sequence-valued transfinite sequence is defined by:

(Def. 14) $OSV\,x = \begin{cases} x, \text{ if } x \text{ is an ordinal-sequence-valued transfinite sequence,} \\ \quad \text{the ordinal-sequence-valued transfinite sequence, otherwise.} \end{cases}$

Let us consider U. The functor U-Veblen yields an ordinal-sequence-valued transfinite sequence and is defined by the conditions (Def. 15).

(Def. 15)(i) $\mathrm{dom}(U\text{-Veblen}) = \mathrm{On}\,U$,

(ii) $U\text{-Veblen}(0) = U\exp\omega$,

(iii)   for every β such that $\mathrm{succ}\,\beta \in \mathrm{On}\,U$ holds $U\text{-Veblen}(\mathrm{succ}\,\beta) = \mathrm{criticals}\,U\text{-Veblen}(\beta)$, and

(iv)   for every λ such that $\lambda \in \mathrm{On}\,U$ holds $U\text{-Veblen}(\lambda) = \mathrm{criticals}(U\text{-Veblen}{\restriction}\lambda)$.

Let us note that there exists a universal class which is uncountable.

One can prove the following propositions:

(57)   For every universal class U holds U is uncountable iff $\omega \in U$.

(58)   If $\alpha \in \beta$ and $\beta,\ \omega \in U$ and $\gamma \in \mathrm{dom}\,U\text{-Veblen}(\beta)$, then $U\text{-Veblen}(\beta)(\gamma)$ is a fixpoint of $U\text{-Veblen}(\alpha)$.

(59)   If $\lambda \in U$ and for every γ such that $\gamma \in \lambda$ holds α is a fixpoint of $U\text{-Veblen}(\gamma)$, then α is a fixpoint of $\mathrm{lims}(U\text{-Veblen}{\restriction}\lambda)$.

(60)   If $\alpha \subseteq \beta$ and $\beta,\ \omega \in U$ and $\gamma \in \mathrm{dom}\,U\text{-Veblen}(\beta)$ and for every γ such that $\gamma \in \beta$ holds $U\text{-Veblen}(\gamma)$ is normal, then $U\text{-Veblen}(\alpha)(\gamma) \subseteq U\text{-Veblen}(\beta)(\gamma)$.

(61)   Suppose $\lambda,\ \alpha \in U$ and $\beta \in \lambda$ and for every γ such that $\gamma \in \lambda$ holds $U\text{-Veblen}(\gamma)$ is a normal transfinite sequence of ordinals of U. Then $(\mathrm{lims}(U\text{-Veblen}{\restriction}\lambda))(\alpha)$ is a fixpoint of $U\text{-Veblen}(\beta)$.

(62)   If $\omega, \alpha \in U$, then $U\text{-Veblen}(\alpha)$ is a normal transfinite sequence of ordinals of U.

(63)   If $\omega \in U$ and $U \subseteq W$ and $\alpha \in U$, then $U\text{-Veblen}(\alpha) \subseteq W\text{-Veblen}(\alpha)$.

(64)   If $\omega,\ \alpha,\ \beta \in U$ and $\omega,\ \alpha,\ \beta \in W$, then $U\text{-Veblen}(\beta)(\alpha) = W\text{-Veblen}(\beta)(\alpha)$.

(65)   Suppose $\lambda \in U$ and for every α such that $\alpha \in \lambda$ holds $U\text{-Veblen}(\alpha)$ is a normal transfinite sequence of ordinals of U. Then $\mathrm{lims}(U\text{-Veblen}{\restriction}\lambda)$ is a non-decreasing continuous transfinite sequence of ordinals of U.

Let us consider α. Note that $\mathbf{T}(\alpha \cup \omega)$ is uncountable.

Let us consider $\alpha,\ \beta$. The functor $\varphi_\alpha(\beta)$ yields an ordinal number and is defined by:

(Def. 16) $\varphi_\alpha(\beta) = \mathbf{T}(\alpha \cup \beta \cup \omega)\text{-Veblen}(\alpha)(\beta)$.

Let us consider $n,\ \beta$. Then $\varphi_n(\beta)$ is an ordinal number and it can be characterized by the condition:

(Def. 17) $\varphi_n(\beta) = \mathbf{T}(\beta \cup \omega)\text{-Veblen}(n)(\beta)$.

One can prove the following propositions:

(66) $\alpha \in \mathbf{T}(\alpha \cup \beta \cup \gamma)$.

(67)   If ω, α, $\beta \in U$, then $\varphi_\beta(\alpha) = U\text{-Veblen}(\beta)(\alpha)$.

(68) $\varphi_0(\alpha) = \omega^\alpha$.

(69) $\varphi_\beta(\varphi_{\operatorname{succ}\beta}(\alpha)) = \varphi_{\operatorname{succ}\beta}(\alpha)$.

(70)   If $\beta \in \gamma$, then $\varphi_\beta(\varphi_\gamma(\alpha)) = \varphi_\gamma(\alpha)$.

(71) $\alpha \subseteq \beta$ iff $\varphi_\gamma(\alpha) \subseteq \varphi_\gamma(\beta)$.

(72) $\alpha \in \beta$ iff $\varphi_\gamma(\alpha) \in \varphi_\gamma(\beta)$.

(73) $\varphi_\alpha(\beta) \in \varphi_\gamma(\delta)$ iff $\alpha = \gamma$ and $\beta \in \delta$ or $\alpha \in \gamma$ and $\beta \in \varphi_\gamma(\delta)$ or $\gamma \in \alpha$ and $\varphi_\alpha(\beta) \in \delta$.

6. Epsilon Numbers

In the sequel U is an uncountable universal class.

Next we state four propositions:

(74) $U\text{-Veblen}(1) = \operatorname{criticals}(U \exp \omega)$.

(75) $\varphi_1(\alpha)$ is epsilon.

(76)   For every epsilon ordinal number e there exists α such that $e = \varphi_1(\alpha)$.

(77) $\varphi_1(\alpha) = \varepsilon_\alpha$.

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