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Second-Order Partial Differentiation of Real Ternary Functions

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Summary. In this article, we shall extend the result of [17] to discuss second-order partial differentiation of real ternary functions (refer to [7] and [14] for partial differentiation).

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The notation and terminology used here have been introduced in the following papers: [6], [11], [12], [1], [2], [3], [4], [5], [7], [16], [17], [13], [8], [15], [10], and [9].

1. Second-order Partial Derivatives

For simplicity, we use the following convention: x, x_0, y, y_0, z, z_0, r denote real numbers, u, u_0 denote elements of $\mathcal{R}^3$, f, f_1, f_2 denote partial functions from $\mathcal{R}^3$ to $\mathbb{R}$, R denotes a rest, and L denotes a linear function.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. We say that f is partial differentiable on 1st-1st coordinate in u if and only if the condition (Def. 1) is satisfied.

(Def. 1) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u))(x) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u))(x_0) = L(x - x_0) + R(x - x_0)$.

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We say that f is partial differentiable on 1st-2nd coordinate in u if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u))(y) - (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u))(y_0) = L(y - y_0) + R(y - y_0)$.

We say that f is partial differentiable on 1st-3rd coordinate in u if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of z_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u))(z) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u))(z_0) = L(z - z_0) + R(z - z_0)$.

We say that f is partial differentiable on 2nd-1st coordinate in u if and only if the condition (Def. 4) is satisfied.

(Def. 4) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u))(x) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u))(x_0) = L(x - x_0) + R(x - x_0)$.

We say that f is partial differentiable on 2nd-2nd coordinate in u if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u))(y) - (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u))(y_0) = L(y - y_0) + R(y - y_0)$.

We say that f is partial differentiable on 2nd-3rd coordinate in u if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of z_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u))(z) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u))(z_0) = L(z - z_0) + R(z - z_0)$.

We say that f is partial differentiable on 3rd-1st coordinate in u if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exist real numbers x_0, y_0, z_0 such that
(i) $u = \langle x_0, y_0, z_0 \rangle$, and
(ii) there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u))(x) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u))(x_0) = L(x - x_0) + R(x - x_0)$.

We say that f is partial differentiable on 3rd-2nd coordinate in u if and only if the condition (Def. 8) is satisfied.

(Def. 8) There exist real numbers x_0, y_0, z_0 such that
(i) $u = \langle x_0, y_0, z_0 \rangle$, and
(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u))(y) - (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u))(y_0) = L(y - y_0) + R(y - y_0)$.

We say that f is partial differentiable on 3rd-3rd coordinate in u if and only if the condition (Def. 9) is satisfied.

(Def. 9) There exist real numbers x_0, y_0, z_0 such that
(i) $u = \langle x_0, y_0, z_0 \rangle$, and
(ii) there exists a neighbourhood N of z_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u))(z) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u))(z_0) = L(z - z_0) + R(z - z_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 1st-1st coordinate in u. The functor hpartdiff11(f, u) yielding a real number is defined by the condition (Def. 10).

(Def. 10) There exist real numbers x_0, y_0, z_0 such that
(i) $u = \langle x_0, y_0, z_0 \rangle$, and
(ii) there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u)$ and there exist L, R such that $\mathrm{hpartdiff11}(f, u) = L(1)$ and for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u))(x) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 1st-2nd coordinate in u. The functor hpartdiff12(f, u) yielding a real number is defined by the condition (Def. 11).

(Def. 11) There exist real numbers x_0, y_0, z_0 such that
(i) $u = \langle x_0, y_0, z_0 \rangle$, and
(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u)$ and there exist L, R such that $\mathrm{hpartdiff12}(f, u) =$

$L(1)$ and for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u))(y) - (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 1st-3rd coordinate in u. The functor hpartdiff13(f, u) yielding a real number is defined by the condition (Def. 12).

(Def. 12) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of z_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u)$ and there exist L, R such that hpartdiff13$(f, u) = L(1)$ and for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u))(z) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u))(z_0) = L(z - z_0) + R(z - z_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 2nd-1st coordinate in u. The functor hpartdiff21(f, u) yielding a real number is defined by the condition (Def. 13).

(Def. 13) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u)$ and there exist L, R such that hpartdiff21$(f, u) = L(1)$ and for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u))(x) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 2nd-2nd coordinate in u. The functor hpartdiff22(f, u) yielding a real number is defined by the condition (Def. 14).

(Def. 14) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u)$ and there exist L, R such that hpartdiff22$(f, u) = L(1)$ and for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u))(y) - (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 2nd-3rd coordinate in u. The functor hpartdiff23(f, u) yielding a real number is defined by the condition (Def. 15).

