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Several Classes of BCK-algebras and their Properties

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Summary. In this article the general theory of Commutative BCK-algebras and BCI-algebras and several classes of BCK-algebras are given according to [2].

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The articles [3] and [1] provide the notation and terminology for this paper.

1. The Basics of General Theory of Commutative BCK-algebras

Let I_1 be a non empty BCI structure with 0. We say that I_1 is commutative if and only if:

(Def. 1) For all elements x, y of I_1 holds $x \setminus (x \setminus y) = y \setminus (y \setminus x)$.

Let us observe that BCI-EXAMPLE is commutative.

Let us note that there exists a BCK-algebra which is commutative.

In the sequel X denotes a BCK-algebra and I_1 denotes a non empty subset of X.

We now state a number of propositions:

(1) X is a commutative BCK-algebra iff for all elements x, y of X holds $x \setminus (x \setminus y) \leq y \setminus (y \setminus x)$.

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(2) For every BCK-algebra X and for all elements x, y of X holds $x \setminus (x \setminus y) \leq y$ and $x \setminus (x \setminus y) \leq x$.

(3) X is a commutative BCK-algebra iff for all elements x, y of X holds $x \setminus y = x \setminus (y \setminus (y \setminus x))$.

(4) X is a commutative BCK-algebra iff for all elements x, y of X holds $x \setminus (x \setminus y) = y \setminus (y \setminus (x \setminus (x \setminus y)))$.

(5) X is a commutative BCK-algebra iff for all elements x, y of X such that $x \leq y$ holds $x = y \setminus (y \setminus x)$.

(6) Let X be a non empty BCI structure with 0. Then X is a commutative BCK-algebra if and only if for all elements x, y, z of X holds $x \setminus (0_X \setminus y) = x$ and $(x \setminus z) \setminus (x \setminus y) = y \setminus z \setminus (y \setminus x)$.

(7) If X is a commutative BCK-algebra, then for all elements x, y of X such that $x \setminus y = x$ holds $y \setminus x = y$.

(8) If X is a commutative BCK-algebra, then for all elements x, y, a of X such that $y \leq a$ holds $a \setminus x \setminus (a \setminus y) = y \setminus x$.

(9) If X is a commutative BCK-algebra, then for all elements x, y of X holds $x \setminus y = x$ iff $y \setminus (y \setminus x) = 0_X$.

(10) If X is a commutative BCK-algebra, then for all elements x, y of X holds $x \setminus (y \setminus (y \setminus x)) = x \setminus y$ and $x \setminus y \setminus (x \setminus y \setminus x) = x \setminus y$.

(11) Suppose X is a commutative BCK-algebra. Let x, y, a be elements of X. If $x \leq a$, then $(a \setminus y) \setminus (a \setminus y \setminus (a \setminus x)) = a \setminus y \setminus (x \setminus y)$.

Let X be a BCI-algebra and let a be an element of X. We say that a is greatest if and only if:

(Def. 2) For every element x of X holds $x \setminus a = 0_X$.

We say that a is positive if and only if:

(Def. 3) $0_X \setminus a = 0_X$.

2. The Basics of General Theory of Commutative BCI-algebras

Let I_1 be a BCI-algebra. We say that I_1 is BCI-commutative if and only if:

(Def. 4) For all elements x, y of I_1 such that $x \setminus y = 0_{(I_1)}$ holds $x = y \setminus (y \setminus x)$.

We say that I_1 is BCI-weakly-commutative if and only if:

(Def. 5) For all elements x, y of I_1 holds $(x \setminus (x \setminus y)) \setminus (0_{(I_1)} \setminus (x \setminus y)) = y \setminus (y \setminus x)$.

One can check that BCI-EXAMPLE is BCI-commutative and BCI-weakly-commutative.

Let us note that there exists a BCI-algebra which is BCI-commutative and BCI-weakly-commutative.

The following propositions are true:

(12) For every BCI-algebra X such that there exists an element of X which is greatest holds X is a BCK-algebra.

(13) Let X be a BCI-algebra. Suppose X is p-semisimple. Then X is BCI-commutative and BCI-weakly-commutative.

(14) Every commutative BCK-algebra is a BCI-commutative BCI-algebra and a BCI-weakly-commutative BCI-algebra.

