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Introduction to Matroids[1]

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Summary. The paper includes elements of the theory of matroids [23]. The formalization is done according to [12].

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The articles [7], [22], [17], [15], [8], [5], [6], [19], [9], [3], [2], [4], [1], [21], [11], [20], [18], [16], [10], [13], and [14] provide the terminology and notation for this paper.

1. Definition by Independent Sets

A subset family structure is a topological structure.

Let M be a subset family structure and let A be a subset of M. We introduce A is independent as a synonym of A is open. We introduce A is dependent as an antonym of A is open.

Let M be a subset family structure. The family of M yielding a family of subsets of M is defined as follows:

(Def. 1) The family of $M =$ the topology of M.

Let M be a subset family structure and let A be a subset of M. Let us observe that A is independent if and only if:

(Def. 2) $A \in$ the family of M.

Let M be a subset family structure. We say that M is subset-closed if and only if:

(Def. 3) The family of M is subset-closed.

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We say that M has exchange property if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let A, B be finite subsets of M. Suppose $A \in$ the family of M and $B \in$ the family of M and $\operatorname{card} B = \operatorname{card} A + 1$. Then there exists an element e of M such that $e \in B \setminus A$ and $A \cup \{e\} \in$ the family of M.

One can check that there exists a subset family structure which is strict, non empty, non void, finite, and subset-closed and has exchange property.

Let M be a non void subset family structure. One can verify that there exists a subset of M which is independent.

Let M be a subset-closed subset family structure. One can verify that the family of M is subset-closed.

We now state the proposition

(1) Let M be a non void subset-closed subset family structure, A be an independent subset of M, and B be a set. If $B \subseteq A$, then B is an independent subset of M.

Let M be a non void subset-closed subset family structure. Note that there exists a subset of M which is finite and independent.

A matroid is a non empty non void subset-closed subset family structure with exchange property.

One can prove the following proposition

(2) For every subset-closed subset family structure M holds M is non void iff $\emptyset \in$ the family of M.

Let M be a non void subset-closed subset family structure. Note that $\emptyset_{\text{the carrier of } M}$ is independent.

The following proposition is true

(3) Let M be a non void subset family structure. Then M is subset-closed if and only if for all subsets A, B of M such that A is independent and $B \subseteq A$ holds B is independent.

Let M be a non void subset-closed subset family structure, let A be an independent subset of M, and let B be a set. One can check the following observations:

* $A \cap B$ is independent,
* $B \cap A$ is independent, and
* $A \setminus B$ is independent.

Next we state the proposition

(4) Let M be a non void non empty subset family structure. Then M has exchange property if and only if for all finite subsets A, B of M such that A is independent and B is independent and $\operatorname{card} B = \operatorname{card} A + 1$ there exists an element e of M such that $e \in B \setminus A$ and $A \cup \{e\}$ is independent.

Let A be a set. We introduce A is finite-membered as a synonym of A has finite elements.

Let A be a set. Let us observe that A is finite-membered if and only if:

(Def. 5) For every set B such that $B \in A$ holds B is finite.

Let M be a subset family structure. We say that M is finite-membered if and only if:

(Def. 6) The family of M is finite-membered.

Let M be a subset family structure. We say that M is finite-degree if and only if the conditions (Def. 7) are satisfied.

(Def. 7)(i) M is finite-membered, and

(ii) there exists a natural number n such that for every finite subset A of M such that A is independent holds $\operatorname{card} A \leq n$.

Let us note that every subset family structure which is finite-degree is also finite-membered and every subset family structure which is finite is also finite-degree.

2. Examples

Let us note that there exists a set which is mutually-disjoint and non empty and has non empty elements.

The following propositions are true:

(5) For all finite sets A, B such that $\operatorname{card} A < \operatorname{card} B$ there exists a set x such that $x \in B \setminus A$.

(6) For every mutually-disjoint non empty set P with non empty elements holds every choice function of P is one-to-one.

Let us mention that every discrete subset family structure is non void and subset-closed and has exchange property.

Next we state the proposition

(7) Every non empty discrete topological structure is a matroid.

Let P be a set. The functor $\operatorname{ProdMatroid} P$ yields a strict subset family structure and is defined by the conditions (Def. 8).

(Def. 8)(i) The carrier of $\operatorname{ProdMatroid} P = \bigcup P$, and

(ii) the family of $\operatorname{ProdMatroid} P = \{A \subseteq \bigcup P : \bigwedge_{D : \mathrm{set}} (D \in P \Rightarrow \bigvee_{d : \mathrm{set}} A \cap D \subseteq \{d\})\}$.

Let P be a non empty set with non empty elements. One can verify that $\operatorname{ProdMatroid} P$ is non empty.

