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Sorting by Exchanging

Grzegorz Bancerek
Białystok Technical University
Poland

Summary. We show that exchanging of pairs in an array which are in incorrect order leads to sorted array. It justifies correctness of Bubble Sort, Insertion Sort, and Quicksort.

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The notation and terminology used here have been introduced in the following papers: [20], [6], [11], [1], [8], [16], [12], [13], [10], [9], [17], [18], [3], [4], [2], [7], [14], [21], [22], [19], [5], and [15].

1. Preliminaries

We adopt the following convention: α, β, γ, δ denote ordinal numbers, k denotes a natural number, and x, y, z, t, X, Y, Z denote sets.

The following propositions are true:

(1) $x \in (\alpha + \beta) \setminus \alpha$ iff there exists γ such that $x = \alpha + \gamma$ and $\gamma \in \beta$.

(2) Suppose $\alpha \in \beta$ and $\gamma \in \delta$. Then $\gamma \neq \alpha$ and $\gamma \neq \beta$ and $\delta \neq \alpha$ and $\delta \neq \beta$ or $\gamma \in \alpha$ and $\delta = \alpha$ or $\gamma \in \alpha$ and $\delta = \beta$ or $\gamma = \alpha$ and $\delta \in \beta$ or $\gamma = \alpha$ and $\delta = \beta$ or $\gamma = \alpha$ and $\beta \in \delta$ or $\alpha \in \gamma$ and $\delta = \beta$ or $\gamma = \beta$ and $\beta \in \delta$.

(3) If $x \notin y$, then $(y \cup \{x\}) \setminus y = \{x\}$.

(4) $\operatorname{succ} x \setminus x = \{x\}$.

(5) Let f be a function, r be a binary relation, and given x. Then $x \in f^{\circ} r$ if and only if there exist y, z such that $\langle y, z \rangle \in r$ and $\langle y, z \rangle \in \operatorname{dom} f$ and $f(y, z) = x$.

(6) If $\alpha \setminus \beta \neq \emptyset$, then $\inf(\alpha \setminus \beta) = \beta$ and $\sup(\alpha \setminus \beta) = \alpha$ and $\bigcup(\alpha \setminus \beta) = \bigcup \alpha$.

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(7) If $\alpha \setminus \beta$ is non empty and finite, then there exists a natural number n such that $\alpha = \beta + n$.

2. Arrays

Let f be a set. We say that f is segmental if and only if:

(Def. 1) There exist α, β such that $\pi_1(f) = \alpha \setminus \beta$.

In the sequel f, g denote functions.

The following two propositions are true:

(8) If $\operatorname{dom} f = \operatorname{dom} g$ and f is segmental, then g is segmental.

(9) If f is segmental, then for all α, β, γ such that $\alpha \subseteq \beta \subseteq \gamma$ and α, $\gamma \in \operatorname{dom} f$ holds $\beta \in \operatorname{dom} f$.

Let us observe that every function which is transfinite sequence-like is also segmental and every function which is finite sequence-like is also segmental.

Let us consider α and let s be a set. We say that s is α-based if and only if:

(Def. 2) If $\beta \in \pi_1(s)$, then $\alpha \in \pi_1(s)$ and $\alpha \subseteq \beta$.

We say that s is α-limited if and only if:

(Def. 3) $\alpha = \sup \pi_1(s)$.

Next we state two propositions:

(10) f is α-based and segmental iff there exists β such that $\operatorname{dom} f = \beta \setminus \alpha$ and $\alpha \subseteq \beta$.

(11) f is β-limited, non empty, and segmental iff there exists α such that $\operatorname{dom} f = \beta \setminus \alpha$ and $\alpha \in \beta$.

Let us observe that every function which is transfinite sequence-like is also 0-based and every function which is finite sequence-like is also 1-based.

The following three propositions are true:

(12) f is $\inf \operatorname{dom} f$-based.

(13) f is $\sup \operatorname{dom} f$-limited.

(14) If f is β-limited and $\alpha \in \operatorname{dom} f$, then $\alpha \in \beta$.

Let us consider f. The functor $\operatorname{base} f$ yielding an ordinal number is defined as follows:

(Def. 4)(i) f is $\operatorname{base} f$-based if there exists α such that $\alpha \in \operatorname{dom} f$,

(ii) $\operatorname{base} f = 0$, otherwise.

