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Congruences and Quotient Algebras of BCI-algebras

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Summary. We have formalized the BCI-algebras closely following the book [7] pp.16-19 and pp.58-65. Firstly, the article focuses on the properties of the element and then the definition and properties of congruences and quotient algebras are given. Quotient algebras are the basic tools for exploring the structures of BCI-algebras.

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The articles [11], [5], [12], [10], [13], [2], [3], [1], [8], [14], [6], [15], [4], and [9] provide the terminology and notation for this paper.

1. Basic Properties of the Element

For simplicity, we adopt the following convention: X is a BCI-algebra, I is an ideal of X, a, x, y, z, u are elements of X, f is a function from $\mathbb{N}$ into the carrier of X, and j, i, k, n, m are elements of $\mathbb{N}$.

Let us consider X, x, y and let n be an element of $\mathbb{N}$. The functor $(x \setminus y)^n$ yielding an element of X is defined by:

(Def. 1) There exists f such that $(x \setminus y)^n = f(n)$ and $f(0) = x$ and for every element j of $\mathbb{N}$ such that $j < n$ holds $f(j+1) = f(j) \setminus y$.

One can prove the following propositions:

(1) $(x \setminus y)^0 = x$.

(2) $(x \setminus y)^1 = x \setminus y$.

(3) $(x \setminus y)^2 = x \setminus y \setminus y$.

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(4) $(x \setminus y)^{n+1} = ((x \setminus y)^n) \setminus y$.
(5) $(x \setminus 0_X)^{n+1} = x$.
(6) $(0_X \setminus 0_X)^n = 0_X$.
(7) $((x \setminus y)^n) \setminus z = ((x \setminus z) \setminus y)^n$.
(8) $(x \setminus (x \setminus (x \setminus y)))^n = (x \setminus y)^n$.
(9) $((0_X \setminus x)^n)^{\mathrm{c}} = (0_X \setminus x^{\mathrm{c}})^n$.
(10) $((x \setminus y)^n \setminus y)^m = (x \setminus y)^{n+m}$.
(11) $((x \setminus y)^n \setminus z)^m = ((x \setminus z)^m \setminus y)^n$.
(12) $(((0_X \setminus x)^n)^{\mathrm{c}})^{\mathrm{c}} = (0_X \setminus x)^n$.
(13) $(0_X \setminus x)^{n+m} = ((0_X \setminus x)^n) \setminus ((0_X \setminus x)^m)^{\mathrm{c}}$.
(14) $((0_X \setminus x)^{m+n})^{\mathrm{c}} = ((0_X \setminus x)^m)^{\mathrm{c}} \setminus ((0_X \setminus x)^n)$.
(15) $((0_X \setminus ((0_X \setminus x)^m))^n)^{\mathrm{c}} = (0_X \setminus x)^{m \cdot n}$.
(16) If $(0_X \setminus x)^m = 0_X$, then $(0_X \setminus x)^{m \cdot n} = 0_X$.
(17) If $x \setminus y = x$, then $(x \setminus y)^n = x$.
(18) $(0_X \setminus (x \setminus y))^n = ((0_X \setminus x)^n) \setminus ((0_X \setminus y)^n)$.
(19) If $x \leq y$, then $(x \setminus z)^n \leq (y \setminus z)^n$.
(20) If $x \leq y$, then $(z \setminus y)^n \leq (z \setminus x)^n$.
(21) $((x \setminus z)^n) \setminus ((y \setminus z)^n) \leq x \setminus y$.
(22) $((x \setminus (x \setminus y))^n \setminus (y \setminus x))^n \leq x$.

Let us consider X, a. We introduce a is minimal as a synonym of a is atom.

Let us consider X, a. We say that a is positive if and only if:

(Def. 2) $0_X \leq a$.

We say that a is least if and only if:

(Def. 3) For every x holds $a \leq x$.

We say that a is maximal if and only if:

(Def. 4) For every x such that $a \leq x$ holds $x = a$.

We say that a is greatest if and only if:

(Def. 5) For every x holds $x \leq a$.

Let us consider X. Observe that there exists an element of X which is positive.

Let us consider X. Note that 0_X is positive and minimal.

