Linear Transformations of Euclidean Topological Spaces

Karol Pąk
Institute of Informatics
University of Białystok
Poland

Summary. We introduce linear transformations of Euclidean topological spaces given by a transformation matrix. Next, we prove selected properties and basic arithmetic operations on these linear transformations. Finally, we show that a linear transformation given by an invertible matrix is a homeomorphism.

The papers [2], [12], [6], [26], [7], [8], [30], [21], [22], [23], [15], [31], [29], [19], [24], [3], [4], [9], [16], [5], [20], [18], [1], [14], [28], [13], [10], [25], [27], [11], and [17] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: X, Y denote sets, n, m, k, i denote natural numbers, r denotes a real number, R denotes an element of $\mathbb{R}_{\mathrm{F}}$, K denotes a field, f, f_1, f_2, g_1, g_2 denote finite sequences, r_1, r_2, r_3 denote real-valued finite sequences, c_1, c_2 denote complex-valued finite sequences, and F denotes a function.

Let us consider X, Y and let F be a positive yielding partial function from X to $\mathbb{R}$. One can check that $F{\restriction}Y$ is positive yielding.

Let us consider X, Y and let F be a negative yielding partial function from X to $\mathbb{R}$. One can verify that $F{\restriction}Y$ is negative yielding.

Let us consider X, Y and let F be a non-positive yielding partial function from X to $\mathbb{R}$. Note that $F{\restriction}Y$ is non-positive yielding.

Let us consider X, Y and let F be a non-negative yielding partial function from X to $\mathbb{R}$. Note that $F{\upharpoonright}Y$ is non-negative yielding.

Let us consider r_1. One can check that $\sqrt{r_1}$ is finite sequence-like.

Let us consider r_1. The functor ${}^{@}r_1$ yielding a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ is defined by:

(Def. 1) ${}^{@}r_1 = r_1$.

Let p be a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$. The functor ${}^{@}p$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 2) ${}^{@}p = p$.

We now state several propositions:

(1) $({}^{@}r_2) + {}^{@}r_3 = r_2 + r_3$.

(2) $\sqrt{r_2 \frown r_3} = \sqrt{r_2} \frown \sqrt{r_3}$.

(3) $\sqrt{\langle r \rangle} = \langle \sqrt{r} \rangle$.

(4) $\sqrt{r_1{}^2} = |r_1|$.

(5)   If r_1 is non-negative yielding, then $\sqrt{\sum r_1} \leq \sum \sqrt{r_1}$.

(6)   There exists X such that $X \subseteq \operatorname{dom} F$ and $\operatorname{rng} F = \operatorname{rng}(F{\upharpoonright}X)$ and $F{\upharpoonright}X$ is one-to-one.

Let us consider c_1, X. Observe that $c_1 - X$ is complex-valued.

Let us consider r_1, X. Observe that $r_1 - X$ is real-valued.

Let c_1 be a complex-valued finite subsequence. Note that $\operatorname{Seq} c_1$ is complex-valued.

Let r_1 be a real-valued finite subsequence. Observe that $\operatorname{Seq} r_1$ is real-valued.

One can prove the following propositions:

(7)   For every permutation P of $\operatorname{dom} f$ such that $f_1 = f \cdot P$ there exists a permutation Q of $\operatorname{dom}(f - X)$ such that $f_1 - X = (f - X) \cdot Q$.

(8)   For every permutation P of $\operatorname{dom} c_1$ such that $c_2 = c_1 \cdot P$ holds $\sum(c_2 - X) = \sum(c_1 - X)$.

(9)   Let f, f_1 be finite subsequences and P be a permutation of $\operatorname{dom} f$. If $f_1 = f \cdot P$, then there exists a permutation Q of $\operatorname{dom} \operatorname{Seq}(f_1{\upharpoonright}P^{-1}(X))$ such that $\operatorname{Seq}(f{\upharpoonright}X) = \operatorname{Seq}(f_1{\upharpoonright}P^{-1}(X)) \cdot Q$.

(10)   Let c_1, c_2 be complex-valued finite subsequences and P be a permutation of $\operatorname{dom} c_1$. If $c_2 = c_1 \cdot P$, then $\sum \operatorname{Seq}(c_1{\upharpoonright}X) = \sum \operatorname{Seq}(c_2{\upharpoonright}P^{-1}(X))$.

(11)   Let f be a finite subsequence and n be an element of $\mathbb{N}$. If for every i holds $i + n \in X$ iff $i \in Y$, then $\operatorname{Shift}^n f{\upharpoonright}X = \operatorname{Shift}^n(f{\upharpoonright}Y)$.