Next we state the proposition

(8) Let P be a set and A be a subset of $\operatorname{ProdMatroid} P$. Then A is independent if and only if for every element D of P there exists an element d of D such that $A \cap D \subseteq \{d\}$.

Let P be a set. One can verify that ProdMatroid P is non void and subset-closed.

Next we state two propositions:

(9) Let P be a mutually-disjoint set and x be a subset of ProdMatroid P. Then there exists a function f from x into P such that for every set a such that $a \in x$ holds $a \in f(a)$.

(10) Let P be a mutually-disjoint set, x be a subset of ProdMatroid P, and f be a function from x into P. Suppose that for every set a such that $a \in x$ holds $a \in f(a)$. Then x is independent if and only if f is one-to-one.

Let P be a mutually-disjoint set. Observe that ProdMatroid P has exchange property.

Let X be a finite set and let P be a subset of 2^X. One can check that ProdMatroid P is finite.

Let X be a set. Observe that every partition of X is mutually-disjoint.

One can check that there exists a matroid which is finite and strict.

Let M be a finite-membered non void subset family structure. Observe that every independent subset of M is finite.

Let F be a field and let V be a vector space over F. The matroid of linearly independent subsets of V is a strict subset family structure and is defined by the conditions (Def. 9).

(Def. 9)(i) The carrier of the matroid of linearly independent subsets of V = the carrier of V, and

(ii) the family of the matroid of linearly independent subsets of $V = \{A \subseteq V\colon A$ is linearly independent$\}$.

Let F be a field and let V be a vector space over F. Note that the matroid of linearly independent subsets of V is non empty, non void, and subset-closed.

Let F be a field and let V be a vector space over F. Observe that there exists a subset of V which is linearly independent and empty.

The following three propositions are true:

(11) Let F be a field, V be a vector space over F, and A be a subset of the matroid of linearly independent subsets of V. Then A is independent if and only if A is a linearly independent subset of V.

(12) Let F be a field, V be a vector space over F, and A, B be finite subsets of V. Suppose $B \subseteq A$. Let v be a vector of V. Suppose $v \in \mathrm{Lin}(A)$ and $v \notin \mathrm{Lin}(B)$. Then there exists a vector w of V such that $w \in A \setminus B$ and $w \in \mathrm{Lin}((A \setminus \{w\}) \cup \{v\})$.

(13) Let F be a field, V be a vector space over F, and A be a subset of V. Suppose A is linearly independent. Let a be an element of V. If $a \notin$ the carrier of $\mathrm{Lin}(A)$, then $A \cup \{a\}$ is linearly independent.

Let F be a field and let V be a vector space over F. Observe that the matroid of linearly independent subsets of V has exchange property.

Let F be a field and let V be a finite dimensional vector space over F. Note that the matroid of linearly independent subsets of V is finite-membered.

3. Maximal Independent Subsets, Ranks, and Basis

Let M be a subset family structure and let A, C be subsets of M. We say that A is maximal independent in C if and only if:

(Def. 10) A is independent and $A \subseteq C$ and for every subset B of M such that B is independent and $B \subseteq C$ and $A \subseteq B$ holds $A = B$.

The following propositions are true:

(14) Let M be a non void finite-degree subset family structure and C, A be subsets of M. Suppose $A \subseteq C$ and A is independent. Then there exists an independent subset B of M such that $A \subseteq B$ and B is maximal independent in C.

(15) Let M be a non void finite-degree subset-closed subset family structure and C be a subset of M. Then there exists an independent subset of M which is maximal independent in C.

(16) Let M be a non empty non void subset-closed finite-degree subset family structure. Then M is a matroid if and only if for every subset C of M and for all independent subsets A, B of M such that A is maximal independent in C and B is maximal independent in C holds $\operatorname{card} A = \operatorname{card} B$.

Let M be a finite-degree matroid and let C be a subset of M. The functor $\operatorname{Rnk} C$ yields a natural number and is defined by:

(Def. 11) $\operatorname{Rnk} C = \bigcup\{\operatorname{card} A; A$ ranges over independent subsets of M: $A \subseteq C\}$.

One can prove the following propositions:

(17) Let M be a finite-degree matroid, C be a subset of M, and A be an independent subset of M. If $A \subseteq C$, then $\operatorname{card} A \leq \operatorname{Rnk} C$.

(18) Let M be a finite-degree matroid and C be a subset of M. Then there exists an independent subset A of M such that $A \subseteq C$ and $\operatorname{card} A = \operatorname{Rnk} C$.

(19) Let M be a finite-degree matroid, C be a subset of M, and A be an independent subset of M. Then A is maximal independent in C if and only if $A \subseteq C$ and $\operatorname{card} A = \operatorname{Rnk} C$.

(20) For every finite-degree matroid M and for every finite subset C of M holds $\operatorname{Rnk} C \leq \operatorname{card} C$.

