

Dilworth's Decomposition Theorem for Posets¹

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Summary. The following theorem is due to Dilworth [8]: Let P be a partially ordered set. If the maximal number of elements in an independent subset (anti-chain) of P is k , then P is the union of k chains (cliques).

In this article we formalize an elegant proof of the above theorem for finite posets by Perles [13]. The result is then used in proving the case of infinite posets following the original proof of Dilworth [8].

A dual of Dilworth's theorem also holds: a poset with maximum clique m is a union of m independent sets. The proof of this dual fact is considerably easier; we follow the proof by Mirsky [11]. Mirsky states also a corollary that a poset of $r \times s + 1$ elements possesses a clique of size $r + 1$ or an independent set of size $s + 1$, or both. This corollary is then used to prove the result of Erdős and Szekeres [9].

Instead of using posets, we drop reflexivity and state the facts about anti-symmetric and transitive relations.

MML identifier: DILWORTH, version: 7.11.04 4.130.1076

The articles [1], [15], [14], [7], [2], [16], [3], [12], [17], [5], [10], [4], and [6] provide the notation and terminology for this paper.

1. PRELIMINARIES

The scheme *FraenkelFinCard1* deals with a finite non empty set $\mathcal{A}$, a finite set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

¹This work has been partially supported by the NSERC grant OGP 9207.

$$\overline{\mathcal{B}} \leq \overline{\mathcal{A}}$$

provided the following condition is satisfied:

- $\mathcal{B} = \{\mathcal{F}(w); w \text{ ranges over elements of } \mathcal{A} : \mathcal{P}[w]\}.$

Next we state the proposition

- (1) For all sets X, Y, x such that $x \notin X$ holds $X \setminus (Y \cup \{x\}) = X \setminus Y$.

Let us note that every set which is empty is also $\subseteq$ -linear and there exists a set which is empty and $\subseteq$ -linear.

Let X be a $\subseteq$ -linear set. Note that every subset of X is $\subseteq$ -linear.

One can prove the following four propositions:

- (2) Let X, Y be sets, F be a family of subsets of X , and G be a family of subsets of Y . Then $F \cup G$ is a family of subsets of $X \cup Y$.
- (3) Let X, Y be sets, F be a partition of X , and G be a partition of Y . If X misses Y , then $F \cup G$ is a partition of $X \cup Y$.
- (4) For all sets X, Y and for every partition F of Y such that $Y \subset X$ holds $F \cup \{X \setminus Y\}$ is a partition of X .
- (5) For every infinite set X and for every natural number n there exists a finite subset Y of X such that $\overline{Y} > n$.

2. CLIQUES AND STABLE SETS

Let R be a relational structure and let S be a subset of R . We say that S is connected if and only if:

- (Def. 1) The internal relation of R is connected in S .

Let R be a relational structure and let S be a subset of R . We introduce S is a clique as a synonym of S is connected.

Let R be a relational structure. Note that every subset of R which is trivial is also a clique.

Let R be a relational structure. One can check that there exists a subset of R which is a clique.

Let R be a relational structure. A clique of R is a clique subset of R .

We now state the proposition

- (6) Let R be a relational structure and S be a subset of R . Then S is a clique of R if and only if for all elements a, b of R such that $a, b \in S$ and $a \neq b$ holds $a \leq b$ or $b \leq a$.

Let R be a relational structure. Observe that there exists a clique of R which is finite.

Let R be a reflexive relational structure. One can check that every subset of R which is connected is also strongly connected.

Let R be a non empty relational structure. Observe that there exists a clique of R which is finite and non empty.

One can prove the following propositions:

- (7) Let R be a non empty relational structure and a_1, a_2 be elements of R .
If $a_1 \neq a_2$ and $\{a_1, a_2\}$ is a clique of R , then $a_1 \leq a_2$ or $a_2 \leq a_1$.
- (8) Let R be a non empty relational structure and a_1, a_2 be elements of R .
If $a_1 \leq a_2$ or $a_2 \leq a_1$, then $\{a_1, a_2\}$ is a clique of R .
- (9) For every relational structure R and for every clique C of R holds every subset of C is a clique of R .
- (10) Let R be a relational structure, C be a finite clique of R , and n be a natural number. If $n \leq \overline{C}$, then there exists a finite clique B of R such that $\overline{B} = n$.
- (11) Let R be a transitive relational structure, C be a clique of R , and x, y be elements of R . If x is maximal in C and $x \leq y$, then $C \cup \{y\}$ is a clique of R .
- (12) Let R be a transitive relational structure, C be a clique of R , and x, y be elements of R . If x is minimal in C and $y \leq x$, then $C \cup \{y\}$ is a clique of R .

Let R be a relational structure and let S be a subset of R . We say that S is stable if and only if:

- (Def. 2) For all elements x, y of R such that $x, y \in S$ and $x \neq y$ holds $x \not\leq y$ and $y \not\leq x$.