(12)   There exists a subset Y of $\mathbb{N}$ such that $\operatorname{Seq}((f_1 \frown f_2){\upharpoonright}X) = (\operatorname{Seq}(f_1{\upharpoonright}X)) \frown \operatorname{Seq}(f_2{\upharpoonright}Y)$ and for every n such that $n > 0$ holds $n \in Y$ iff $n + \operatorname{len} f_1 \in X \cap \operatorname{dom}(f_1 \frown f_2)$.

(13)   If $\operatorname{len} g_1 = \operatorname{len} f_1$ and $\operatorname{len} g_2 \leq \operatorname{len} f_2$, then $\operatorname{Seq}((f_1 \frown f_2){\upharpoonright}(g_1 \frown g_2)^{-1}(X)) = (\operatorname{Seq}(f_1{\upharpoonright}g_1{}^{-1}(X))) \frown \operatorname{Seq}(f_2{\upharpoonright}g_2{}^{-1}(X))$.

(14) Let D be a non empty set and M be a matrix over D of dimension $n \times m$. Then $M - X$ is a matrix over D of dimension $n -' \overline{\overline{M^{-1}(X)}} \times m$.

(15) Let F be a function from $\operatorname{Seg} n$ into $\operatorname{Seg} n$, D be a non empty set, M be a matrix over D of dimension $n \times m$, and given i. If $i \in \operatorname{Seg} \operatorname{width} M$, then $(M \cdot F)_{\square,i} = M_{\square,i} \cdot F$.

(16) Let A be a matrix over K of dimension $n \times m$. Suppose $\operatorname{rk}(A) = n$. Then there exists a matrix B over K of dimension $m -' n \times m$ such that $\operatorname{rk}(A ^\frown B) = m$.

(17) Let A be a matrix over K of dimension $n \times m$. Suppose $\operatorname{rk}(A) = m$. Then there exists a matrix B over K of dimension $n \times n -' m$ such that $\operatorname{rk}(A ^\frown B) = n$.

For simplicity, we adopt the following convention: f, f_1, f_2 denote n-element real-valued finite sequences, p, p_1, p_2 denote points of $\mathcal{E}_{\mathrm{T}}^n$, M, M_1, M_2 denote matrices over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$, and A, B denote square matrices over $\mathbb{R}_{\mathrm{F}}$ of dimension n.

2. Linear Transformations of Euclidean Topological Spaces Given by a Transformation Matrix

Let us consider n, m, M. The functor $\operatorname{Mx2Tran} M$ yielding a function from $\mathcal{E}_{\mathrm{T}}^n$ into $\mathcal{E}_{\mathrm{T}}^m$ is defined by:

(Def. 3)(i) $(\operatorname{Mx2Tran} M)(f) = \operatorname{Line}(\operatorname{LineVec2Mx}(^@ f) \cdot M, 1)$ if $n \neq 0$,

(ii) $(\operatorname{Mx2Tran} M)(f) = 0_{\mathcal{E}_{\mathrm{T}}^m}$, otherwise.

Let us consider n, m, M and let x be a set. One can check that $(\operatorname{Mx2Tran} M)(x)$ is function-like and relation-like and $(\operatorname{Mx2Tran} M)(x)$ is real-valued and finite sequence-like.

Let us consider n, m, M, f. One can check that $(\operatorname{Mx2Tran} M)(f)$ is m-element.

One can prove the following propositions:

(18) If $1 \leq i \leq m$ and $n \neq 0$, then $(\operatorname{Mx2Tran} M)(f)(i) = (^@ f) \cdot M_{\square,i}$.

(19) $\operatorname{len} \operatorname{MX2FinS}(I_K^{n \times n}) = n$.

(20) Let B_1 be an ordered basis of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$ and B_2 be an ordered basis of the m-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. Suppose $B_1 = \operatorname{MX2FinS}(I_{\mathbb{R}_{\mathrm{F}}}^{n \times n})$ and $B_2 = \operatorname{MX2FinS}(I_{\mathbb{R}_{\mathrm{F}}}^{m \times m})$. Let M_1 be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $\operatorname{len} B_1 \times \operatorname{len} B_2$. If $M_1 = M$, then $\operatorname{Mx2Tran} M = \operatorname{Mx2Tran}(M_1, B_1, B_2)$.

(21) For every permutation P of $\operatorname{Seg} n$ holds $(\operatorname{Mx2Tran} M)(f) = (\operatorname{Mx2Tran}(M \cdot P))(f \cdot P)$ and $f \cdot P$ is an n-element finite sequence of elements of $\mathbb{R}$.