The functor $\operatorname{limit} f$ yields an ordinal number and is defined as follows:

(Def. 5)(i) f is $\operatorname{limit} f$-limited if there exists α such that $\alpha \in \operatorname{dom} f$,

(ii) $\operatorname{limit} f = 0$, otherwise.

Let us consider f. The functor $\operatorname{length} f$ yielding an ordinal number is defined as follows:

(Def. 6) length f = limit f − base f.

We now state four propositions:

(15) base $\emptyset = 0$ and limit $\emptyset = 0$ and length $\emptyset = 0$.

(16) limit f = sup dom f.

(17) f is limit f-limited.

(18) Every empty set is α-based.

Let us consider α, X, Y. Note that there exists a transfinite sequence which is Y-defined, X-valued, α-based, segmental, finite, and empty.

An array is a segmental function.

Let A be an array. Observe that dom A is ordinal-membered.

We now state the proposition

(19) For every array f holds f is 0-limited iff f is empty.

Let us mention that every array which is 0-based is also transfinite sequence-like.

Let us consider X. An array of X is an X-valued array.

Let X be a 1-sorted structure. An array of X is an array of the carrier of X.

Let us consider α, X. An array of α, X is an α-defined array of X.

In the sequel A, B, C denote arrays.

Next we state several propositions:

(20) base f = inf dom f.

(21) f is base f-based.

(22) dom A = limit $A \setminus$ base A.

(23) If dom $A = \alpha \setminus \beta$ and A is non empty, then base $A = \beta$ and limit $A = \alpha$.

(24) For every transfinite sequence f holds base $f = 0$ and limit f = dom f and length f = dom f.

Let us consider α, β, X. Note that there exists an array of α, X which is β-based, natural-valued, integer-valued, real-valued, complex-valued, and finite.

Let us consider α, x. Note that $\{\langle \alpha, x \rangle\}$ is segmental.

Let us consider α and let x be a natural number. Observe that $\{\langle \alpha, x \rangle\}$ is natural-valued.

Let us consider α and let x be a real number. One can verify that $\{\langle \alpha, x \rangle\}$ is real-valued.

Let us consider α, let X be a non empty set, and let x be an element of X. One can check that $\{\langle \alpha, x \rangle\}$ is X-valued.

Let us consider α, x. One can check that $\{\langle \alpha, x \rangle\}$ is α-based and succ α-limited.

Let us consider β. Note that there exists an array which is non empty, β-based, natural-valued, integer-valued, real-valued, complex-valued, and finite. Let X be a non empty set. Note that there exists an array which is non empty, β-based, finite, and X-valued.

Let s be a transfinite sequence. We introduce $s\,\mathrm{last}$ as a synonym of $\mathrm{last}\,s$.

Let A be an array. The functor $\mathrm{last}\,A$ is defined by:

(Def. 7) $\mathrm{last}\,A = A(\bigcup \mathrm{dom}\,A)$.

3. Descending Sequences

Let f be a function. We say that f is descending if and only if:

(Def. 8) For all α, β such that α, $\beta \in \mathrm{dom}\,f$ and $\alpha \in \beta$ holds $f(\beta) \subset f(\alpha)$.

We now state four propositions:

(25) For every finite array f such that for every α such that α, $\mathrm{succ}\,\alpha \in \mathrm{dom}\,f$ holds $f(\mathrm{succ}\,\alpha) \subset f(\alpha)$ holds f is descending.

(26) For every array f such that $\mathrm{length}\,f = \omega$ and for every α such that α, $\mathrm{succ}\,\alpha \in \mathrm{dom}\,f$ holds $f(\mathrm{succ}\,\alpha) \subset f(\alpha)$ holds f is descending.

(27) For every transfinite sequence f such that f is descending and $f(0)$ is finite holds f is finite.

(28) Let f be a transfinite sequence. Suppose f is descending and $f(0)$ is finite and for every α such that $f(\alpha) \neq \emptyset$ holds $\mathrm{succ}\,\alpha \in \mathrm{dom}\,f$. Then $\mathrm{last}\,f = \emptyset$.

The scheme A deals with a transfinite sequence $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\mathcal{A}$ is finite

provided the parameters meet the following requirements:

- $\mathcal{F}(\mathcal{A}(0))$ is finite, and
- For every α such that $\mathrm{succ}\,\alpha \in \mathrm{dom}\,\mathcal{A}$ and $\mathcal{F}(\mathcal{A}(\alpha))$ is finite holds $\mathcal{F}(\mathcal{A}(\mathrm{succ}\,\alpha)) \subset \mathcal{F}(\mathcal{A}(\alpha))$.