Let R be a relational structure. One can check that every subset of R which is trivial is also stable. Let R be a relational structure. Note that there exists a subset of R which is stable.

Let R be a relational structure. A stable set of R is a stable subset of R .

Let R be a relational structure. Note that there exists a stable set of R which is finite.

Let R be a non empty relational structure. Observe that there exists a stable set of R which is finite and non empty.

The following propositions are true:

- (13) Let R be a non empty relational structure and a_1, a_2 be elements of R .
If $a_1 \neq a_2$ and $\{a_1, a_2\}$ is a stable set of R , then $a_1 \not\leq a_2$ and $a_2 \not\leq a_1$.
- (14) Let R be a non empty relational structure and a_1, a_2 be elements of R .
If $a_1 \not\leq a_2$ and $a_2 \not\leq a_1$, then $\{a_1, a_2\}$ is a stable set of R .
- (15) Let R be a relational structure, C be a clique of R , A be a stable set of R , and a, b be sets. If $a, b \in A$ and $a, b \in C$, then $a = b$.
- (16) For every relational structure R and for every stable set A of R holds every subset of A is a stable set of R .
- (17) Let R be a relational structure, A be a finite stable set of R , and n be a natural number. If $n \leq \overline{A}$, then there exists a finite stable set B of R such that $\overline{B} = n$.

3. CLIQUE NUMBER AND STABILITY NUMBER

Let R be a relational structure. We say that R has finite clique number if and only if:

- (Def. 3) There exists a finite clique C of R such that for every finite clique D of R holds $\overline{D} \leq \overline{C}$.

Let us observe that every relational structure which is finite has also finite clique number and there exists a relational structure which is non empty, antisymmetric, and transitive and has finite clique number.

Let R be a relational structure with finite clique number. Observe that every clique of R is finite.

Let R be a relational structure with finite clique number. The functor $\omega(R)$ yields a natural number and is defined as follows:

- (Def. 4) There exists a finite clique C of R such that $\overline{C} = \omega(R)$ and for every finite clique T of R holds $\overline{T} \leq \omega(R)$.

Let R be an empty relational structure. Note that $\omega(R)$ is empty.

Let R be a non empty relational structure with finite clique number. Observe that $\omega(R)$ is positive.

Next we state two propositions:

- (18) For every non empty relational structure R with finite clique number such that Ω_R is a stable set of R holds $\omega(R) = 1$.
 (19) For every relational structure R with finite clique number such that $\omega(R) = 1$ holds Ω_R is a stable set of R .

Let R be a relational structure. We say that R has finite stability number if and only if:

- (Def. 5) There exists a finite stable set A of R such that for every finite stable set B of R holds $\overline{B} \leq \overline{A}$.

One can verify that every relational structure which is finite has also finite stability number and there exists a relational structure which is antisymmetric, transitive, and non empty and has finite stability number.

Let R be a relational structure with finite stability number. Note that every stable set of R is finite.

Let R be a relational structure with finite stability number. The functor $\alpha(R)$ yielding a natural number is defined by:

- (Def. 6) There exists a finite stable set A of R such that $\overline{A} = \alpha(R)$ and for every finite stable set T of R holds $\overline{T} \leq \alpha(R)$.

Let R be an empty relational structure. Observe that $\alpha(R)$ is empty.

Let R be a non empty relational structure with finite stability number. One can verify that $\alpha(R)$ is positive.

We now state two propositions:

- (20) For every non empty relational structure R with finite stability number such that Ω_R is a clique of R holds $\alpha(R) = 1$.
- (21) For every relational structure R with finite stability number such that $\alpha(R) = 1$ holds Ω_R is a clique of R .

Let us mention that every relational structure which has finite clique number and finite stability number is also finite.

4. LOWER AND UPPER SETS, MINIMAL AND MAXIMAL ELEMENTS

Let R be a relational structure and let X be a subset of R . The functor Lower X yields a subset of R and is defined by:

(Def. 7) Lower $X = X \cup \downarrow X$.

The functor Upper X yielding a subset of R is defined as follows:

(Def. 8) Upper $X = X \cup \uparrow X$.

One can prove the following propositions:

- (22) Let R be an antisymmetric transitive relational structure, A be a stable set of R , and z be a set. If $z \in \text{Upper } A$ and $z \in \text{Lower } A$, then $z \in A$.
- (23) Let R be a relational structure with finite stability number and A be a stable set of R . If $\overline{A} = \alpha(R)$, then $\text{Upper } A \cup \text{Lower } A = \Omega_R$.
- (24) Let R be a transitive relational structure, x be an element of R , and S be a subset of R . If x is minimal in $\text{Lower } S$, then x is minimal in Ω_R .
- (25) Let R be a transitive relational structure, x be an element of R , and S be a subset of R . If x is maximal in $\text{Upper } S$, then x is maximal in Ω_R .

