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Fatou's Lemma and the Lebesgue's Convergence Theorem

Noboru Endou
Gifu National College of Technology
Japan

Keiko Narita
Hirosaki-city
Aomori, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article we prove the Fatou's Lemma and Lebesgue's Convergence Theorem [10].

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The articles [15], [1], [16], [14], [11], [5], [12], [2], [3], [4], [8], [9], [13], [6], [7], and [17] provide the terminology and notation for this paper.

1. Fatou's Lemma

For simplicity, we adopt the following rules: X denotes a non empty set, F denotes a sequence of partial functions from X into $\overline{\mathbb{R}}$ with the same dom, s_1, s_2, s_3 denote sequences of extended reals, x denotes an element of X, a, r denote extended real numbers, and n, m, k denote natural numbers.

We now state several propositions:

(1) If for every natural number n holds $s_2(n) \leq s_3(n)$, then $\inf \operatorname{rng} s_2 \leq \inf \operatorname{rng} s_3$.

(2) Suppose that for every natural number n holds $s_2(n) \leq s_3(n)$. Then

(i) (the inferior real sequence of s_2)$(k) \leq$ (the inferior real sequence of s_3)(k), and

(ii) (the superior real sequence of s_2)$(k) \leq$ (the superior real sequence of s_3)(k).

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(3) If for every natural number n holds $s_2(n) \leq s_3(n)$, then $\liminf s_2 \leq \liminf s_3$ and $\limsup s_2 \leq \limsup s_3$.

(4) If for every natural number n holds $s_1(n) \geq a$, then $\inf s_1 \geq a$.

(5) If for every natural number n holds $s_1(n) \leq a$, then $\sup s_1 \leq a$.

(6) For every element n of $\mathbb{N}$ such that $x \in \mathrm{dom} \inf(F \uparrow n)$ holds $(\inf(F \uparrow n))(x) = \inf((F\#x) \uparrow n)$.

In the sequel S is a σ-field of subsets of X, M is a σ-measure on S, and E is an element of S.

We now state the proposition

(7) Suppose $E = \mathrm{dom}\, F(0)$ and for every n holds $F(n)$ is non-negative and $F(n)$ is measurable on E. Then there exists a sequence I of extended reals such that for every n holds $I(n) = \int F(n)\,\mathrm{d}M$ and $\int \liminf F\,\mathrm{d}M \leq \liminf I$.

2. Lebesgue's Convergence Theorem

We now state three propositions:

(8) For all non empty subsets X, Y of $\overline{\mathbb{R}}$ and for every extended real number r such that $X = \{r\}$ and $r \in \mathbb{R}$ holds $\sup(X + Y) = \sup X + \sup Y$.

(9) For all non empty subsets X, Y of $\overline{\mathbb{R}}$ and for every extended real number r such that $X = \{r\}$ and $r \in \mathbb{R}$ holds $\inf(X + Y) = \inf X + \inf Y$.

(10) If $r \in \mathbb{R}$ and for every natural number n holds $s_2(n) = r + s_3(n)$, then $\liminf s_2 = r + \liminf s_3$ and $\limsup s_2 = r + \limsup s_3$.

We follow the rules: F_1, F_2 are sequences of partial functions from X into $\overline{\mathbb{R}}$ and f, g, P are partial functions from X to $\overline{\mathbb{R}}$.

We now state several propositions:

(11) Suppose that

(i) $\mathrm{dom}\, F_1(0) = \mathrm{dom}\, F_2(0)$,

(ii) F_1 has the same dom,

(iii) F_2 has the same dom,

(iv) $f^{-1}(\{+\infty\}) = \emptyset$,

(v) $f^{-1}(\{-\infty\}) = \emptyset$, and

(vi) for every natural number n holds $F_1(n) = f + F_2(n)$.

Then $\liminf F_1 = f + \liminf F_2$ and $\limsup F_1 = f + \limsup F_2$.

(12) $s_1 \uparrow 0 = s_1$.

(13) If f is integrable on M and g is integrable on M, then $f - g$ is integrable on M.

(14) Suppose f is integrable on M and g is integrable on M. Then there exists an element E of S such that $E = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and $\int f - g\,\mathrm{d}M = \int f{\restriction}E\,\mathrm{d}M + \int (-g){\restriction}E\,\mathrm{d}M$.

(15) If for every natural number n holds $s_2(n) = -s_3(n)$, then $\liminf s_3 = -\limsup s_2$ and $\limsup s_3 = -\liminf s_2$.

(16) Suppose $\operatorname{dom} F_1(0) = \operatorname{dom} F_2(0)$ and F_1 has the same dom and F_2 has the same dom and for every natural number n holds $F_1(n) = -F_2(n)$. Then $\liminf F_1 = -\limsup F_2$ and $\limsup F_1 = -\liminf F_2$.

(17) Suppose that

(i) $E = \operatorname{dom} F(0)$,
(ii) $E = \operatorname{dom} P$,
(iii) for every natural number n holds $F(n)$ is measurable on E,
(iv) P is integrable on M,
(v) P is non-negative, and
(vi) for every element x of X and for every natural number n such that $x \in E$ holds $|F(n)|(x) \leq P(x)$.

