FORMALIZED MATHEMATICS
Volume 14, Number 4, Pages 161–169
University of Białystok, 2006

The Quaternion Numbers

Xiquan Liang
Qingdao University of Science and Technology
China

Fuguo Ge
Qingdao University of Science and Technology
China

Summary. In this article, we define the set $\mathbb{H}$ of quaternion numbers as the set of all ordered sequences $q = \langle x, y, w, z \rangle$ where x,y,w and z are real numbers. The addition, difference and multiplication of the quaternion numbers are also defined. We define the real and imaginary parts of q and denote this by $x = \Re(q)$, $y = \Im_1(q)$, $w = \Im_2(q)$, $z = \Im_3(q)$. We define the addition, difference, multiplication again and denote this operation by real and three imaginary parts. We define the conjugate of q denoted by $q*'$ and the absolute value of q denoted by $|q|$. We also give some properties of quaternion numbers.

MML identifier: `QUATERNI`, version: `7.8.03 4.75.958`

The articles [14], [16], [2], [1], [12], [17], [4], [5], [6], [13], [3], [11], [7], [8], [15], [18], [9], and [10] provide the terminology and notation for this paper.

We use the following convention: a, b, c, d, x, y, w, z, x_1, x_2, x_3, x_4 denote sets and A denotes a non empty set.

The functor $\mathbb{H}$ is defined by:

(Def. 1) $\mathbb{H} = (\mathbb{R}^4 \setminus \{x; x$ ranges over elements of $\mathbb{R}^4$: $x(2) = 0 \wedge x(3) = 0\}) \cup \mathbb{C}$.

Let x be a number. We say that x is quaternion if and only if:

(Def. 2) $x \in \mathbb{H}$.

Let us observe that $\mathbb{H}$ is non empty.

Let us consider x, y, w, z, a, b, c, d. The functor $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d]$ yields a set and is defined as follows:

(Def. 3) $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] = [x \longmapsto a, y \longmapsto b] {+\cdot} [w \longmapsto c, z \longmapsto d]$.

Let us consider x, y, w, z, a, b, c, d. Note that $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d]$ is function-like and relation-like.

Next we state several propositions:

ISSN 1426–2630

(1) $\mathrm{dom}[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] = \{x, y, w, z\}$.

(2) $\mathrm{rng}[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] \subseteq \{a, b, c, d\}$.

(3) Suppose x, y, w, z are mutually different. Then $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](x) = a$ and $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](y) = b$ and $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](w) = c$ and $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d](z) = d$.

(4) If x, y, w, z are mutually different, then $\mathrm{rng}[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d] = \{a, b, c, d\}$.

(5) $\{x_1, x_2, x_3, x_4\} \subseteq X$ iff $x_1 \in X$ and $x_2 \in X$ and $x_3 \in X$ and $x_4 \in X$.

Let us consider A, x, y, w, z and let a, b, c, d be elements of A. Then $[x \mapsto a, y \mapsto b, w \mapsto c, z \mapsto d]$ is a function from $\{x, y, w, z\}$ into A.

The functor j is defined by:

(Def. 4) $j = [0 \mapsto 0, 1 \mapsto 0, 2 \mapsto 1, 3 \mapsto 0]$.

The functor k is defined by:

(Def. 5) $k = [0 \mapsto 0, 1 \mapsto 0, 2 \mapsto 0, 3 \mapsto 1]$.

One can check the following observations:

* i is quaternion,
* j is quaternion, and
* k is quaternion.

Let us observe that there exists a number which is quaternion.

Let us mention that every element of $\mathbb{H}$ is quaternion.

Let x, y, w, z be elements of $\mathbb{R}$. The functor $\langle x, y, w, z\rangle_{\mathbb{H}}$ yields an element of $\mathbb{H}$ and is defined as follows:

(Def. 6) $\langle x, y, w, z\rangle_{\mathbb{H}} = \begin{cases} x + yi, & \text{if } w = 0 \text{ and } z = 0, \\ [0 \mapsto x, 1 \mapsto y, 2 \mapsto w, 3 \mapsto z], & \text{otherwise.} \end{cases}$

Next we state three propositions:

(6) Let a, b, c, d, e, i, j, k be sets and g be a function. Suppose $a \neq b$ and $c \neq d$ and $\mathrm{dom}\, g = \{a, b, c, d\}$ and $g(a) = e$ and $g(b) = i$ and $g(c) = j$ and $g(d) = k$. Then $g = [a \mapsto e, b \mapsto i, c \mapsto j, d \mapsto k]$.

