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Counting Derangements, Non Bijective Functions and the Birthday Problem[1]

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Summary. The article provides counting derangements of finite sets and counting non bijective functions. We provide a recursive formula for the number of derangements of a finite set, together with an explicit formula involving the number e. We count the number of non-one-to-one functions between to finite sets and perform a computation to give explicitely a formalization of the birthday problem. The article is an extension of [10].

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The notation and terminology used here have been introduced in the following papers: [13], [16], [9], [1], [4], [7], [5], [6], [14], [2], [8], [3], [11], [12], [17], [18], and [15].

1. Preliminaries

In this paper x denotes a set.

One can verify that every finite 0-sequence of $\mathbb{Z}$ is integer-valued.

Let n be a natural number. Observe that $n!$ is natural.

Let n be a natural number. One can check that $n!$ is positive.

Let c be a real number. One can verify that $\exp c$ is positive.

Let us observe that e is positive.

The following two propositions are true:

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(1) $\mathrm{id}_{\emptyset}$ has no fixpoint.

(2) For every real number c such that $c < 0$ holds $\exp c < 1$.

2. Rounding

Let n be a real number. The functor round n yielding an integer is defined by:

(Def. 1) $\mathrm{round}\, n = \lfloor n + \frac{1}{2} \rfloor$.

One can prove the following two propositions:

(3) For every integer a holds $\mathrm{round}\, a = a$.

(4) For every integer a and for every real number b such that $|a - b| < \frac{1}{2}$ holds $a = \mathrm{round}\, b$.

3. Counting Derangements

Next we state two propositions:

(5) Let n be a natural number and a, b be real numbers. Suppose $a < b$. Then there exists a real number c such that $c \in]a, b[$ and $\exp a = (\sum_{\alpha=0}^{\kappa}(\mathrm{Taylor}(\text{the function exp}, \Omega_{\mathbb{R}}, b, a))(\alpha))_{\kappa \in \mathbb{N}}(n) + \frac{\exp c \cdot (a-b)^{n+1}}{(n+1)!}$.

(6) For every positive natural number n and for every real number c such that $c < 0$ holds $|-n! \cdot \frac{\exp c \cdot (-1)^{n+1}}{(n+1)!}| < \frac{1}{2}$.

Let s be a set. The functor derangements s is defined as follows:

(Def. 2) derangements $s = \{f; f$ ranges over permutations of s: f has no fixpoint$\}$.

Let s be a finite set. Observe that derangements s is finite.

Next we state several propositions:

(7) Let s be a finite set. Then derangements $s = \{h : s \to s$: h is one-to-one $\wedge \bigwedge_x (x \in s \Rightarrow h(x) \neq x)\}$.

(8) For every non empty finite set s there exists a real number c such that $c \in]-1, 0[$ and $\overline{\overline{\text{derangements}\, s}} - \frac{\overline{\overline{s}}!}{e} = -\overline{\overline{s}}! \cdot \frac{\exp c \cdot (-1)^{\overline{\overline{s}}+1}}{(\overline{\overline{s}}+1)!}$.

(9) For every non empty finite set s holds $|\overline{\overline{\text{derangements}\, s}} - \frac{\overline{\overline{s}}!}{e}| < \frac{1}{2}$.

(10) For every non empty finite set s holds $\overline{\overline{\text{derangements}\, s}} = \mathrm{round}(\frac{\overline{\overline{s}}!}{e})$.

(11) derangements $\emptyset = \{\emptyset\}$.

(12) derangements$\{x\} = \emptyset$.

The function der seq from $\mathbb{N}$ into $\mathbb{Z}$ is defined as follows:

(Def. 3) $(\text{der seq})(0) = 1$ and $(\text{der seq})(1) = 0$ and for every natural number n holds $(\text{der seq})(n+2) = (n+1) \cdot ((\text{der seq})(n) + (\text{der seq})(n+1))$.

Let c be an integer and let F be a finite 0-sequence of $\mathbb{Z}$. Observe that $c\,F$ is finite, integer-valued, and transfinite sequence-like.

Let c be a complex number and let F be an empty function. One can check that $c\,F$ is empty.

Next we state three propositions:

(13) For every finite 0-sequence F of $\mathbb{Z}$ and for every integer c holds $c \cdot \sum F = \sum((c\,F){\restriction}(\mathrm{len}\,F -' 1)) + c \cdot F(\mathrm{len}\,F -' 1)$.

(14) Let X, N be finite 0-sequences of $\mathbb{Z}$. Suppose $\mathrm{len}\,N = \mathrm{len}\,X + 1$. Let c be an integer. If $N{\restriction}\mathrm{len}\,X = c\,X$, then $\sum N = c \cdot \sum X + N(\mathrm{len}\,X)$.

(15) For every finite set s holds $(\mathrm{der\ seq})(\overline{\overline{s}}) = \overline{\overline{\mathrm{derangements}\,s}}$.

4. Counting not-one-to-one Functions and the Birthday Problem

Let s, t be sets. The functor not-one-to-one(s,t) yields a subset of t^s and is defined by:

(Def. 4) not-one-to-one$(s,t) = \{f : s \to t\colon f$ is not one-to-one$\}$.

Let s, t be finite sets. Observe that not-one-to-one(s,t) is finite.

The scheme *FraenkelDiff* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a unary predicate $\mathcal{P}$, and states that:

$$\{f : \mathcal{A} \to \mathcal{B} : \text{not } \mathcal{P}[f]\} = \mathcal{B}^{\mathcal{A}} \setminus \{f : \mathcal{A} \to \mathcal{B} : \mathcal{P}[f]\}$$

provided the following requirement is met:

- If $\mathcal{B} = \emptyset$, then $\mathcal{A} = \emptyset$.

We now state three propositions:

(16) For all finite sets s, t such that $\overline{\overline{s}} \leq \overline{\overline{t}}$ holds $\overline{\overline{\text{not-one-to-one}(s,t)}} = \overline{\overline{t}}^{\overline{\overline{s}}} - \frac{\overline{\overline{t}}!}{(\overline{\overline{t}} -' \overline{\overline{s}})!}$.

(17) For every finite set s and for every non empty finite set t such that $\overline{\overline{s}} = 23$ and $\overline{\overline{t}} = 365$ holds $2 \cdot \overline{\overline{\text{not-one-to-one}(s,t)}} > \overline{\overline{t^s}}$.

(18) For all non empty finite sets s, t such that $\overline{\overline{s}} = 23$ and $\overline{\overline{t}} = 365$ holds $\mathrm{P}(\text{not-one-to-one}(s,t)) > \frac{1}{2}$.

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