(15) If X is a BCI-weakly-commutative BCI-algebra, then X is BCI-commutative.

(16) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for all elements x, y of X holds $x \setminus (x \setminus y) = y \setminus (y \setminus (x \setminus (x \setminus y)))$.

(17) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for all elements x, y of X holds $(x \setminus (x \setminus y)) \setminus (y \setminus (y \setminus x)) = 0_X \setminus (x \setminus y)$.

(18) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for every element a of $\mathrm{AtomSet}\, X$ and for all elements x, y of $\mathrm{BranchV}\, a$ holds $x \setminus (x \setminus y) = y \setminus (y \setminus x)$.

(19) Let X be a non empty BCI structure with 0. Then X is a BCI-commutative BCI-algebra if and only if for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) \setminus (z \setminus y) = 0_X$ and $x \setminus 0_X = x$ and $x \setminus (x \setminus y) = y \setminus (y \setminus (x \setminus (x \setminus y)))$.

(20) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for all elements x, y, z of X such that $x \leq z$ and $z \setminus y \leq z \setminus x$ holds $x \leq y$.

(21) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for all elements x, y, z of X such that $x \leq y$ and $x \leq z$ holds $x \leq y \setminus (y \setminus z)$.

3. Bounded BCK-algebras

Let I_1 be a BCK-algebra. We say that I_1 is bounded if and only if:

(Def. 6) There exists an element of I_1 which is greatest.

Let us note that BCI-EXAMPLE is bounded.

One can verify that there exists a BCK-algebra which is bounded and commutative.

Let I_1 be a bounded BCK-algebra. We say that I_1 is involutory if and only if:

(Def. 7) For every element a of I_1 such that a is greatest and for every element x of I_1 holds $a \setminus (a \setminus x) = x$.

Next we state three propositions:

(22) Let X be a bounded BCK-algebra. Then X is involutory if and only if for every element a of X such that a is greatest and for all elements x, y of X holds $x \setminus y = a \setminus y \setminus (a \setminus x)$.

(23) Let X be a bounded BCK-algebra. Then X is involutory if and only if for every element a of X such that a is greatest and for all elements x, y of X holds $x \setminus (a \setminus y) = y \setminus (a \setminus x)$.

(24) Let X be a bounded BCK-algebra. Then X is involutory if and only if for every element a of X such that a is greatest and for all elements x, y of X such that $x \leq a \setminus y$ holds $y \leq a \setminus x$.

Let I_1 be a BCK-algebra and let a be an element of I_1. We say that a is Iseki if and only if:

(Def. 8) For every element x of I_1 holds $x \setminus a = 0_{(I_1)}$ and $a \setminus x = a$.

Let I_1 be a BCK-algebra. We say that I_1 is Iseki-extension if and only if:

(Def. 9) There exists an element of I_1 which is Iseki.

Let us observe that BCI-EXAMPLE is Iseki-extension.

Let X be a BCK-algebra. A non empty subset of X is said to be a commutative-ideal of X if:

(Def. 10) $0_X \in$ it and for all elements x, y, z of X such that $x \setminus y \setminus z \in$ it and $z \in$ it holds $x \setminus (y \setminus (y \setminus x)) \in$ it.

The following three propositions are true:

(25) If I_1 is a commutative-ideal of X, then for all elements x, y of X such that $x \setminus y \in I_1$ holds $x \setminus (y \setminus (y \setminus x)) \in I_1$.

(26) For every BCK-algebra X such that I_1 is a commutative-ideal of X holds I_1 is an ideal of X.

(27) If I_1 is a commutative-ideal of X, then for all elements x, y of X such that $x \setminus (x \setminus y) \in I_1$ holds $y \setminus (y \setminus x) \setminus (x \setminus y) \in I_1$.

4. Implicative and Positive-Implicative BCK-algebras

Let I_1 be a BCK-algebra. We say that I_1 is BCK-positive-implicative if and only if:

(Def. 11) For all elements x, y, z of I_1 holds $(x \setminus y) \setminus z = x \setminus z \setminus (y \setminus z)$.

We say that I_1 is BCK-implicative if and only if:

(Def. 12) For all elements x, y of I_1 holds $x \setminus (y \setminus x) = x$.

Let us observe that BCI-EXAMPLE is BCK-positive-implicative and BCK-implicative.