(7) For every element g of $\mathbb{H}$ there exist elements r, s, t, u of $\mathbb{R}$ such that $g = \langle r, s, t, u\rangle_{\mathbb{H}}$.

(8) If a, c, x, w are mutually different, then $[a \mapsto b, c \mapsto d, x \mapsto y, w \mapsto z] = \{\langle a, b\rangle, \langle c, d\rangle, \langle x, y\rangle, \langle w, z\rangle\}$.

We adopt the following convention: a, b, c, d are elements of $\mathbb{R}$ and r, s, t are elements of $\mathbb{Q}_+$.

One can prove the following four propositions:

(9) Let A be a subset of $\mathbb{Q}_+$. Suppose there exists t such that $t \in A$ and $t \neq \emptyset$ and for all r, s such that $r \in A$ and $s \leq r$ holds $s \in A$. Then there exist elements r_1, r_2, r_3, r_4, r_5 of $\mathbb{Q}_+$ such that

$r_1 \in A$ and $r_2 \in A$ and $r_3 \in A$ and $r_4 \in A$ and $r_5 \in A$ and $r_1 \neq r_2$ and $r_1 \neq r_3$ and $r_1 \neq r_4$ and $r_1 \neq r_5$ and $r_2 \neq r_3$ and $r_2 \neq r_4$ and $r_2 \neq r_5$ and $r_3 \neq r_4$ and $r_3 \neq r_5$ and $r_4 \neq r_5$.

(10) $[0 \mapsto a, 1 \mapsto b, 2 \mapsto c, 3 \mapsto d] \notin \mathbb{C}$.

(11) Let a, b, c, d, x, y, z, w, x', y', z', w' be sets. Suppose a, b, c, d are mutually different and $[a \mapsto x, b \mapsto y, c \mapsto z, d \mapsto w] = [a \mapsto x', b \mapsto y', c \mapsto z', d \mapsto w']$. Then $x = x'$ and $y = y'$ and $z = z'$ and $w = w'$.

(12) For all elements x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $\langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{H}} = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ holds $x_1 = y_1$ and $x_2 = y_2$ and $x_3 = y_3$ and $x_4 = y_4$.

Let x, y be quaternion numbers. The functor $x + y$ is defined by:

(Def. 7) There exist elements x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $x = \langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{H}}$ and $y = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ and $x + y = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4 \rangle_{\mathbb{H}}$.

Let us observe that the functor $x + y$ is commutative.

Let z be a quaternion number. The functor $-z$ yields a quaternion number and is defined by:

(Def. 8) $z + -z = 0$.

Let us observe that the functor $-z$ is involutive.

Let x, y be quaternion numbers. The functor $x - y$ is defined as follows:

(Def. 9) $x - y = x + -y$.

Let x, y be quaternion numbers. The functor $x \cdot y$ is defined by the condition (Def. 10).

(Def. 10) There exist elements x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $x = \langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{H}}$ and $y = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ and $x \cdot y = \langle x_1 \cdot y_1 - x_2 \cdot y_2 - x_3 \cdot y_3 - x_4 \cdot y_4, (x_1 \cdot y_2 + x_2 \cdot y_1 + x_3 \cdot y_4) - x_4 \cdot y_3, (x_1 \cdot y_3 + y_1 \cdot x_3 + y_2 \cdot x_4) - y_4 \cdot x_2, (x_1 \cdot y_4 + x_4 \cdot y_1 + x_2 \cdot y_3) - x_3 \cdot y_2 \rangle_{\mathbb{H}}$.

Let z, z' be quaternion numbers. One can verify the following observations:

* $z + z'$ is quaternion,
* $z \cdot z'$ is quaternion, and
* $z - z'$ is quaternion.

j Is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 11) $j = \langle 0, 0, 1, 0 \rangle_{\mathbb{H}}$.

Then k is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 12) $k = \langle 0, 0, 0, 1 \rangle_{\mathbb{H}}$.