(22) $(\operatorname{Mx2Tran} M)(f_1 + f_2) = (\operatorname{Mx2Tran} M)(f_1) + (\operatorname{Mx2Tran} M)(f_2)$.

(23) $(\mathrm{Mx2Tran}\,M)(r \cdot f) = r \cdot (\mathrm{Mx2Tran}\,M)(f)$.

(24) $(\mathrm{Mx2Tran}\,M)(f_1 - f_2) = (\mathrm{Mx2Tran}\,M)(f_1) - (\mathrm{Mx2Tran}\,M)(f_2)$.

(25) $(\mathrm{Mx2Tran}(M_1 + M_2))(f) = (\mathrm{Mx2Tran}\,M_1)(f) + (\mathrm{Mx2Tran}\,M_2)(f)$.

(26) $(R) \cdot (\mathrm{Mx2Tran}\,M)(f) = (\mathrm{Mx2Tran}(R \cdot M))(f)$.

(27) $(\mathrm{Mx2Tran}\,M)(p_1 + p_2) = (\mathrm{Mx2Tran}\,M)(p_1) + (\mathrm{Mx2Tran}\,M)(p_2)$.

(28) $(\mathrm{Mx2Tran}\,M)(p_1 - p_2) = (\mathrm{Mx2Tran}\,M)(p_1) - (\mathrm{Mx2Tran}\,M)(p_2)$.

(29) $(\mathrm{Mx2Tran}\,M)(0_{\mathcal{E}_\mathrm{T}^n}) = 0_{\mathcal{E}_\mathrm{T}^m}$.

(30) For every subset A of $\mathcal{E}_\mathrm{T}^n$ holds $(\mathrm{Mx2Tran}\,M)^\circ(p + A) = (\mathrm{Mx2Tran}\,M)(p) + (\mathrm{Mx2Tran}\,M)^\circ A$.

(31) For every subset A of $\mathcal{E}_\mathrm{T}^m$ holds $(\mathrm{Mx2Tran}\,M)^{-1}((\mathrm{Mx2Tran}\,M)(p)+A) = p + (\mathrm{Mx2Tran}\,M)^{-1}(A)$.

(32) Let A be a matrix over $\mathbb{R}_\mathrm{F}$ of dimension $n \times m$ and B be a matrix over $\mathbb{R}_\mathrm{F}$ of dimension width $A \times k$. If if $n = 0$, then $m = 0$ and if $m = 0$, then $k = 0$, then $\mathrm{Mx2Tran}\,B \cdot \mathrm{Mx2Tran}\,A = \mathrm{Mx2Tran}(A \cdot B)$.

(33) $\mathrm{Mx2Tran}(I_{\mathbb{R}_\mathrm{F}}^{n \times n}) = \mathrm{id}_{\mathcal{E}_\mathrm{T}^n}$.

(34) If $\mathrm{Mx2Tran}\,M_1 = \mathrm{Mx2Tran}\,M_2$, then $M_1 = M_2$.

(35) Let A be a matrix over $\mathbb{R}_\mathrm{F}$ of dimension $n \times m$ and B be a matrix over $\mathbb{R}_\mathrm{F}$ of dimension $k \times m$. Then $(\mathrm{Mx2Tran}(A \frown B))(f \frown (k \mapsto 0)) = (\mathrm{Mx2Tran}\,A)(f)$ and $(\mathrm{Mx2Tran}(B \frown A))((k \mapsto 0) \frown f) = (\mathrm{Mx2Tran}\,A)(f)$.

(36) Let A be a matrix over $\mathbb{R}_\mathrm{F}$ of dimension $n \times m$, B be a matrix over $\mathbb{R}_\mathrm{F}$ of dimension $k \times m$, and g be a k-element real-valued finite sequence. Then $(\mathrm{Mx2Tran}(A \frown B))(f \frown g) = (\mathrm{Mx2Tran}\,A)(f) + (\mathrm{Mx2Tran}\,B)(g)$.

(37) Let A be a matrix over $\mathbb{R}_\mathrm{F}$ of dimension $n \times k$ and B be a matrix over $\mathbb{R}_\mathrm{F}$ of dimension $n \times m$ such that if $n = 0$, then $k + m = 0$. Then $(\mathrm{Mx2Tran}(A \frown B))(f) = (\mathrm{Mx2Tran}\,A)(f) \frown (\mathrm{Mx2Tran}\,B)(f)$.

