FORMALIZED MATHEMATICS
Vol. 17, No. 2, Pages 61–65, 2009

Some Operations on Quaternion Numbers

Bo Li
Qingdao University of Science and Technology
China

Pan Wang
Qingdao University of Science and Technology
China

Xiquan Liang
Qingdao University of Science and Technology
China

Yanping Zhuang
Qingdao University of Science and Technology
China

Summary. In this article, we give some equality and basic theorems about quaternion numbers, and some special operations.

MML identifier: `QUATERN3`, version: `7.11.02 4.120.1050`

The articles [11], [1], [12], [3], [4], [9], [2], [5], [8], [7], [10], [13], and [6] provide the notation and terminology for this paper.

In this paper z_1, z_2, z_3, z_4, z are quaternion numbers.

The following propositions are true:

(1) $\Re(z_1 \cdot z_2) = \Re(z_2 \cdot z_1)$.

(2) If z is a real number, then $z + z_3 = \Re(z) + \Re(z_3) + \Im_1(z_3) \cdot i + \Im_2(z_3) \cdot j + \Im_3(z_3) \cdot k$.

(3) If z is a real number, then $z - z_3 = \langle \Re(z) - \Re(z_3), -\Im_1(z_3), -\Im_2(z_3), -\Im_3(z_3)\rangle_{\mathbb{H}}$.

(4) If z is a real number, then $z \cdot z_3 = \langle \Re(z) \cdot \Re(z_3), \Re(z) \cdot \Im_1(z_3), \Re(z) \cdot \Im_2(z_3), \Re(z) \cdot \Im_3(z_3)\rangle_{\mathbb{H}}$.

(5) If z is a real number, then $z \cdot i = \langle 0, \Re(z), 0, 0\rangle_{\mathbb{H}}$.

(6) If z is a real number, then $z \cdot j = \langle 0, 0, \Re(z), 0\rangle_{\mathbb{H}}$.

(7) If z is a real number, then $z \cdot k = \langle 0, 0, 0, \Re(z)\rangle_{\mathbb{H}}$.

(8) $z - 0_{\mathbb{H}} = z$.

ISSN 1426–2630(p), 1898-9934(e)

(9) If z is a real number, then $z \cdot z_1 = z_1 \cdot z$.

(10) If $\Im_1(z) = 0$ and $\Im_2(z) = 0$ and $\Im_3(z) = 0$, then $z = \Re(z)$.

(11) $|z|^{\mathbf{2}} = (\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}}$.

(12) $|z|^{\mathbf{2}} = |z \cdot \overline{z}|$.

(13) $|z|^{\mathbf{2}} = \Re(z \cdot \overline{z})$.

(14) $2 \cdot \Re(z) = \Re(z + \overline{z})$.

(15) If z is a real number, then $\overline{z \cdot z_1} = \overline{z} \cdot \overline{z_1}$.

(16) $\overline{z_1 \cdot z_2} = \overline{z_2} \cdot \overline{z_1}$.

(17) $|z_1 \cdot z_2|^{\mathbf{2}} = |z_1|^{\mathbf{2}} \cdot |z_2|^{\mathbf{2}}$.

(18) $i \cdot z_1 - z_1 \cdot i = \langle 0, 0, -2 \cdot \Im_3(z_1), 2 \cdot \Im_2(z_1) \rangle_{\mathbb{H}}$.

(19) $i \cdot z_1 + z_1 \cdot i = \langle -2 \cdot \Im_1(z_1), 2 \cdot \Re(z_1), 0, 0 \rangle_{\mathbb{H}}$.

(20) $j \cdot z_1 - z_1 \cdot j = \langle 0, 2 \cdot \Im_3(z_1), 0, -2 \cdot \Im_1(z_1) \rangle_{\mathbb{H}}$.

(21) $j \cdot z_1 + z_1 \cdot j = \langle -2 \cdot \Im_2(z_1), 0, 2 \cdot \Re(z_1), 0 \rangle_{\mathbb{H}}$.

(22) $k \cdot z_1 - z_1 \cdot k = \langle 0, -2 \cdot \Im_2(z_1), 2 \cdot \Im_1(z_1), 0 \rangle_{\mathbb{H}}$.