(Def. 15) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of z_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u)$ and there exist L, R such that hpartdiff23$(f, u) = L(1)$ and for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u))(z) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u))(z_0) = L(z - z_0) + R(z - z_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 3rd-1st coordinate in u. The functor hpartdiff31(f,u) yields a real number and is defined by the condition (Def. 16).

(Def. 16) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of x_0 such that $N \subseteq$ dom SVF1(1, pdiff1$(f,3)$, u) and there exist L, R such that hpartdiff31$(f,u) = L(1)$ and for every x such that $x \in N$ holds (SVF1(1, pdiff1$(f,3)$, u))(x) − (SVF1(1, pdiff1$(f,3)$, u))$(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 3rd-2nd coordinate in u. The functor hpartdiff32(f,u) yielding a real number is defined by the condition (Def. 17).

(Def. 17) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of y_0 such that $N \subseteq$ dom SVF1(2, pdiff1$(f,3)$, u) and there exist L, R such that hpartdiff32$(f,u) = L(1)$ and for every y such that $y \in N$ holds (SVF1(2, pdiff1$(f,3)$, u))(y) − (SVF1(2, pdiff1$(f,3)$, u))$(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let u be an element of $\mathcal{R}^3$. Let us assume that f is partial differentiable on 3rd-3rd coordinate in u. The functor hpartdiff33(f,u) yielding a real number is defined by the condition (Def. 18).

(Def. 18) There exist real numbers x_0, y_0, z_0 such that

(i) $u = \langle x_0, y_0, z_0 \rangle$, and

(ii) there exists a neighbourhood N of z_0 such that $N \subseteq$ dom SVF1(3, pdiff1$(f,3)$, u) and there exist L, R such that hpartdiff33$(f,u) = L(1)$ and for every z such that $z \in N$ holds (SVF1(3, pdiff1$(f,3)$, u))(z) − (SVF1(3, pdiff1$(f,3)$, u))$(z_0) = L(z - z_0) + R(z - z_0)$.

Next we state a number of propositions:

(1) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-1st coordinate in u, then SVF1(1, pdiff1$(f,1)$, u) is differentiable in x_0.

(2) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-2nd coordinate in u, then SVF1(2, pdiff1$(f,1)$, u) is differentiable in y_0.

(3) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-3rd coordinate in u, then SVF1(3, pdiff1$(f,1)$, u) is differentiable in z_0.

(4) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-1st coordinate in u, then SVF1(1, pdiff1$(f,2)$, u) is differentiable in x_0.

(5) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-2nd coordinate in u, then SVF1(2, pdiff1$(f,2)$, u) is differentiable in y_0.

(6) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-3rd coordinate in u, then SVF1(3, pdiff1$(f,2)$, u) is differentiable in z_0.

(7) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-1st coordinate in u, then SVF1(1, pdiff1$(f,3)$, u) is differentiable in x_0.

(8) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-2nd coordinate in u, then $\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u)$ is differentiable in y_0.

(9) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-3rd coordinate in u, then $\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u)$ is differentiable in z_0.

(10) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-1st coordinate in u, then $\mathrm{hpartdiff11}(f, u) = (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u))'(x_0)$.

(11) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-2nd coordinate in u, then $\mathrm{hpartdiff12}(f, u) = (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u))'(y_0)$.

(12) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 1st-3rd coordinate in u, then $\mathrm{hpartdiff13}(f, u) = (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u))'(z_0)$.

(13) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-1st coordinate in u, then $\mathrm{hpartdiff21}(f, u) = (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u))'(x_0)$.

(14) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-2nd coordinate in u, then $\mathrm{hpartdiff22}(f, u) = (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u))'(y_0)$.

(15) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 2nd-3rd coordinate in u, then $\mathrm{hpartdiff23}(f, u) = (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u))'(z_0)$.

(16) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-1st coordinate in u, then $\mathrm{hpartdiff31}(f, u) = (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u))'(x_0)$.

(17) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-2nd coordinate in u, then $\mathrm{hpartdiff32}(f, u) = (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u))'(y_0)$.

(18) If $u = \langle x_0, y_0, z_0 \rangle$ and f is partial differentiable on 3rd-3rd coordinate in u, then $\mathrm{hpartdiff33}(f, u) = (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u))'(z_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. We say that f is partial differentiable on 1st-1st coordinate on D if and only if:

(Def. 19) $D \subseteq \operatorname{dom} f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partial differentiable on 1st-1st coordinate in u.