Let us mention that there exists a BCK-algebra which is Iseki-extension, BCK-positive-implicative, BCK-implicative, bounded, and commutative.

The following propositions are true:

(28) X is a BCK-positive-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus y = x \setminus y \setminus y$.

(29) X is a BCK-positive-implicative BCK-algebra if and only if for all elements x, y of X holds $(x \setminus (x \setminus y)) \setminus (y \setminus x) = x \setminus (x \setminus (y \setminus (y \setminus x)))$.

(30) X is a BCK-positive-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus y = x \setminus y \setminus (x \setminus (x \setminus y))$.

(31) X is a BCK-positive-implicative BCK-algebra if and only if for all elements x, y, z of X holds $x \setminus z \setminus (y \setminus z) \leq (x \setminus y) \setminus z$.

(32) X is a BCK-positive-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus y \leq x \setminus y \setminus y$.

(33) X is a BCK-positive-implicative BCK-algebra if and only if for all elements x, y of X holds $x \setminus (x \setminus (y \setminus (y \setminus x))) \leq (x \setminus (x \setminus y)) \setminus (y \setminus x)$.

(34) X is a BCK-implicative BCK-algebra if and only if X is a commutative BCK-algebra and a BCK-positive-implicative BCK-algebra.

(35) X is a BCK-implicative BCK-algebra iff for all elements x, y of X holds $(x \setminus (x \setminus y)) \setminus (x \setminus y) = y \setminus (y \setminus x)$.

(36) Let X be a non empty BCI structure with 0. Then X is a BCK-implicative BCK-algebra if and only if for all elements x, y, z of X holds $x \setminus (0_X \setminus y) = x$ and $(x \setminus z) \setminus (x \setminus y) = y \setminus z \setminus (y \setminus x) \setminus (x \setminus y)$.

(37) Let X be a bounded BCK-algebra and a be an element of X. Suppose a is greatest. Then X is BCK-implicative if and only if X is involutory and BCK-positive-implicative.

(38) X is a BCK-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus (x \setminus (y \setminus x)) = 0_X$.

(39) X is a BCK-implicative BCK-algebra iff for all elements x, y of X holds $(x \setminus (x \setminus y)) \setminus (x \setminus y) = y \setminus (y \setminus (x \setminus (x \setminus y)))$.

(40) X is a BCK-implicative BCK-algebra iff for all elements x, y, z of X holds $(x \setminus z) \setminus (x \setminus y) = y \setminus z \setminus (y \setminus x \setminus z)$.

(41) X is a BCK-implicative BCK-algebra iff for all elements x, y, z of X holds $x \setminus (x \setminus (y \setminus z)) = (y \setminus z) \setminus (y \setminus z \setminus (x \setminus z))$.

(42) X is a BCK-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus (x \setminus y) = (y \setminus (y \setminus x)) \setminus (x \setminus y)$.

(43) Let X be a bounded commutative BCK-algebra and a be an element of X. Suppose a is greatest. Then X is BCK-implicative if and only if for every element x of X holds $a \setminus x \setminus (a \setminus x \setminus x) = 0_X$.

(44) Let X be a bounded commutative BCK-algebra and a be an element of X. Suppose a is greatest. Then X is BCK-implicative if and only if for every element x of X holds $x \setminus (a \setminus x) = x$.

(45) Let X be a bounded commutative BCK-algebra and a be an element of X. Suppose a is greatest. Then X is BCK-implicative if and only if for every element x of X holds $a \setminus x \setminus x = a \setminus x$.

(46) Let X be a bounded commutative BCK-algebra and a be an element of X. Suppose a is greatest. Then X is BCK-implicative if and only if for all elements x, y of X holds $a \setminus y \setminus (a \setminus y \setminus x) = x \setminus y$.

(47) Let X be a bounded commutative BCK-algebra and a be an element of X. Suppose a is greatest. Then X is BCK-implicative if and only if for all elements x, y of X holds $y \setminus (y \setminus x) = x \setminus (a \setminus y)$.

(48) Let X be a bounded commutative BCK-algebra and a be an element of X. Suppose a is greatest. Then X is BCK-implicative if and only if for all elements x, y, z of X holds $(x \setminus (y \setminus z)) \setminus (x \setminus y) \leq x \setminus (a \setminus z)$.

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