(23) $k \cdot z_1 + z_1 \cdot k = \langle -2 \cdot \Im_3(z_1), 0, 0, 2 \cdot \Re(z_1) \rangle_{\mathbb{H}}$.

(24) $\Re(\frac{1}{|z|^{\mathbf{2}}} \cdot \overline{z}) = \frac{1}{|z|^{\mathbf{2}}} \cdot \Re(z)$.

(25) $\Im_1(\frac{1}{|z|^{\mathbf{2}}} \cdot \overline{z}) = -\frac{1}{|z|^{\mathbf{2}}} \cdot \Im_1(z)$.

(26) $\Im_2(\frac{1}{|z|^{\mathbf{2}}} \cdot \overline{z}) = -\frac{1}{|z|^{\mathbf{2}}} \cdot \Im_2(z)$.

(27) $\Im_3(\frac{1}{|z|^{\mathbf{2}}} \cdot \overline{z}) = -\frac{1}{|z|^{\mathbf{2}}} \cdot \Im_3(z)$.

(28) $\frac{1}{|z|^{\mathbf{2}}} \cdot \overline{z} = \langle \frac{1}{|z|^{\mathbf{2}}} \cdot \Re(z), -\frac{1}{|z|^{\mathbf{2}}} \cdot \Im_1(z), -\frac{1}{|z|^{\mathbf{2}}} \cdot \Im_2(z), -\frac{1}{|z|^{\mathbf{2}}} \cdot \Im_3(z) \rangle_{\mathbb{H}}$.

(29) $z \cdot (\frac{1}{|z|^{\mathbf{2}}} \cdot \overline{z}) = \langle \frac{|z|^{\mathbf{2}}}{|z|^{\mathbf{2}}}, 0, 0, 0 \rangle_{\mathbb{H}}$.

(30) $\Re(z_1 \cdot z_2) = \Re(z_1) \cdot \Re(z_2) - \Im_1(z_1) \cdot \Im_1(z_2) - \Im_2(z_1) \cdot \Im_2(z_2) - \Im_3(z_1) \cdot \Im_3(z_2)$.

(31) $\Im_1(z_1 \cdot z_2) = (\Re(z_1) \cdot \Im_1(z_2) + \Im_1(z_1) \cdot \Re(z_2) + \Im_2(z_1) \cdot \Im_3(z_2)) - \Im_3(z_1) \cdot \Im_2(z_2)$.

(32) $\Im_2(z_1 \cdot z_2) = (\Re(z_1) \cdot \Im_2(z_2) + \Im_2(z_1) \cdot \Re(z_2) + \Im_3(z_1) \cdot \Im_1(z_2)) - \Im_1(z_1) \cdot \Im_3(z_2)$.

(33) $\Im_3(z_1 \cdot z_2) = (\Re(z_1) \cdot \Im_3(z_2) + \Im_3(z_1) \cdot \Re(z_2) + \Im_1(z_1) \cdot \Im_2(z_2)) - \Im_2(z_1) \cdot \Im_1(z_2)$.

(34) $|z_1 \cdot z_2 \cdot z_3|^{\mathbf{2}} = |z_1|^{\mathbf{2}} \cdot |z_2|^{\mathbf{2}} \cdot |z_3|^{\mathbf{2}}$.

(35) $\Re(z_1 \cdot z_2 \cdot z_3) = \Re(z_3 \cdot z_1 \cdot z_2)$.

(36) $|z \cdot z| = |\overline{z} \cdot \overline{z}|$.

(37) $|\overline{z} \cdot \overline{z}| = |z|^{\mathbf{2}}$.

(38) $|z_1 \cdot z_2 \cdot z_3| = |z_1| \cdot |z_2| \cdot |z_3|$.

(39) $|z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$.

(40) $|(z_1 + z_2) - z_3| \leq |z_1| + |z_2| + |z_3|$.

(41) $|z_1 - z_2 - z_3| \leq |z_1| + |z_2| + |z_3|$.

(42) $|z_1| - |z_2| \leq \frac{|z_1+z_2|+|z_1-z_2|}{2}$.

(43) $|z_1| - |z_2| \leq \frac{|z_1+z_2|+|z_2-z_1|}{2}$.

(44) $||z_1| - |z_2|| \leq |z_2 - z_1|$.

(45) $||z_1| - |z_2|| \leq |z_1| + |z_2|$.