We say that f is partial differentiable on 1st-2nd coordinate on D if and only if:

(Def. 20) $D \subseteq \operatorname{dom} f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partial differentiable on 1st-2nd coordinate in u.

We say that f is partial differentiable on 1st-3rd coordinate on D if and only if:

(Def. 21) $D \subseteq \operatorname{dom} f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partial differentiable on 1st-3rd coordinate in u.

We say that f is partial differentiable on 2nd-1st coordinate on D if and only if:

(Def. 22) $D \subseteq \operatorname{dom} f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partial differentiable on 2nd-1st coordinate in u.

We say that f is partial differentiable on 2nd-2nd coordinate on D if and only if:

(Def. 23) $D \subseteq \operatorname{dom} f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partial differentiable on 2nd-2nd coordinate in u.

We say that f is partial differentiable on 2nd-3rd coordinate on D if and only if:

(Def. 24) $D \subseteq \mathrm{dom}\, f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partial differentiable on 2nd-3rd coordinate in u.

We say that f is partial differentiable on 3rd-1st coordinate on D if and only if:

(Def. 25) $D \subseteq \mathrm{dom}\, f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partial differentiable on 3rd-1st coordinate in u.

We say that f is partial differentiable on 3rd-2nd coordinate on D if and only if:

(Def. 26) $D \subseteq \mathrm{dom}\, f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partial differentiable on 3rd-2nd coordinate in u.

We say that f is partial differentiable on 3rd-3rd coordinate on D if and only if:

(Def. 27) $D \subseteq \mathrm{dom}\, f$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partial differentiable on 3rd-3rd coordinate in u.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 1st-1st coordinate on D. The functor $f{\restriction}_D^{\mathrm{1st-1st}}$ yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined by:

(Def. 28) $\mathrm{dom}(f{\restriction}_D^{\mathrm{1st-1st}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\mathrm{1st-1st}}(u) = \mathrm{hpartdiff11}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 1st-2nd coordinate on D. The functor $f{\restriction}_D^{\mathrm{1st-2nd}}$ yielding a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ is defined by:

(Def. 29) $\mathrm{dom}(f{\restriction}_D^{\mathrm{1st-2nd}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\mathrm{1st-2nd}}(u) = \mathrm{hpartdiff12}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 1st-3rd coordinate on D. The functor $f{\restriction}_D^{\mathrm{1st-3rd}}$ yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined by:

(Def. 30) $\mathrm{dom}(f{\restriction}_D^{\mathrm{1st-3rd}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\mathrm{1st-3rd}}(u) = \mathrm{hpartdiff13}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 2nd-1st coordinate on D. The functor $f{\restriction}_D^{\mathrm{2nd-1st}}$ yielding a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ is defined as follows:

(Def. 31) $\mathrm{dom}(f{\restriction}_D^{\mathrm{2nd-1st}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\mathrm{2nd-1st}}(u) = \mathrm{hpartdiff21}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 2nd-2nd coordinate on D. The functor $f{\restriction}_D^{\mathrm{2nd-2nd}}$ yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined by:

(Def. 32) $\mathrm{dom}(f{\restriction}_D^{\mathrm{2nd-2nd}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\mathrm{2nd-2nd}}(u) = \mathrm{hpartdiff22}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 2nd-3rd coordinate on D. The functor $f{\restriction}_D^{\mathrm{2nd-3rd}}$

yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined by:

(Def. 33) $\mathrm{dom}(f{\restriction}_D^{\mathrm{2nd-3rd}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\mathrm{2nd-3rd}}(u) = \mathrm{hpartdiff23}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 3rd-1st coordinate on D. The functor $f{\restriction}_D^{\mathrm{3rd-1st}}$ yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined as follows:

(Def. 34) $\mathrm{dom}(f{\restriction}_D^{\mathrm{3rd-1st}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\mathrm{3rd-1st}}(u) = \mathrm{hpartdiff31}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 3rd-2nd coordinate on D. The functor $f{\restriction}_D^{\mathrm{3rd-2nd}}$ yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined by:

(Def. 35) $\mathrm{dom}(f{\restriction}_D^{\mathrm{3rd-2nd}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\mathrm{3rd-2nd}}(u) = \mathrm{hpartdiff32}(f, u)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partial differentiable on 3rd-3rd coordinate on D. The functor $f{\restriction}_D^{\mathrm{3rd-3rd}}$ yielding a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ is defined by:

(Def. 36) $\mathrm{dom}(f{\restriction}_D^{\mathrm{3rd-3rd}}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}_D^{\mathrm{3rd-3rd}}(u) = \mathrm{hpartdiff33}(f, u)$.