Next we state several propositions:

(23) a is minimal iff for every x holds $a \setminus x = x^{\mathrm{c}} \setminus a^{\mathrm{c}}$.
(24) x^{c} is minimal iff for every y such that $y \leq x$ holds $x^{\mathrm{c}} = y^{\mathrm{c}}$.
(25) x^{c} is minimal iff for all y, z holds $((x \setminus z \setminus (y \setminus z))^{\mathrm{c}})^{\mathrm{c}} = y^{\mathrm{c}} \setminus x^{\mathrm{c}}$.
(26) If 0_X is maximal, then every a is minimal.
(27) If there exists x which is greatest, then every a is positive.

(28) $x \setminus (x^{c})^{c}$ is a positive element of X.

(29) a is minimal iff $(a^{c})^{c} = a$.

(30) a is minimal iff there exists x such that $a = x^{c}$.

Let us consider X, x. We say that x is nilpotent if and only if:

(Def. 6) There exists a non empty element k of $\mathbb{N}$ such that $(0_X \setminus x)^k = 0_X$.

Let us consider X. We say that X is nilpotent if and only if:

(Def. 7) Every element of X is nilpotent.

Let us consider X, x. Let us assume that x is nilpotent. The functor $\mathrm{ord}(x)$ yielding a non empty element of $\mathbb{N}$ is defined by:

(Def. 8) $(0_X \setminus x)^{\mathrm{ord}(x)} = 0_X$ and for every element m of $\mathbb{N}$ such that $(0_X \setminus x)^m = 0_X$ and $m \neq 0$ holds $\mathrm{ord}(x) \leq m$.

Let us consider X. One can verify that 0_X is nilpotent.

We now state four propositions:

(31) x is a positive element of X iff x is nilpotent and $\mathrm{ord}(x) = 1$.

(32) X is a BCK-algebra iff for every x holds $\mathrm{ord}(x) = 1$ and x is nilpotent.

(33) $(0_X \setminus x^{c})^n$ is minimal.

(34) If x is nilpotent, then $\mathrm{ord}(x) = \mathrm{ord}(x^{c})$.

2. Congruences and Quotient Algebras

Let X be a BCI-algebra. An equivalence relation of X is said to be a congruence of X if:

(Def. 9) For all elements x, y, u, v of X such that $\langle x, y \rangle \in$ it and $\langle u, v \rangle \in$ it holds $\langle x \setminus u, y \setminus v \rangle \in$ it.

Let X be a BCI-algebra. An equivalence relation of X is said to be an L-congruence of X if:

(Def. 10) For all elements x, y of X such that $\langle x, y \rangle \in$ it and for every element u of X holds $\langle u \setminus x, u \setminus y \rangle \in$ it.

Let X be a BCI-algebra. An equivalence relation of X is said to be an R-congruence of X if:

(Def. 11) For all elements x, y of X such that $\langle x, y \rangle \in$ it and for every element u of X holds $\langle x \setminus u, y \setminus u \rangle \in$ it.

Let X be a BCI-algebra and let A be an ideal of X. A binary relation on X is said to be an I-congruence of X by A if:

(Def. 12) For all elements x, y of X holds $\langle x, y \rangle \in$ it iff $x \setminus y \in A$ and $y \setminus x \in A$.

Let X be a BCI-algebra and let A be an ideal of X. Note that every I-congruence of X by A is total, symmetric, and transitive.

Let X be a BCI-algebra. The functor $\mathrm{IConSet}\, X$ is defined as follows:

(Def. 13) For every set A_1 holds $A_1 \in \mathrm{IConSet}\, X$ iff there exists an ideal I of X such that A_1 is an I-congruence of X by I.

Let X be a BCI-algebra. The functor $\mathrm{ConSet}\, X$ is defined as follows:

(Def. 14) $\mathrm{ConSet}\, X = \{R : R$ ranges over congruences of $X\}$.

The functor $\mathrm{LConSet}\, X$ is defined by:

(Def. 15) $\mathrm{LConSet}\, X = \{R : R$ ranges over L-congruences of $X\}$.

The functor $\mathrm{RConSet}\, X$ is defined as follows:

(Def. 16) $\mathrm{RConSet}\, X = \{R : R$ ranges over R-congruences of $X\}$.

For simplicity, we adopt the following rules: R is an equivalence relation of X, R_1 is an I-congruence of X by I, E is a congruence of X, R_2 is an R-congruence of X, and L_1 is an L-congruence of X.