4. Swap

Let us consider X, let f be an X-defined function, and let α, β be sets. Note that $\mathrm{Swap}(f, \alpha, \beta)$ is X-defined.

Let X be a set, let f be an X-valued function, and let x, y be sets. Note that $\mathrm{Swap}(f, x, y)$ is X-valued.

The following propositions are true:

(29) If x, $y \in \mathrm{dom}\,f$, then $(\mathrm{Swap}(f, x, y))(x) = f(y)$.

(30) For every array f of X such that x, $y \in \mathrm{dom}\,f$ holds $(\mathrm{Swap}(f, x, y))_x = f_y$.

(31) If x, $y \in \mathrm{dom}\,f$, then $(\mathrm{Swap}(f, x, y))(y) = f(x)$.

(32) For every array f of X such that x, $y \in \mathrm{dom}\,f$ holds $(\mathrm{Swap}(f, x, y))_y = f_x$.

(33) If $z \neq x$ and $z \neq y$, then $(\mathrm{Swap}(f, x, y))(z) = f(z)$.

(34) For every array f of X such that $z \in \mathrm{dom}\, f$ and $z \neq x$ and $z \neq y$ holds $(\mathrm{Swap}(f, x, y))_z = f_z$.

(35) If x, $y \in \mathrm{dom}\, f$, then $\mathrm{Swap}(f, x, y) = \mathrm{Swap}(f, y, x)$.

Let X be a non empty set. Observe that there exists an X-valued non empty function which is onto.

Let X be a non empty set, let f be an onto X-valued non empty function, and let x, y be elements of $\mathrm{dom}\, f$. Note that $\mathrm{Swap}(f, x, y)$ is onto.

Let us consider A and let us consider x, y. Note that $\mathrm{Swap}(A, x, y)$ is segmental.

We now state the proposition

(36) If x, $y \in \mathrm{dom}\, A$, then $\mathrm{Swap}(\mathrm{Swap}(A, x, y), x, y) = A$.

Let A be a real-valued array and let us consider x, y. One can verify that $\mathrm{Swap}(A, x, y)$ is real-valued.

5. Permutations

Let A be an array. An array is called a permutation of A if:

(Def. 9) There exists a permutation f of $\mathrm{dom}\, A$ such that $\mathrm{it} = A \cdot f$.

We now state several propositions:

(37) For every permutation B of A holds $\mathrm{dom}\, B = \mathrm{dom}\, A$ and $\mathrm{rng}\, B = \mathrm{rng}\, A$.

(38) A is a permutation of A.

(39) If A is a permutation of B, then B is a permutation of A.

(40) If A is a permutation of B and B is a permutation of C, then A is a permutation of C.

(41) $\mathrm{Swap}(\mathrm{id}_X, x, y)$ is one-to-one.

Let X be a non empty set and let x, y be elements of X.

Note that $\mathrm{Swap}(\mathrm{id}_X, x, y)$ is one-to-one, X-valued, and X-defined.

Let X be a non empty set and let x, y be elements of X.

Note that $\mathrm{Swap}(\mathrm{id}_X, x, y)$ is onto and total.

Let X, Y be non empty sets, let f be a function from X into Y, and let x, y be elements of X. Then $\mathrm{Swap}(f, x, y)$ is a function from X into Y.

Next we state three propositions:

(42) If x, $y \in X$ and $f = \mathrm{Swap}(\mathrm{id}_X, x, y)$ and $X = \mathrm{dom}\, A$, then $\mathrm{Swap}(A, x, y) = A \cdot f$.

(43) If x, $y \in \mathrm{dom}\, A$, then $\mathrm{Swap}(A, x, y)$ is a permutation of A and A is a permutation of $\mathrm{Swap}(A, x, y)$.

(44) Suppose x, $y \in \mathrm{dom}\, A$ and A is a permutation of B. Then $\mathrm{Swap}(A, x, y)$ is a permutation of B and A is a permutation of $\mathrm{Swap}(B, x, y)$.

6. Exchanging

Let O be a relational structure and let A be an array of O. We say that A is ascending if and only if:

(Def. 10) For all α, β such that α, $\beta \in \operatorname{dom} A$ and $\alpha \in \beta$ holds $A_\alpha \leq A_\beta$.

The functor inversions A is defined by:

(Def. 11) $\operatorname{inversions} A = \{\langle\alpha, \beta\rangle; \alpha$ ranges over elements of $\operatorname{dom} A, \beta$ ranges over elements of $\operatorname{dom} A : \alpha \in \beta \ \wedge\ A_\alpha \not\leq A_\beta\}$.