(21) Let M be a finite-degree matroid and C be a finite subset of M. Then C is independent if and only if $\operatorname{card} C = \operatorname{Rnk} C$.

Let M be a finite-degree matroid. The functor $\operatorname{Rnk} M$ yielding a natural number is defined by:

(Def. 12) $\mathrm{Rnk}\, M = \mathrm{Rnk}(\Omega_M)$.

Let M be a non void finite-degree subset family structure. An independent subset of M is said to be a basis of M if:

(Def. 13) It is maximal independent in Ω_M.

One can prove the following propositions:

(22) For every finite-degree matroid M and for all bases B_1, B_2 of M holds $\mathrm{card}\, B_1 = \mathrm{card}\, B_2$.

(23) For every finite-degree matroid M and for every independent subset A of M there exists a basis B of M such that $A \subseteq B$.

We follow the rules: M is a finite-degree matroid, A, B, C are subsets of M, and e, f are elements of M.

Next we state four propositions:

(24) If $A \subseteq B$, then $\mathrm{Rnk}\, A \leq \mathrm{Rnk}\, B$.

(25) $\mathrm{Rnk}(A \cup B) + \mathrm{Rnk}(A \cap B) \leq \mathrm{Rnk}\, A + \mathrm{Rnk}\, B$.

(26) $\mathrm{Rnk}\, A \leq \mathrm{Rnk}(A \cup B)$ and $\mathrm{Rnk}(A \cup \{e\}) \leq \mathrm{Rnk}\, A + 1$.

(27) If $\mathrm{Rnk}(A \cup \{e\}) = \mathrm{Rnk}(A \cup \{f\})$ and $\mathrm{Rnk}(A \cup \{f\}) = \mathrm{Rnk}\, A$, then $\mathrm{Rnk}(A \cup \{e, f\}) = \mathrm{Rnk}\, A$.

4. Dependence on a Set, Spans, and Cycles

Let M be a finite-degree matroid, let e be an element of M, and let A be a subset of M. We say that e is dependent on A if and only if:

(Def. 14) $\mathrm{Rnk}(A \cup \{e\}) = \mathrm{Rnk}\, A$.

We now state two propositions:

(28) If $e \in A$, then e is dependent on A.

(29) If $A \subseteq B$ and e is dependent on A, then e is dependent on B.

Let M be a finite-degree matroid and let A be a subset of M. The functor $\mathrm{Span}\, A$ yielding a subset of M is defined as follows:

(Def. 15) $\mathrm{Span}\, A = \{e \in M\colon e \text{ is dependent on } A\}$.

Next we state several propositions:

(30) $e \in \mathrm{Span}\, A$ iff $\mathrm{Rnk}(A \cup \{e\}) = \mathrm{Rnk}\, A$.

(31) $A \subseteq \mathrm{Span}\, A$.

(32) If $A \subseteq B$, then $\mathrm{Span}\, A \subseteq \mathrm{Span}\, B$.

(33) $\mathrm{Rnk}\, \mathrm{Span}\, A = \mathrm{Rnk}\, A$.

(34) If e is dependent on $\mathrm{Span}\, A$, then e is dependent on A.

(35) $\mathrm{Span}\, \mathrm{Span}\, A = \mathrm{Span}\, A$.

(36) If $f \notin \mathrm{Span}\, A$ and $f \in \mathrm{Span}(A \cup \{e\})$, then $e \in \mathrm{Span}(A \cup \{f\})$.

Let M be a subset family structure and let A be a subset of M. We say that A is cycle if and only if:

(Def. 16) A is dependent and for every element e of M such that $e \in A$ holds $A \setminus \{e\}$ is independent.

Next we state the proposition

(37) If A is cycle, then A is non empty and finite.

Let us consider M. Note that every subset of M which is cycle is also non empty and finite.

One can prove the following propositions:

(38) A is cycle iff A is non empty and for every e such that $e \in A$ holds $A \setminus \{e\}$ is maximal independent in A.

(39) If A is cycle, then $\operatorname{Rnk} A + 1 = \overline{\overline{A}}$.

(40) If A is cycle and $e \in A$, then e is dependent on $A \setminus \{e\}$.

(41) If A is cycle and B is cycle and $A \subseteq B$, then $A = B$.

(42) If for every B such that $B \subseteq A$ holds B is not cycle, then A is independent.

(43) If A is cycle and B is cycle and $A \neq B$ and $e \in A \cap B$, then there exists C such that C is cycle and $C \subseteq (A \cup B) \setminus \{e\}$.

(44) If A is independent and B is cycle and C is cycle and $B \subseteq A \cup \{e\}$ and $C \subseteq A \cup \{e\}$, then $B = C$.

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