2. Main Properties of Second-order Partial Derivatives

Next we state a number of propositions:

(19) f is partial differentiable on 1st-1st coordinate in u if and only if $\mathrm{pdiff1}(f, 1)$ is partially differentiable in u w.r.t. 1.

(20) f is partial differentiable on 1st-2nd coordinate in u if and only if $\mathrm{pdiff1}(f, 1)$ is partially differentiable in u w.r.t. 2.

(21) f is partial differentiable on 1st-3rd coordinate in u if and only if $\mathrm{pdiff1}(f, 1)$ is partially differentiable in u w.r.t. 3.

(22) f is partial differentiable on 2nd-1st coordinate in u if and only if $\mathrm{pdiff1}(f, 2)$ is partially differentiable in u w.r.t. 1.

(23) f is partial differentiable on 2nd-2nd coordinate in u if and only if $\mathrm{pdiff1}(f, 2)$ is partially differentiable in u w.r.t. 2.

(24) f is partial differentiable on 2nd-3rd coordinate in u if and only if $\mathrm{pdiff1}(f, 2)$ is partially differentiable in u w.r.t. 3.

(25) f is partial differentiable on 3rd-1st coordinate in u if and only if $\mathrm{pdiff1}(f, 3)$ is partially differentiable in u w.r.t. 1.

(26) f is partial differentiable on 3rd-2nd coordinate in u if and only if $\mathrm{pdiff1}(f, 3)$ is partially differentiable in u w.r.t. 2.

(27) f is partial differentiable on 3rd-3rd coordinate in u if and only if $\mathrm{pdiff1}(f,3)$ is partially differentiable in u w.r.t. 3.

(28) If f is partial differentiable on 1st-1st coordinate in u, then $\mathrm{hpartdiff11}(f,u) = \mathrm{partdiff}(\mathrm{pdiff1}(f,1),u,1)$.

(29) If f is partial differentiable on 1st-2nd coordinate in u, then $\mathrm{hpartdiff12}(f,u) = \mathrm{partdiff}(\mathrm{pdiff1}(f,1),u,2)$.

(30) If f is partial differentiable on 1st-3rd coordinate in u, then $\mathrm{hpartdiff13}(f,u) = \mathrm{partdiff}(\mathrm{pdiff1}(f,1),u,3)$.

(31) If f is partial differentiable on 2nd-1st coordinate in u, then $\mathrm{hpartdiff21}(f,u) = \mathrm{partdiff}(\mathrm{pdiff1}(f,2),u,1)$.

(32) If f is partial differentiable on 2nd-2nd coordinate in u, then $\mathrm{hpartdiff22}(f,u) = \mathrm{partdiff}(\mathrm{pdiff1}(f,2),u,2)$.

(33) If f is partial differentiable on 2nd-3rd coordinate in u, then $\mathrm{hpartdiff23}(f,u) = \mathrm{partdiff}(\mathrm{pdiff1}(f,2),u,3)$.

(34) If f is partial differentiable on 3rd-1st coordinate in u, then $\mathrm{hpartdiff31}(f,u) = \mathrm{partdiff}(\mathrm{pdiff1}(f,3),u,1)$.

(35) If f is partial differentiable on 3rd-2nd coordinate in u, then $\mathrm{hpartdiff32}(f,u) = \mathrm{partdiff}(\mathrm{pdiff1}(f,3),u,2)$.

(36) If f is partial differentiable on 3rd-3rd coordinate in u, then $\mathrm{hpartdiff33}(f,u) = \mathrm{partdiff}(\mathrm{pdiff1}(f,3),u,3)$.

(37) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\mathrm{proj}(1,3))(u_0)$. Suppose f is partial differentiable on 1st-1st coordinate in u_0 and $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(1,\mathrm{pdiff1}(f,1),u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\mathrm{rng}\,c = \{(\mathrm{proj}(1,3))(u_0)\}$ and $\mathrm{rng}(h+c) \subseteq N$. Then $h^{-1}\,((\mathrm{SVF1}(1,\mathrm{pdiff1}(f,1),u_0)_*(h+c)) - (\mathrm{SVF1}(1,\mathrm{pdiff1}(f,1),u_0)_*c))$ is convergent and $\mathrm{hpartdiff11}(f,u_0) = \lim(h^{-1}\,((\mathrm{SVF1}(1,\mathrm{pdiff1}(f,1),u_0)_*(h+c)) - (\mathrm{SVF1}(1,\mathrm{pdiff1}(f,1),u_0)_*c)))$.