(46) $|z_1| - |z_2| \leq |z_1 - z| + |z - z_2|$.

(47) If $|z_1| - |z_2| \neq 0$, then $(|z_1|^{\mathbf{2}} + |z_2|^{\mathbf{2}}) - 2 \cdot |z_1| \cdot |z_2| > 0$.

(48) $|z_1| + |z_2| \geq \frac{|z_1+z_2|+|z_2-z_1|}{2}$.

(49) $|z_1| + |z_2| \geq \frac{|z_1+z_2|+|z_1-z_2|}{2}$.

(50) $(z_1 \cdot z_2)^{-1} = {z_2}^{-1} \cdot {z_1}^{-1}$.

(51) $\overline{z}^{-1} = \overline{z^{-1}}$.

(52) $(1_{\mathbb{H}})^{-1} = 1_{\mathbb{H}}$.

(53) If $|z_1| = |z_2|$ and $|z_1| \neq 0$ and ${z_1}^{-1} = {z_2}^{-1}$, then $z_1 = z_2$.

(54) $(z_1 - z_2) \cdot (z_3 + z_4) = ((z_1 \cdot z_3 - z_2 \cdot z_3) + z_1 \cdot z_4) - z_2 \cdot z_4$.

(55) $(z_1 + z_2) \cdot (z_3 + z_4) = z_1 \cdot z_3 + z_2 \cdot z_3 + z_1 \cdot z_4 + z_2 \cdot z_4$.

(56) $-(z_1 + z_2) = -z_1 - z_2$.

(57) $-(z_1 - z_2) = -z_1 + z_2$.

(58) $z - (z_1 + z_2) = z - z_1 - z_2$.

(59) $z - (z_1 - z_2) = (z - z_1) + z_2$.

(60) $(z_1 + z_2) \cdot (z_3 - z_4) = (z_1 \cdot z_3 + z_2 \cdot z_3) - z_1 \cdot z_4 - z_2 \cdot z_4$.

(61) $(z_1 - z_2) \cdot (z_3 - z_4) = (z_1 \cdot z_3 - z_2 \cdot z_3 - z_1 \cdot z_4) + z_2 \cdot z_4$.

(62) $-(z_1 + z_2 + z_3) = -z_1 - z_2 - z_3$.

(63) $-(z_1 - z_2 - z_3) = -z_1 + z_2 + z_3$.

(64) $-((z_1 - z_2) + z_3) = (-z_1 + z_2) - z_3$.

(65) $-((z_1 + z_2) - z_3) = (-z_1 - z_2) + z_3$.

(66) If $z_1 + z = z_2 + z$, then $z_1 = z_2$.

(67) If $z_1 - z = z_2 - z$, then $z_1 = z_2$.

(68) $((z_1 + z_2) - z_3) \cdot z_4 = (z_1 \cdot z_4 + z_2 \cdot z_4) - z_3 \cdot z_4$.

(69) $((z_1 - z_2) + z_3) \cdot z_4 = (z_1 \cdot z_4 - z_2 \cdot z_4) + z_3 \cdot z_4$.

(70) $(z_1 - z_2 - z_3) \cdot z_4 = z_1 \cdot z_4 - z_2 \cdot z_4 - z_3 \cdot z_4$.

(71) $(z_1 + z_2 + z_3) \cdot z_4 = z_1 \cdot z_4 + z_2 \cdot z_4 + z_3 \cdot z_4$.

(72) $(z_1 - z_2) \cdot z_3 = (z_2 - z_1) \cdot -z_3$.

(73) $z_3 \cdot (z_1 - z_2) = (-z_3) \cdot (z_2 - z_1)$.

(74) $(z_1 - z_2 - z_3) + z_4 = (z_4 - z_3 - z_2) + z_1$.

(75) $(z_1 - z_2) \cdot (z_3 - z_4) = (z_2 - z_1) \cdot (z_4 - z_3)$.

(76) $z - z_1 - z_2 = z - z_2 - z_1$.

(77) $z^{-1} = \langle \frac{\Re(z)}{|z|^{\mathbf{2}}}, -\frac{\Im_1(z)}{|z|^{\mathbf{2}}}, -\frac{\Im_2(z)}{|z|^{\mathbf{2}}}, -\frac{\Im_3(z)}{|z|^{\mathbf{2}}} \rangle_{\mathbb{H}}$.