(38) $(\mathrm{Mx2Tran}(I_{\mathbb{R}_\mathrm{F}}^{m \times m} {\restriction} n))(f){\restriction}n = f$.

3. Selected Properties of the Mx2Tran Operator

Next we state several propositions:

(39) $\mathrm{Mx2Tran}\,M$ is one-to-one iff $\mathrm{rk}(M) = n$.

(40) $\mathrm{Mx2Tran}\,A$ is one-to-one iff $\mathrm{Det}\,A \neq 0_{\mathbb{R}_\mathrm{F}}$.

(41) $\mathrm{Mx2Tran}\,M$ is onto iff $\mathrm{rk}(M) = m$.

(42) $\mathrm{Mx2Tran}\,A$ is onto iff $\mathrm{Det}\,A \neq 0_{\mathbb{R}_\mathrm{F}}$.

(43) For all A, B such that $\mathrm{Det}\,A \neq 0_{\mathbb{R}_\mathrm{F}}$ holds $(\mathrm{Mx2Tran}\,A)^{-1} = \mathrm{Mx2Tran}\,B$ iff $A^\smile = B$.

(44) There exists an m-element finite sequence L of elements of $\mathbb{R}$ such that for every i such that $i \in \mathrm{dom}\,L$ holds $L(i) = |^{@}(M_{\square,i})|$ and for every f holds $|(\mathrm{Mx2Tran}\,M)(f)| \leq \sum L \cdot |f|$.

(45)   There exists a real number L such that $L > 0$ and for every f holds $|(\mathrm{Mx2Tran}\, M)(f)| \leq L \cdot |f|$.

(46)   If $\mathrm{rk}(M) = n$, then there exists a real number L such that $L > 0$ and for every f holds $|f| \leq L \cdot |(\mathrm{Mx2Tran}\, M)(f)|$.

(47) $\mathrm{Mx2Tran}\, M$ is continuous.

Let us consider n, K. One can check that there exists a square matrix over K of dimension n which is invertible.

Let us consider n and let A be an invertible square matrix over $\mathbb{R}_\mathrm{F}$ of dimension n. Note that $\mathrm{Mx2Tran}\, A$ is homeomorphism.

References

[1] Jesse Alama. The rank+nullity theorem. *Formalized Mathematics*, 15(**3**):137–142, 2007, doi:10.2478/v10037-007-0015-6.

[2] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[4] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[5] Grzegorz Bancerek, Mitsuru Aoki, Akio Matsumoto, and Yasunari Shidama. Processes in Petri nets. *Formalized Mathematics*, 11(**1**):125–132, 2003.

[6] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[10] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.

[11] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.

[12] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[13] Noboru Endou, Takashi Mitsuishi, and Yasunari Shidama. Dimension of real unitary space. *Formalized Mathematics*, 11(**1**):23–28, 2003.

[14] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[15] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.

[16] Artur Korniłowicz. On the real valued functions. *Formalized Mathematics*, 13(**1**):181–187, 2005.

[17] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[18] Anna Lango and Grzegorz Bancerek. Product of families of groups and vector spaces. *Formalized Mathematics*, 3(**2**):235–240, 1992.

[19] Robert Milewski. Associated matrix of linear map. *Formalized Mathematics*, 5(**3**):339–345, 1996.

[20] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[21] Karol Pąk. Basic properties of determinants of square matrices over a field. *Formalized Mathematics*, 15(**1**):17–25, 2007, doi:10.2478/v10037-007-0003-x.

[22] Karol Pąk. Basic properties of the rank of matrices over a field. *Formalized Mathematics*, 15(**4**):199–211, 2007, doi:10.2478/v10037-007-0024-5.

[23] Karol Pąk. Solutions of linear equations. *Formalized Mathematics*, 16(**1**):81–90, 2008, doi:10.2478/v10037-008-0012-4.

[24] Karol Pąk. Linear map of matrices. *Formalized Mathematics*, 16(**3**):269–275, 2008, doi:10.2478/v10037-008-0032-0.

[25] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[26] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.

[27] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[28] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[29] Xiaopeng Yue, Xiquan Liang, and Zhongpin Sun. Some properties of some special matrices. *Formalized Mathematics*, 13(**4**):541–547, 2005.

[30] Katarzyna Zawadzka. The sum and product of finite sequences of elements of a field. *Formalized Mathematics*, 3(**2**):205–211, 1992.

[31] Katarzyna Zawadzka. The product and the determinant of matrices with entries in a field. *Formalized Mathematics*, 4(**1**):1–8, 1993.