Let R be a relational structure. The functor $\text{minimals}(R)$ yielding a subset of R is defined as follows:

- (Def. 9)(i) For every element x of R holds $x \in \text{minimals}(R)$ iff x is minimal in Ω_R if R is non empty,
- (ii) $\text{minimals}(R) = \emptyset$, otherwise.

The functor $\text{maximals}(R)$ yielding a subset of R is defined as follows:

- (Def. 10)(i) For every element x of R holds $x \in \text{maximals}(R)$ iff x is maximal in Ω_R if R is non empty,
- (ii) $\text{maximals}(R) = \emptyset$, otherwise.

Let R be a non empty antisymmetric transitive relational structure with finite clique number. One can verify that $\text{maximals}(R)$ is non empty and $\text{minimals}(R)$ is non empty.

Let R be a relational structure. Note that $\text{minimals}(R)$ is stable and $\text{maximals}(R)$ is stable.

The following two propositions are true:

- (26) For every relational structure R and for every stable set A of R such that $\text{minimals}(R) \not\subseteq A$ holds $\text{minimals}(R) \not\subseteq \text{Upper } A$.
- (27) For every relational structure R and for every stable set A of R such that $\text{maximals}(R) \not\subseteq A$ holds $\text{maximals}(R) \not\subseteq \text{Lower } A$.

5. SUBSTRUCTURES

Let R be a relational structure and let X be a finite subset of R . Observe that $\text{sub}(X)$ is finite.

One can prove the following propositions:

- (28) For every relational structure R and for every subset S of R holds every clique of $\text{sub}(S)$ is a clique of R .
- (29) Let R be a relational structure, S be a subset of R , and C be a clique of R . Then $C \cap S$ is a clique of $\text{sub}(S)$.
- (30) For every relational structure R and for every subset S of R holds every stable set of $\text{sub}(S)$ is a stable set of R .
- (31) Let R be a relational structure, S be a subset of R , and A be a stable set of R . Then $A \cap S$ is a stable set of $\text{sub}(S)$.
- (32) Let R be a relational structure, S be a subset of R , B be a subset of $\text{sub}(S)$, x be an element of $\text{sub}(S)$, and y be an element of R . If $x = y$ and x is maximal in B , then y is maximal in B .
- (33) Let R be a relational structure, S be a subset of R , B be a subset of $\text{sub}(S)$, x be an element of $\text{sub}(S)$, and y be an element of R . If $x = y$ and x is minimal in B , then y is minimal in B .
- (34) Let R be a transitive relational structure, A be a stable set of R , C be a clique of $\text{sub}(\text{Lower } A)$, and a, b be elements of R . If $a \in A$ and $a, b \in C$, then $a = b$ or $b \leq a$.
- (35) Let R be a transitive relational structure, A be a stable set of R , C be a clique of $\text{sub}(\text{Upper } A)$, and a, b be elements of R . If $a \in A$ and $a, b \in C$, then $a = b$ or $a \leq b$.

Let R be a relational structure with finite clique number and let S be a subset of R . One can verify that $\text{sub}(S)$ has finite clique number.

Let R be a relational structure with finite stability number and let S be a subset of R . One can verify that $\text{sub}(S)$ has finite stability number.

The following propositions are true:

- (36) Let R be a non empty antisymmetric transitive relational structure with finite clique number and x be an element of R . Then there exists an element y of R such that y is minimal in Ω_R but $y = x$ or $y < x$.
- (37) For every antisymmetric transitive relational structure R with finite clique number holds $\text{Upper } \text{minimals}(R) = \Omega_R$.

- (38) Let R be a non empty antisymmetric transitive relational structure with finite clique number and x be an element of R . Then there exists an element y of R such that y is maximal in Ω_R but $y = x$ or $x < y$.
- (39) For every antisymmetric transitive relational structure R with finite clique number holds $\text{Lower maximals}(R) = \Omega_R$.
- (40) Let R be an antisymmetric transitive relational structure with finite clique number and A be a stable set of R . If $\text{minimals}(R) \subseteq A$, then $A = \text{minimals}(R)$.
- (41) Let R be an antisymmetric transitive relational structure with finite clique number and A be a stable set of R . If $\text{maximals}(R) \subseteq A$, then $A = \text{maximals}(R)$.
- (42) For every relational structure R with finite clique number and for every subset S of R holds $\omega(\text{sub}(S)) \leq \omega(R)$.
- (43) Let R be a relational structure with finite clique number, C be a clique of R , and S be a subset of R . If $\overline{C} = \omega(R)$ and $C \subseteq S$, then $\omega(\text{sub}(S)) = \omega(R)$.
- (44) For every relational structure R with finite stability number and for every subset S of R holds $\alpha(\text{sub}(S)) \leq \alpha(R)$.
- (45) Let R be a relational structure with finite stability number, A be a stable set of R , and S be a subset of R . If $\overline{A} = \alpha(R)$ and $A \subseteq S$, then $\alpha(\text{sub}(S)) = \alpha(R)$.