We now state three propositions:

(35) For all X, E holds $[0_X]_E$ is a closed ideal of X.

(36) R is a congruence of X iff R is an R-congruence of X and an L-congruence of X.

(37) R_1 is a congruence of X.

Let X be a BCI-algebra and let I be an ideal of X. We see that the I-congruence of X by I is a congruence of X.

One can prove the following propositions:

(38) $[0_X]_{(R_1)} \subseteq I$.

(39) I is closed iff $I = [0_X]_{(R_1)}$.

(40) If $\langle x, y\rangle \in E$, then $x \setminus y \in [0_X]_E$ and $y \setminus x \in [0_X]_E$.

(41) Let A, I be ideals of X, I_1 be an I-congruence of X by A, and I_2 be an I-congruence of X by I. If $[0_X]_{(I_1)} = [0_X]_{(I_2)}$, then $I_1 = I_2$.

(42) If $\langle x, y\rangle \in E$ and $u \in [0_X]_E$, then $\langle x, (y \setminus u)^k\rangle \in E$.

(43) Suppose that for all X, x, y there exist i, j, m, n such that $((x \setminus (x \setminus y))^i \setminus (y \setminus x))^j = ((y \setminus (y \setminus x))^m \setminus (x \setminus y))^n$. Let given E, I. If $I = [0_X]_E$, then E is an I-congruence of X by I.

(44) $\mathrm{IConSet}\, X \subseteq \mathrm{ConSet}\, X$.

(45) $\mathrm{ConSet}\, X \subseteq \mathrm{LConSet}\, X$.

(46) $\mathrm{ConSet}\, X \subseteq \mathrm{RConSet}\, X$.

(47) $\mathrm{ConSet}\, X = \mathrm{LConSet}\, X \cap \mathrm{RConSet}\, X$.

(48) If every L_1 is an I-congruence of X by I, then $E = R_1$.

(49) If every R_2 is an I-congruence of X by I, then $E = R_1$.

(50) $[0_X]_{(L_1)}$ is a closed ideal of X.

In the sequel E denotes a congruence of X and R_1 denotes an I-congruence of X by I.

Let us consider X, E. Note that $\mathrm{Classes}\, E$ is non empty.

Let us consider X, E. The functor EqClaOp E yielding a binary operation on Classes E is defined by:

(Def. 17) For all elements W_1, W_2 of Classes E and for all x, y such that $W_1 = [x]_E$ and $W_2 = [y]_E$ holds $(\mathrm{EqClaOp}\,E)(W_1, W_2) = [x \setminus y]_E$.

Let us consider X, E. The functor zeroEqC E yields an element of Classes E and is defined as follows:

(Def. 18) $\mathrm{zeroEqC}\,E = [0_X]_E$.

Let us consider X, E. The functor ${}^X/_E$ yielding a BCI structure with 0 is defined by:

(Def. 19) ${}^X/_E = \langle \mathrm{Classes}\,E, \mathrm{EqClaOp}\,E, \mathrm{zeroEqC}\,E \rangle$.

Let us consider X and let E be a congruence of X. One can check that ${}^X/_E$ is non empty.

In the sequel W_1, W_2 denote elements of Classes E.

Let us consider X, E, W_1, W_2. The functor $W_1 \setminus W_2$ yielding an element of Classes E is defined by:

(Def. 20) $W_1 \setminus W_2 = (\mathrm{EqClaOp}\,E)(W_1, W_2)$.

Next we state the proposition

(51) ${}^X/_{R_1}$ is a BCI-algebra.

Let us consider X, I, R_1. Note that ${}^X/_{R_1}$ is strict, B, C, I, and BCI-4.

Next we state the proposition

(52) For all X, I such that $I = \text{BCK-part}\,X$ and for every I-congruence R_1 of X by I holds ${}^X/_{R_1}$ is a p-semisimple BCI-algebra.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[5] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[6] Yuzhong Ding. Several classes of BCI-algebras and their properties. *Formalized Mathematics*, 15(**1**):1–9, 2007.

[7] Yisheng Huang. *BCI-algebras.* Science Press, 2006.

[8] Library Committee of the Association of Mizar Users. Binary operations on numbers. *To appear in Formalized Mathematics.*

[9] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.

[10] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[12] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[13] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[14] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

[15] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(**1**):85–89, 1990.

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