(38) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\mathrm{proj}(2,3))(u_0)$. Suppose f is partial differentiable on 1st-2nd coordinate in u_0 and $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(2,\mathrm{pdiff1}(f,1),u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\mathrm{rng}\,c = \{(\mathrm{proj}(2,3))(u_0)\}$ and $\mathrm{rng}(h+c) \subseteq N$. Then $h^{-1}\,((\mathrm{SVF1}(2,\mathrm{pdiff1}(f,1),u_0)_*(h+c)) - (\mathrm{SVF1}(2,\mathrm{pdiff1}(f,1),u_0)_*c))$ is convergent and $\mathrm{hpartdiff12}(f,u_0) = \lim(h^{-1}\,((\mathrm{SVF1}(2,\mathrm{pdiff1}(f,1),u_0)_*(h+c)) - (\mathrm{SVF1}(2,\mathrm{pdiff1}(f,1),u_0)_*c)))$.

(39) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\mathrm{proj}(3,3))(u_0)$. Suppose f is partial differentiable on 1st-3rd coordinate in u_0 and $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(3,\mathrm{pdiff1}(f,1),u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real num-

bers. Suppose $\operatorname{rng} c = \{(\operatorname{proj}(3,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}\,((\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u_0)_*(h+c)) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u_0)_*c))$ is convergent and $\mathrm{hpartdiff13}(f, u_0) = \lim(h^{-1}\,((\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u_0)_*(h + c)) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u_0)_*c)))$.

(40) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\operatorname{proj}(1,3))(u_0)$. Suppose f is partial differentiable on 2nd-1st coordinate in u_0 and $N \subseteq \operatorname{dom} \mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\operatorname{proj}(1,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}\,((\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u_0)_*(h+c)) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u_0)_*c))$ is convergent and $\mathrm{hpartdiff21}(f, u_0) = \lim(h^{-1}\,((\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u_0)_*(h + c)) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u_0)_*c)))$.

(41) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\operatorname{proj}(2,3))(u_0)$. Suppose f is partial differentiable on 2nd-2nd coordinate in u_0 and $N \subseteq \operatorname{dom} \mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\operatorname{proj}(2,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}\,((\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u_0)_*(h+c)) - (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u_0)_*c))$ is convergent and $\mathrm{hpartdiff22}(f, u_0) = \lim(h^{-1}\,((\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u_0)_*(h + c)) - (\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u_0)_*c)))$.

(42) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\operatorname{proj}(3,3))(u_0)$. Suppose f is partial differentiable on 2nd-3rd coordinate in u_0 and $N \subseteq \operatorname{dom} \mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\operatorname{proj}(3,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}\,((\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u_0)_*(h+c)) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u_0)_*c))$ is convergent and $\mathrm{hpartdiff23}(f, u_0) = \lim(h^{-1}\,((\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u_0)_*(h + c)) - (\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u_0)_*c)))$.

(43) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\operatorname{proj}(1,3))(u_0)$. Suppose f is partial differentiable on 3rd-1st coordinate in u_0 and $N \subseteq \operatorname{dom} \mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\operatorname{proj}(1,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}\,((\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u_0)_*(h+c)) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u_0)_*c))$ is convergent and $\mathrm{hpartdiff31}(f, u_0) = \lim(h^{-1}\,((\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u_0)_*(h + c)) - (\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u_0)_*c)))$.

(44) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\operatorname{proj}(2,3))(u_0)$. Suppose f is partial differentiable on 3rd-2nd coordinate in u_0 and $N \subseteq \operatorname{dom} \mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\operatorname{proj}(2,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then

h^{-1} ((SVF1(2, pdiff1(f, 3), u_0)$_*$(h+c))−(SVF1(2, pdiff1(f, 3), u_0)$_*c$)) is convergent and hpartdiff32(f, u_0) = lim(h^{-1} ((SVF1(2, pdiff1(f, 3), u_0)$_*$(h + c)) − (SVF1(2, pdiff1(f, 3), u_0)$_*c$))).

(45) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of (proj(3, 3))(u_0). Suppose f is partial differentiable on 3rd-3rd coordinate in u_0 and $N \subseteq$ dom SVF1(3, pdiff1(f, 3), u_0). Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose rng c = {(proj(3, 3))(u_0)} and rng($h + c$) $\subseteq N$. Then h^{-1} ((SVF1(3, pdiff1(f, 3), u_0)$_*$(h+c))−(SVF1(3, pdiff1(f, 3), u_0)$_*c$)) is convergent and hpartdiff33(f, u_0) = lim(h^{-1} ((SVF1(3, pdiff1(f, 3), u_0)$_*$(h + c)) − (SVF1(3, pdiff1(f, 3), u_0)$_*c$))).