(78) $\frac{z_1}{z_2} = \langle \frac{\Re(z_2)\cdot\Re(z_1)+\Im_1(z_1)\cdot\Im_1(z_2)+\Im_2(z_2)\cdot\Im_2(z_1)+\Im_3(z_2)\cdot\Im_3(z_1)}{|z_2|^{\mathbf{2}}}, \frac{(\Re(z_2)\cdot\Im_1(z_1)-\Im_1(z_2)\cdot\Re(z_1)-\Im_2(z_2)\cdot\Im_3(z_1))+\Im_3(z_2)\cdot\Im_2(z_1)}{|z_2|^{\mathbf{2}}}, \frac{(\Re(z_2)\cdot\Im_2(z_1)+\Im_1(z_2)\cdot\Im_3(z_1))-\Im_2(z_2)\cdot\Re(z_1)-\Im_3(z_2)\cdot\Im_1(z_1)}{|z_2|^{\mathbf{2}}}, \frac{((\Re(z_2)\cdot\Im_3(z_1)-\Im_1(z_2)\cdot\Im_2(z_1))+\Im_2(z_2)\cdot\Im_1(z_1))-\Im_3(z_2)\cdot\Re(z_1)}{|z_2|^{\mathbf{2}}}\rangle_{\mathbb{H}}$.

(79) $(i)^{-1} = -i$.

(80) $(j)^{-1} = -j$.

(81) $(k)^{-1} = -k$.

Let z be a quaternion number. The functor $z^{\mathbf{2}}$ is defined by:

(Def. 1) $z^{\mathbf{2}} = z \cdot z$.

Let z be a quaternion number. One can verify that $z^{\mathbf{2}}$ is quaternion.

Let z be an element of $\mathbb{H}$. Then $z^{\mathbf{2}}$ is an element of $\mathbb{H}$.

One can prove the following four propositions:

(82) $z^{\mathbf{2}} = \langle (\Re(z))^{\mathbf{2}} - (\Im_1(z))^{\mathbf{2}} - (\Im_2(z))^{\mathbf{2}} - (\Im_3(z))^{\mathbf{2}}, 2\cdot(\Re(z)\cdot\Im_1(z)), 2\cdot(\Re(z)\cdot\Im_2(z)), 2\cdot(\Re(z)\cdot\Im_3(z))\rangle_{\mathbb{H}}$.

(83) $(0_{\mathbb{H}})^{\mathbf{2}} = 0$.

(84) $(1_{\mathbb{H}})^{\mathbf{2}} = 1$.

(85) $z^{\mathbf{2}} = (-z)^{\mathbf{2}}$.

Let z be a quaternion number. The functor $z^{\mathbf{3}}$ is defined as follows:

(Def. 2) $z^{\mathbf{3}} = z \cdot z \cdot z$.

Let z be a quaternion number. Observe that $z^{\mathbf{3}}$ is quaternion.

Let z be an element of $\mathbb{H}$. Then $z^{\mathbf{3}}$ is an element of $\mathbb{H}$.

Next we state several propositions:

(86) $(0_{\mathbb{H}})^{\mathbf{3}} = 0$.

(87) $(1_{\mathbb{H}})^{\mathbf{3}} = 1$.

(88) $(i)^{\mathbf{3}} = -i$.

(89) $(j)^{\mathbf{3}} = -j$.

(90) $(k)^{\mathbf{3}} = -k$.

(91) $(-1_{\mathbb{H}})^{\mathbf{2}} = 1$.

(92) $(-1_{\mathbb{H}})^{\mathbf{3}} = -1$.

(93) $z^{\mathbf{3}} = -(-z)^{\mathbf{3}}$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[5] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[6] Fuguo Ge. Inner products, group, ring of quaternion numbers. *Formalized Mathematics*, 16(**2**):135–139, 2008, doi:10.2478/v10037-008-0019-x.
[7] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[8] Xiquan Liang and Fuguo Ge. The quaternion numbers. *Formalized Mathematics*, 14(**4**):161–169, 2006, doi:10.2478/v10037-006-0020-1.
[9] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[10] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[11] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[12] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[13] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received October 14, 2008