Let O be a relational structure. One can verify that every empty array of O is ascending. Let A be an empty array of O. One can verify that inversions A is empty.

In the sequel O is a connected non empty poset and R, Q are arrays of O.

We now state the proposition

(45) For every O and for all elements x, y of O holds $x > y$ iff $x \not\leq y$.

Let us consider O, R. Then inversions R can be characterized by the condition:

(Def. 12) $\operatorname{inversions} R = \{\langle\alpha, \beta\rangle; \alpha$ ranges over elements of $\operatorname{dom} R, \beta$ ranges over elements of $\operatorname{dom} R : \alpha \in \beta \ \wedge\ R_\alpha > R_\beta\}$.

Next we state two propositions:

(46) $\langle x, y\rangle \in \operatorname{inversions} R$ iff x, $y \in \operatorname{dom} R$ and $x \in y$ and $R_x > R_y$.

(47) $\operatorname{inversions} R \subseteq \operatorname{dom} R \times \operatorname{dom} R$.

Let us consider O and let R be a finite array of O. Observe that inversions R is finite.

We now state three propositions:

(48) R is ascending iff $\operatorname{inversions} R = \emptyset$.

(49) If $\langle x, y\rangle \in \operatorname{inversions} R$, then $\langle y, x\rangle \notin \operatorname{inversions} R$.

(50) If $\langle x, y\rangle$, $\langle y, z\rangle \in \operatorname{inversions} R$, then $\langle x, z\rangle \in \operatorname{inversions} R$.

Let us consider O, R. Note that inversions R is relation-like.

Let us consider O, R. Note that inversions R is asymmetric and transitive.

Let us consider O and let α, β be elements of O. Let us note that the predicate $\alpha < \beta$ is antisymmetric.

Next we state several propositions:

(51) If $\langle x, y\rangle \in \operatorname{inversions} R$, then $\langle x, y\rangle \notin \operatorname{inversions} \operatorname{Swap}(R, x, y)$.

(52) If x, $y \in \operatorname{dom} R$ and $z \neq x$ and $z \neq y$ and $t \neq x$ and $t \neq y$, then $\langle z, t\rangle \in \operatorname{inversions} R$ iff $\langle z, t\rangle \in \operatorname{inversions} \operatorname{Swap}(R, x, y)$.

(53) If $\langle x, y\rangle \in \operatorname{inversions} R$, then $\langle z, y\rangle \in \operatorname{inversions} R$ and $z \in x$ iff $\langle z, x\rangle \in \operatorname{inversions} \operatorname{Swap}(R, x, y)$.

(54) If $\langle x, y\rangle \in \operatorname{inversions} R$, then $\langle z, x\rangle \in \operatorname{inversions} R$ iff $z \in x$ and $\langle z, y\rangle \in \operatorname{inversions} \operatorname{Swap}(R, x, y)$.

(55) If $\langle x, y\rangle \in \operatorname{inversions} R$ and $z \in y$ and $\langle x, z\rangle \in \operatorname{inversions} \operatorname{Swap}(R, x, y)$, then $\langle x, z\rangle \in \operatorname{inversions} R$.

(56) If $\langle x, y\rangle \in \operatorname{inversions} R$ and $x \in z$ and $\langle z, y\rangle \in \operatorname{inversions} \operatorname{Swap}(R, x, y)$, then $\langle z, y\rangle \in \operatorname{inversions} R$.

(57) If $\langle x, y\rangle \in \operatorname{inversions} R$ and $y \in z$ and $\langle x, z\rangle \in \operatorname{inversions} \operatorname{Swap}(R, x, y)$, then $\langle y, z\rangle \in \operatorname{inversions} R$.

(58) If $\langle x, y\rangle \in \operatorname{inversions} R$, then $y \in z$ and $\langle x, z\rangle \in \operatorname{inversions} R$ iff $\langle y, z\rangle \in \operatorname{inversions} \operatorname{Swap}(R, x, y)$.

Let us consider O, R, x, y. The functor $\subseteq^{R}_{x,y}$ yields a function and is defined by:

(Def. 13) $\subseteq^{R}_{x,y} = \operatorname{Swap}(\mathrm{id}_{\operatorname{dom} R}, x, y) \times \operatorname{Swap}(\mathrm{id}_{\operatorname{dom} R}, x, y) +\cdot\, \mathrm{id}_{\{x\} \times (\operatorname{succ} y \setminus x) \cup (\operatorname{succ} y \setminus x) \times \{y\}}$.