6. PARTITIONS INTO CLIQUES AND STABLE SETS

Let R be a relational structure and let P be a partition of the carrier of R . We say that P is clique-wise if and only if:

- (Def. 11) For every set x such that $x \in P$ holds x is a clique of R .

Let R be a relational structure. Observe that there exists a partition of the carrier of R which is clique-wise.

Let R be a relational structure. A clique-partition of R is a clique-wise partition of the carrier of R .

Let R be an empty relational structure. One can verify that every partition of the carrier of R which is empty is also clique-wise.

Next we state four propositions:

- (46) For every finite relational structure R and for every clique-partition C of R holds $\overline{C} \geq \alpha(R)$.
- (47) Let R be a relational structure with finite stability number, A be a stable set of R , and C be a clique-partition of R . Suppose $\text{Card } C = \text{Card } A$. Then there exists a function f from A into C such that f is bijective and for every set x such that $x \in A$ holds $x \in f(x)$.

- (48) Let R be a finite relational structure, A be a stable set of R , and C be a clique-partition of R . Suppose $\overline{C} = \overline{A}$. Let c be a set. If $c \in C$, then there exists an element a of A such that $c \cap A = \{a\}$.
- (49) Let R be an antisymmetric transitive non empty relational structure with finite stability number, A be a stable set of R , U be a clique-partition of $\text{sub}(\text{Upper } A)$, and L be a clique-partition of $\text{sub}(\text{Lower } A)$. Suppose $\overline{A} = \alpha(R)$ and $\text{Card } U = \alpha(R)$ and $\text{Card } L = \alpha(R)$. Then there exists a clique-partition C of R such that $\text{Card } C = \alpha(R)$.

Let R be a relational structure and let P be a partition of the carrier of R . We say that P is stable-wise if and only if:

(Def. 12) For every set x such that $x \in P$ holds x is a stable set of R .

Let R be a relational structure. Observe that there exists a partition of the carrier of R which is stable-wise.

Let R be a relational structure. A coloring of R is a stable-wise partition of the carrier of R .

Let R be an empty relational structure. Note that every partition of the carrier of R is stable-wise.

We now state the proposition

- (50) For every finite relational structure R and for every coloring C of R holds $\overline{C} \geq \omega(R)$.

7. DILWORTH'S THEOREM AND A DUAL

Next we state the proposition

- (51) Let R be a finite antisymmetric transitive relational structure. Then there exists a clique-partition C of R such that $\overline{C} = \alpha(R)$.

Let R be a non empty relational structure with finite stability number and let C be a subset of R . We say that C is strong-chain if and only if the condition (Def. 13) is satisfied.

- (Def. 13) Let S be a finite non empty subset of R . Then there exists a clique-partition P of $\text{sub}(S)$ such that $\overline{P} \leq \alpha(R)$ and there exists a set c such that $c \in P$ and $S \cap C \subseteq c$ and for every set d such that $d \in P$ and $d \neq c$ holds $C \cap d = \emptyset$.

Let R be a non empty relational structure with finite stability number. Note that every subset of R which is strong-chain is also a clique.

Let R be an antisymmetric transitive non empty relational structure with finite stability number. Observe that every subset of R which is trivial and non empty is also strong-chain.

The following propositions are true:

- (52) Let R be a non empty antisymmetric transitive relational structure with finite stability number. Then there exists a non empty subset S of R such that S is strong-chain and it is not true that there exists a subset D of R such that D is strong-chain and $S \subset D$.
- (53) Let R be an antisymmetric transitive relational structure with finite stability number. Then there exists a clique-partition C of R such that $\text{Card } C = \alpha(R)$.
- (54) Let R be an antisymmetric transitive relational structure with finite clique number. Then there exists a coloring A of R such that $\text{Card } A = \omega(R)$.

8. ERDŐS-SZEKERES THEOREM

One can prove the following two propositions:

- (55) Let R be a finite antisymmetric transitive relational structure and r, s be natural numbers. Suppose $\text{Card } R = r \cdot s + 1$. Then there exists a clique C of R such that $\overline{C} \geq r + 1$ or there exists a stable set A of R such that $\overline{A} \geq s + 1$.
- (56) Let f be a real-valued finite sequence and n be a natural number. Suppose $\overline{f} = n^2 + 1$ and f is one-to-one. Then there exists a real-valued finite subsequence g such that $g \subseteq f$ and $\overline{g} \geq n + 1$ and g is increasing or decreasing.

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Received September 17, 2009