(46) Suppose that
- (i) f_1 is partial differentiable on 1st-1st coordinate in u_0, and
- (ii) f_2 is partial differentiable on 1st-1st coordinate in u_0.

Then pdiff1(f_1, 1) + pdiff1(f_2, 1) is partially differentiable in u_0 w.r.t. 1 and partdiff(pdiff1(f_1, 1) + pdiff1(f_2, 1), u_0, 1) = hpartdiff11(f_1, u_0) + hpartdiff11(f_2, u_0).

(47) Suppose that
- (i) f_1 is partial differentiable on 1st-2nd coordinate in u_0, and
- (ii) f_2 is partial differentiable on 1st-2nd coordinate in u_0.

Then pdiff1(f_1, 1) + pdiff1(f_2, 1) is partially differentiable in u_0 w.r.t. 2 and partdiff(pdiff1(f_1, 1) + pdiff1(f_2, 1), u_0, 2) = hpartdiff12(f_1, u_0) + hpartdiff12(f_2, u_0).

(48) Suppose that
- (i) f_1 is partial differentiable on 1st-3rd coordinate in u_0, and
- (ii) f_2 is partial differentiable on 1st-3rd coordinate in u_0.

Then pdiff1(f_1, 1) + pdiff1(f_2, 1) is partially differentiable in u_0 w.r.t. 3 and partdiff(pdiff1(f_1, 1) + pdiff1(f_2, 1), u_0, 3) = hpartdiff13(f_1, u_0) + hpartdiff13(f_2, u_0).

(49) Suppose that
- (i) f_1 is partial differentiable on 2nd-1st coordinate in u_0, and
- (ii) f_2 is partial differentiable on 2nd-1st coordinate in u_0.

Then pdiff1(f_1, 2) + pdiff1(f_2, 2) is partially differentiable in u_0 w.r.t. 1 and partdiff(pdiff1(f_1, 2) + pdiff1(f_2, 2), u_0, 1) = hpartdiff21(f_1, u_0) + hpartdiff21(f_2, u_0).

(50) Suppose that
- (i) f_1 is partial differentiable on 2nd-2nd coordinate in u_0, and
- (ii) f_2 is partial differentiable on 2nd-2nd coordinate in u_0.

Then pdiff1(f_1, 2) + pdiff1(f_2, 2) is partially differentiable in u_0 w.r.t. 2 and partdiff(pdiff1(f_1, 2) + pdiff1(f_2, 2), u_0, 2) = hpartdiff22(f_1, u_0) + hpartdiff22(f_2, u_0).

(51) Suppose that
(i) f_1 is partial differentiable on 2nd-3rd coordinate in u_0, and
(ii) f_2 is partial differentiable on 2nd-3rd coordinate in u_0.
Then pdiff1$(f_1, 2)$ + pdiff1$(f_2, 2)$ is partially differentiable in u_0 w.r.t. 3 and partdiff(pdiff1$(f_1, 2)$ + pdiff1$(f_2, 2), u_0, 3)$ = hpartdiff23(f_1, u_0) + hpartdiff23(f_2, u_0).

(52) Suppose that
(i) f_1 is partial differentiable on 1st-1st coordinate in u_0, and
(ii) f_2 is partial differentiable on 1st-1st coordinate in u_0.
Then pdiff1$(f_1, 1)$ − pdiff1$(f_2, 1)$ is partially differentiable in u_0 w.r.t. 1 and partdiff(pdiff1$(f_1, 1)$ − pdiff1$(f_2, 1), u_0, 1)$ = hpartdiff11(f_1, u_0) − hpartdiff11(f_2, u_0).

(53) Suppose that
(i) f_1 is partial differentiable on 1st-2nd coordinate in u_0, and
(ii) f_2 is partial differentiable on 1st-2nd coordinate in u_0.
Then pdiff1$(f_1, 1)$ − pdiff1$(f_2, 1)$ is partially differentiable in u_0 w.r.t. 2 and partdiff(pdiff1$(f_1, 1)$ − pdiff1$(f_2, 1), u_0, 2)$ = hpartdiff12(f_1, u_0) − hpartdiff12(f_2, u_0).

(54) Suppose that
(i) f_1 is partial differentiable on 1st-3rd coordinate in u_0, and
(ii) f_2 is partial differentiable on 1st-3rd coordinate in u_0.
Then pdiff1$(f_1, 1)$ − pdiff1$(f_2, 1)$ is partially differentiable in u_0 w.r.t. 3 and partdiff(pdiff1$(f_1, 1)$ − pdiff1$(f_2, 1), u_0, 3)$ = hpartdiff13(f_1, u_0) − hpartdiff13(f_2, u_0).