Next we state the proposition

(59) $\gamma \in \operatorname{succ} \beta \setminus \alpha$ iff $\alpha \subseteq \gamma \subseteq \beta$.

We adopt the following convention: T is a non empty array of O and p, q, r, s are elements of $\operatorname{dom} T$.

The following propositions are true:

(60) $\operatorname{succ} q \setminus p \subseteq \operatorname{dom} T$.

(61) $\operatorname{dom} \subseteq^{T}_{p,q} = \operatorname{dom} T \times \operatorname{dom} T$ and $\operatorname{rng} \subseteq^{T}_{p,q} \subseteq \operatorname{dom} T \times \operatorname{dom} T$.

(62) If $p \subseteq r \subseteq q$, then $(\subseteq^{T}_{p,q})(p, r) = \langle p, r\rangle$ and $(\subseteq^{T}_{p,q})(r, q) = \langle r, q\rangle$.

(63) If $r \neq p$ and $s \neq q$ and $f = \operatorname{Swap}(\mathrm{id}_{\operatorname{dom} T}, p, q)$, then $(\subseteq^{T}_{p,q})(r, s) = \langle f(r), f(s)\rangle$.

(64) If $r \in p$ and $f = \operatorname{Swap}(\mathrm{id}_{\operatorname{dom} T}, p, q)$, then $(\subseteq^{T}_{p,q})(r, q) = \langle f(r), f(q)\rangle$ and $(\subseteq^{T}_{p,q})(r, p) = \langle f(r), f(p)\rangle$.

(65) If $q \in r$ and $f = \operatorname{Swap}(\mathrm{id}_{\operatorname{dom} T}, p, q)$, then $(\subseteq^{T}_{p,q})(p, r) = \langle f(p), f(r)\rangle$ and $(\subseteq^{T}_{p,q})(q, r) = \langle f(q), f(r)\rangle$.

(66) If $p \in q$, then $(\subseteq^{T}_{p,q})(p, q) = \langle p, q\rangle$.

(67) If $p \in q$ and $r \neq p$ and $r \neq q$ and $s \neq p$ and $s \neq q$, then $(\subseteq^{T}_{p,q})(r, s) = \langle r, s\rangle$.

(68) If $r \in p$ and $p \in q$, then $(\subseteq^{T}_{p,q})(r, p) = \langle r, q\rangle$ and $(\subseteq^{T}_{p,q})(r, q) = \langle r, p\rangle$.

(69) If $p \in s$ and $s \in q$, then $(\subseteq^{T}_{p,q})(p, s) = \langle p, s\rangle$ and $(\subseteq^{T}_{p,q})(s, q) = \langle s, q\rangle$.

(70) If $p \in q$ and $q \in s$, then $(\subseteq^{T}_{p,q})(p, s) = \langle q, s\rangle$ and $(\subseteq^{T}_{p,q})(q, s) = \langle p, s\rangle$.

(71) If $p \in q$, then $\subseteq^{T}_{p,q} \restriction (\operatorname{inversions} \operatorname{Swap}(T, p, q)$ **qua** set) is one-to-one.

Let us consider O, R, x, y, z. Note that $(\subseteq^{R}_{x,y})^{\circ} z$ is relation-like.

7. Correctness of Sorting by Exchanging

The following proposition is true

(72) If $\langle x, y\rangle \in \text{inversions}\, R$, then $(\subseteq^{R}_{x,y})^{\circ}\, \text{inversions}\, \text{Swap}(R, x, y) \subset \text{inversions}\, R$.

Let R be a finite function and let us consider x, y. One can check that $\text{Swap}(R, x, y)$ is finite.

Next we state two propositions:

(73) For every array R of O such that $\langle x, y\rangle \in \text{inversions}\, R$ and inversions R is finite holds $\overline{\overline{\text{inversions}\, \text{Swap}(R, x, y)}} \in \overline{\overline{\text{inversions}\, R}}$.

(74) For every finite array R of O such that $\langle x, y\rangle \in \text{inversions}\, R$ holds $\overline{\overline{\text{inversions}\, \text{Swap}(R, x, y)}} < \overline{\overline{\text{inversions}\, R}}$.

Let us consider O, R. A non empty array is called a computation of R if it satisfies the conditions (Def. 14).