Then

(vii) for every natural number n holds $|F(n)|$ is integrable on M,
(viii) $|\liminf F|$ is integrable on M, and
(ix) $|\limsup F|$ is integrable on M.

(18) Suppose that

(i) $E = \operatorname{dom} F(0)$,
(ii) $E = \operatorname{dom} P$,
(iii) for every natural number n holds $F(n)$ is measurable on E,
(iv) P is integrable on M,
(v) P is non-negative, and
(vi) for every element x of X and for every natural number n such that $x \in E$ holds $|F(n)|(x) \leq P(x)$.

Then there exists a sequence I of extended reals such that

(vii) for every natural number n holds $I(n) = \int F(n) \, \mathrm{d}M$,
(viii) $\liminf I \geq \int \liminf F \, \mathrm{d}M$,
(ix) $\limsup I \leq \int \limsup F \, \mathrm{d}M$, and
(x) if for every element x of X such that $x \in E$ holds $F\#x$ is convergent, then I is convergent and $\lim I = \int \lim F \, \mathrm{d}M$.

(19) Suppose that

(i) $E = \operatorname{dom} F(0)$,
(ii) for every n holds $F(n)$ is non-negative and $F(n)$ is measurable on E,
(iii) for all x, n, m such that $x \in E$ and $n \leq m$ holds $F(n)(x) \geq F(m)(x)$, and
(iv) $\int F(0){\restriction}E \, \mathrm{d}M < +\infty$.

Then there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int F(n) \, \mathrm{d}M$ and I is convergent and $\lim I = \int \lim F \, \mathrm{d}M$.

Let X be a set and let F be a sequence of partial functions from X into $\overline{\mathbb{R}}$. We say that F is uniformly bounded if and only if:

(Def. 1) There exists a real number K such that for every natural number n and for every set x such that $x \in \operatorname{dom} F(0)$ holds $|F(n)(x)| \leq K$.

Next we state the proposition

(20) Suppose that
  (i) $M(E) < +\infty$,
  (ii) $E = \operatorname{dom} F(0)$,
  (iii) for every natural number n holds $F(n)$ is measurable on E,
  (iv) F is uniformly bounded, and
  (v) for every element x of X such that $x \in E$ holds $F \# x$ is convergent.

  Then
  (vi) for every natural number n holds $F(n)$ is integrable on M,
  (vii) $\lim F$ is integrable on M, and
  (viii) there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int F(n)\,\mathrm{d}M$ and I is convergent and $\lim I = \int \lim F\,\mathrm{d}M$.

Let X be a set, let F be a sequence of partial functions from X into $\overline{\mathbb{R}}$, and let f be a partial function from X to $\overline{\mathbb{R}}$. We say that F is uniformly convergent to f if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) F has the same dom,
  (ii) $\operatorname{dom} F(0) = \operatorname{dom} f$, and
  (iii) for every real number e such that $e > 0$ there exists a natural number N such that for every natural number n and for every set x such that $n \geq N$ and $x \in \operatorname{dom} F(0)$ holds $|F(n)(x) - f(x)| < e$.

One can prove the following two propositions:

(21) Suppose F_1 is uniformly convergent to f. Let x be an element of X. If $x \in \operatorname{dom} F_1(0)$, then $F_1 \# x$ is convergent and $\lim(F_1 \# x) = f(x)$.

(22) Suppose that
  (i) $M(E) < +\infty$,
  (ii) $E = \operatorname{dom} F(0)$,
  (iii) for every natural number n holds $F(n)$ is integrable on M, and
  (iv) F is uniformly convergent to f.

  Then
  (v) f is integrable on M, and
  (vi) there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int F(n)\,\mathrm{d}M$ and I is convergent and $\lim I = \int f\,\mathrm{d}M$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):163–171, 1991.
[3] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.
[4] Józef Białas. The σ-additive measure theory. *Formalized Mathematics*, 2(**2**):263–270, 1991.
[5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[6] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(**2**):53–70, 2006.
[7] Noboru Endou, Yasunari Shidama, and Keiko Narita. Egoroff's theorem. *Formalized Mathematics*, 16(**1**):57–63, 2008.
[8] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Basic properties of extended real numbers. *Formalized Mathematics*, 9(**3**):491–494, 2001.
[9] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definitions and basic properties of measurable functions. *Formalized Mathematics*, 9(**3**):495–500, 2001.
[10] P. R. Halmos. *Measure Theory.* Springer-Verlag, 1987.
[11] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[12] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.
[13] Beata Perkowska. Functional sequence from a domain to a domain. *Formalized Mathematics*, 3(**1**):17–21, 1992.
[14] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[16] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[17] Hiroshi Yamazaki, Noboru Endou, Yasunari Shidama, and Hiroyuki Okazaki. Inferior limit, superior limit and convergence of sequences of extended real numbers. *Formalized Mathematics*, 15(**4**):231–236, 2007.

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