(55) Suppose that
(i) f_1 is partial differentiable on 2nd-1st coordinate in u_0, and
(ii) f_2 is partial differentiable on 2nd-1st coordinate in u_0.
Then pdiff1$(f_1, 2)$ − pdiff1$(f_2, 2)$ is partially differentiable in u_0 w.r.t. 1 and partdiff(pdiff1$(f_1, 2)$ − pdiff1$(f_2, 2), u_0, 1)$ = hpartdiff21(f_1, u_0) − hpartdiff21(f_2, u_0).

(56) Suppose that
(i) f_1 is partial differentiable on 2nd-2nd coordinate in u_0, and
(ii) f_2 is partial differentiable on 2nd-2nd coordinate in u_0.
Then pdiff1$(f_1, 2)$ − pdiff1$(f_2, 2)$ is partially differentiable in u_0 w.r.t. 2 and partdiff(pdiff1$(f_1, 2)$ − pdiff1$(f_2, 2), u_0, 2)$ = hpartdiff22(f_1, u_0) − hpartdiff22(f_2, u_0).

(57) Suppose that
(i) f_1 is partial differentiable on 2nd-3rd coordinate in u_0, and
(ii) f_2 is partial differentiable on 2nd-3rd coordinate in u_0.
Then pdiff1$(f_1, 2)$ − pdiff1$(f_2, 2)$ is partially differentiable in u_0 w.r.t. 3 and partdiff(pdiff1$(f_1, 2)$ − pdiff1$(f_2, 2), u_0, 3)$ = hpartdiff23(f_1, u_0) −

hpartdiff23(f_2, u_0).

(58) Suppose f is partial differentiable on 1st-1st coordinate in u_0. Then r pdiff1($f, 1$) is partially differentiable in u_0 w.r.t. 1 and partdiff(r pdiff1($f, 1$), u_0, 1) $= r \cdot$ hpartdiff11(f, u_0).

(59) Suppose f is partial differentiable on 1st-2nd coordinate in u_0. Then r pdiff1($f, 1$) is partially differentiable in u_0 w.r.t. 2 and partdiff(r pdiff1($f, 1$), u_0, 2) $= r \cdot$ hpartdiff12(f, u_0).

(60) Suppose f is partial differentiable on 1st-3rd coordinate in u_0. Then r pdiff1($f, 1$) is partially differentiable in u_0 w.r.t. 3 and partdiff(r pdiff1($f, 1$), u_0, 3) $= r \cdot$ hpartdiff13(f, u_0).

(61) Suppose f is partial differentiable on 2nd-1st coordinate in u_0. Then r pdiff1($f, 2$) is partially differentiable in u_0 w.r.t. 1 and partdiff(r pdiff1($f, 2$), u_0, 1) $= r \cdot$ hpartdiff21(f, u_0).

(62) Suppose f is partial differentiable on 2nd-2nd coordinate in u_0. Then r pdiff1($f, 2$) is partially differentiable in u_0 w.r.t. 2 and partdiff(r pdiff1($f, 2$), u_0, 2) $= r \cdot$ hpartdiff22(f, u_0).

(63) Suppose f is partial differentiable on 2nd-3rd coordinate in u_0. Then r pdiff1($f, 2$) is partially differentiable in u_0 w.r.t. 3 and partdiff(r pdiff1($f, 2$), u_0, 3) $= r \cdot$ hpartdiff23(f, u_0).

(64) Suppose f is partial differentiable on 3rd-1st coordinate in u_0. Then r pdiff1($f, 3$) is partially differentiable in u_0 w.r.t. 1 and partdiff(r pdiff1($f, 3$), u_0, 1) $= r \cdot$ hpartdiff31(f, u_0).

(65) Suppose f is partial differentiable on 3rd-2nd coordinate in u_0. Then r pdiff1($f, 3$) is partially differentiable in u_0 w.r.t. 2 and partdiff(r pdiff1($f, 3$), u_0, 2) $= r \cdot$ hpartdiff32(f, u_0).

(66) Suppose f is partial differentiable on 3rd-3rd coordinate in u_0. Then r pdiff1($f, 3$) is partially differentiable in u_0 w.r.t. 3 and partdiff(r pdiff1($f, 3$), u_0, 3) $= r \cdot$ hpartdiff33(f, u_0).

(67) Suppose that
  (i) f_1 is partial differentiable on 1st-1st coordinate in u_0, and
  (ii) f_2 is partial differentiable on 1st-1st coordinate in u_0.