(Def. 14)(i) $\text{It}(\text{base}\, \text{it}) = R$,

(ii) for every α such that $\alpha \in \text{dom}\, \text{it}$ holds $\text{it}(\alpha)$ is an array of O, and

(iii) for every α such that α, $\text{succ}\, \alpha \in \text{dom}\, \text{it}$ there exist R, x, y such that $\langle x, y\rangle \in \text{inversions}\, R$ and $\text{it}(\alpha) = R$ and $\text{it}(\text{succ}\, \alpha) = \text{Swap}(R, x, y)$.

We now state the proposition

(75) $\{\langle \alpha, R\rangle\}$ is a computation of R.

Let us consider O, R, α. One can check that there exists a computation of R which is α-based and finite.

Let us consider O, R, let C be a computation of R, and let us consider x. One can check that $C(x)$ is segmental, function-like, and relation-like.

Let us consider O, R, let C be a computation of R, and let us consider x. Observe that $C(x)$ is the carrier of O-valued.

Let us consider O, R and let C be a computation of R. Observe that $\text{last}\, C$ is segmental, relation-like, and function-like.

Let us consider O, R and let C be a computation of R. Observe that $\text{last}\, C$ is the carrier of O-valued.

Let us consider O, R and let C be a computation of R. We say that C is complete if and only if:

(Def. 15) $\text{last}\, C$ is ascending.

One can prove the following three propositions:

(76) For every 0-based computation C of R such that R is a finite array of O holds C is finite.

(77) Let C be a 0-based computation of R. Suppose R is a finite array of O and for every α such that $\text{inversions}\, C(\alpha) \neq \emptyset$ holds $\text{succ}\, \alpha \in \text{dom}\, C$. Then C is complete.

(78) Let C be a finite computation of R. Then $\operatorname{last} C$ is a permutation of R and for every α such that $\alpha \in \operatorname{dom} C$ holds $C(\alpha)$ is a permutation of R.

8. Existence of Complete Computations

Next we state three propositions:

(79) For every 0-based finite array A of X such that $A \neq \emptyset$ holds $\operatorname{last} A \in X$.

(80) $\operatorname{last}\langle x \rangle = x$.

(81) For every 0-based finite array A holds $\operatorname{last}(A \frown \langle x \rangle) = x$.

Let X be a set. Observe that every element of X^{ω} is X-valued.

The scheme A deals with a unary functor $\mathcal{F}$ yielding a set, a non empty set $\mathcal{A}$, a set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a finite 0-based non empty array f and there exists an element k of $\mathcal{A}$ such that

(i) $k = \operatorname{last} f$,

(ii) $\mathcal{F}(k) = \emptyset$,

(iii) $f(0) = \mathcal{B}$, and

(iv) for every α such that $\operatorname{succ} \alpha \in \operatorname{dom} f$ there exist elements x, y of $\mathcal{A}$ such that $x = f(\alpha)$ and $y = f(\operatorname{succ} \alpha)$ and $\mathcal{P}[x, y]$

provided the following requirements are met:

- $\mathcal{B} \in \mathcal{A}$,
- $\mathcal{F}(\mathcal{B})$ is finite, and
- For every element x of $\mathcal{A}$ such that $\mathcal{F}(x) \neq \emptyset$ there exists an element y of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{F}(y) \subset \mathcal{F}(x)$.

In the sequel A is an array and B is a permutation of A.

We now state the proposition

(82) $B \in (\operatorname{rng} A)^{\operatorname{dom} A}$.

Let A be a real-valued array. One can verify that every permutation of A is real-valued.

Let us consider α and let X be a non empty set. Observe that every element of X^{α} is transfinite sequence-like.

Let us consider X and let Y be a real-membered non empty set. One can check that every element of Y^X is real-valued.

Let us consider X and let A be an array of X. One can check that every permutation of A is X-valued.

Let X be a set, let Z be a set, and let Y be a subset of Z. Note that every element of Y^X is Z-valued.

One can prove the following propositions:

(83) Every X-defined Y-valued binary relation is a relation between X and Y.

(84) For every finite ordinal number α and for every x such that $x \in \alpha$ holds $x = 0$ or there exists β such that $x = \operatorname{succ} \beta$.

(85) For every 0-based finite non empty array A of O holds there exists a 0-based computation of A which is complete.

(86) For every 0-based finite non empty array A of O holds there exists a permutation of A which is ascending.

Let us consider O and let A be a 0-based finite array of O. Observe that there exists a permutation of A which is ascending.

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