Then pdiff1($f_1, 1$) pdiff1($f_2, 1$) is partially differentiable in u_0 w.r.t. 1.

(68) Suppose that
  (i) f_1 is partial differentiable on 1st-2nd coordinate in u_0, and
  (ii) f_2 is partial differentiable on 1st-2nd coordinate in u_0.

Then pdiff1($f_1, 1$) pdiff1($f_2, 1$) is partially differentiable in u_0 w.r.t. 2.

(69) Suppose that
  (i) f_1 is partial differentiable on 1st-3rd coordinate in u_0, and
  (ii) f_2 is partial differentiable on 1st-3rd coordinate in u_0.

Then pdiff1($f_1, 1$) pdiff1($f_2, 1$) is partially differentiable in u_0 w.r.t. 3.

(70) Suppose that
 (i) f_1 is partial differentiable on 2nd-1st coordinate in u_0, and
 (ii) f_2 is partial differentiable on 2nd-1st coordinate in u_0.
 Then $\mathrm{pdiff1}(f_1, 2)$ $\mathrm{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 1.

(71) Suppose that
 (i) f_1 is partial differentiable on 2nd-2nd coordinate in u_0, and
 (ii) f_2 is partial differentiable on 2nd-2nd coordinate in u_0.
 Then $\mathrm{pdiff1}(f_1, 2)$ $\mathrm{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 2.

(72) Suppose that
 (i) f_1 is partial differentiable on 2nd-3rd coordinate in u_0, and
 (ii) f_2 is partial differentiable on 2nd-3rd coordinate in u_0.
 Then $\mathrm{pdiff1}(f_1, 2)$ $\mathrm{pdiff1}(f_2, 2)$ is partially differentiable in u_0 w.r.t. 3.

(73) Suppose that
 (i) f_1 is partial differentiable on 3rd-1st coordinate in u_0, and
 (ii) f_2 is partial differentiable on 3rd-1st coordinate in u_0.
 Then $\mathrm{pdiff1}(f_1, 3)$ $\mathrm{pdiff1}(f_2, 3)$ is partially differentiable in u_0 w.r.t. 1.

(74) Suppose that
 (i) f_1 is partial differentiable on 3rd-2nd coordinate in u_0, and
 (ii) f_2 is partial differentiable on 3rd-2nd coordinate in u_0.
 Then $\mathrm{pdiff1}(f_1, 3)$ $\mathrm{pdiff1}(f_2, 3)$ is partially differentiable in u_0 w.r.t. 2.

(75) Suppose that
 (i) f_1 is partial differentiable on 3rd-3rd coordinate in u_0, and
 (ii) f_2 is partial differentiable on 3rd-3rd coordinate in u_0.
 Then $\mathrm{pdiff1}(f_1, 3)$ $\mathrm{pdiff1}(f_2, 3)$ is partially differentiable in u_0 w.r.t. 3.

(76) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 1st-1st coordinate in u_0. Then $\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 1), u_0)$ is continuous in $(\mathrm{proj}(1, 3))(u_0)$.

(77) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 1st-2nd coordinate in u_0. Then $\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 1), u_0)$ is continuous in $(\mathrm{proj}(2, 3))(u_0)$.

(78) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 1st-3rd coordinate in u_0. Then $\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 1), u_0)$ is continuous in $(\mathrm{proj}(3, 3))(u_0)$.

(79) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 2nd-1st coordinate in u_0. Then $\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 2), u_0)$ is continuous in $(\mathrm{proj}(1, 3))(u_0)$.

(80) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 2nd-2nd coordinate in u_0. Then $\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 2), u_0)$ is continuous in $(\mathrm{proj}(2, 3))(u_0)$.

(81) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 2nd-3rd coordinate in u_0. Then $\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 2), u_0)$ is continuous in $(\mathrm{proj}(3, 3))(u_0)$.

(82) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 3rd-1st coordinate in u_0. Then $\mathrm{SVF1}(1, \mathrm{pdiff1}(f, 3), u_0)$ is continuous in $(\mathrm{proj}(1, 3))(u_0)$.

(83) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 3rd-2nd coordinate in u_0. Then $\mathrm{SVF1}(2, \mathrm{pdiff1}(f, 3), u_0)$ is continuous in $(\mathrm{proj}(2, 3))(u_0)$.

(84) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partial differentiable on 3rd-3rd coordinate in u_0. Then $\mathrm{SVF1}(3, \mathrm{pdiff1}(f, 3), u_0)$ is continuous in $(\mathrm{proj}(3, 3))(u_0)$.

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