One can prove the following propositions:

(13) $i \cdot i = -1$.

(14) $j \cdot j = -1$.

(15) $k \cdot k = -1$.

(16) $i \cdot j = k$.

(17) $j \cdot k = i$.

(18) $k \cdot i = j$.

(19) $i \cdot j = -j \cdot i$.

(20) $j \cdot k = -k \cdot j$.

(21) $k \cdot i = -i \cdot k$.

Let z be a quaternion number. The functor $\Re(z)$ is defined as follows:

(Def. 13)(i) There exists a complex number z' such that $z = z'$ and $\Re(z) = \Re(z')$ if $z \in \mathbb{C}$,

(ii) there exists a function f from 4 into $\mathbb{R}$ such that $z = f$ and $\Re(z) = f(0)$, otherwise.

The functor $\Im_1(z)$ is defined by:

(Def. 14)(i) There exists a complex number z' such that $z = z'$ and $\Im_1(z) = \Im(z')$ if $z \in \mathbb{C}$,

(ii) there exists a function f from 4 into $\mathbb{R}$ such that $z = f$ and $\Im_1(z) = f(1)$, otherwise.

The functor $\Im_2(z)$ is defined as follows:

(Def. 15)(i) $\Im_2(z) = 0$ if $z \in \mathbb{C}$,

(ii) there exists a function f from 4 into $\mathbb{R}$ such that $z = f$ and $\Im_2(z) = f(2)$, otherwise.

The functor $\Im_3(z)$ is defined by:

(Def. 16)(i) $\Im_3(z) = 0$ if $z \in \mathbb{C}$,

(ii) there exists a function f from 4 into $\mathbb{R}$ such that $z = f$ and $\Im_3(z) = f(3)$, otherwise.

Let z be a quaternion number. One can check the following observations:

* $\Re(z)$ is real,
* $\Im_1(z)$ is real,
* $\Im_2(z)$ is real, and
* $\Im_3(z)$ is real.

Let z be a quaternion number. Then $\Re(z)$ is a real number. Then $\Im_1(z)$ is a real number. Then $\Im_2(z)$ is a real number. Then $\Im_3(z)$ is a real number.

One can prove the following two propositions:

(22) For every function f from 4 into $\mathbb{R}$ there exist a, b, c, d such that $f = [0 \mapsto a, 1 \mapsto b, 2 \mapsto c, 3 \mapsto d]$.

(23) $\Re(\langle a, b, c, d\rangle_{\mathbb{H}}) = a$ and $\Im_1(\langle a, b, c, d\rangle_{\mathbb{H}}) = b$ and $\Im_2(\langle a, b, c, d\rangle_{\mathbb{H}}) = c$ and $\Im_3(\langle a, b, c, d\rangle_{\mathbb{H}}) = d$.

In the sequel z, z_1, z_2, z_3, z_4 denote quaternion numbers.

Next we state two propositions:

(24) $z = \langle \Re(z), \Im_1(z), \Im_2(z), \Im_3(z) \rangle_{\mathbb{H}}$.

(25) If $\Re(z_1) = \Re(z_2)$ and $\Im_1(z_1) = \Im_1(z_2)$ and $\Im_2(z_1) = \Im_2(z_2)$ and $\Im_3(z_1) = \Im_3(z_2)$, then $z_1 = z_2$.

The quaternion number $0_{\mathbb{H}}$ is defined as follows:

(Def. 17) $0_{\mathbb{H}} = 0$.

The quaternion number $1_{\mathbb{H}}$ is defined as follows:

(Def. 18) $1_{\mathbb{H}} = 1$.

One can prove the following propositions:

(26) If $\Re(z) = 0$ and $\Im_1(z) = 0$ and $\Im_2(z) = 0$ and $\Im_3(z) = 0$, then $z = 0_{\mathbb{H}}$.

(27) If $z = 0$, then $(\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}} = 0$.

(28) If $(\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}} = 0$, then $z = 0_{\mathbb{H}}$.

(29) $\Re(1_{\mathbb{H}}) = 1$ and $\Im_1(1_{\mathbb{H}}) = 0$ and $\Im_2(1_{\mathbb{H}}) = 0$ and $\Im_3(1_{\mathbb{H}}) = 0$.

(30) $\Re(i) = 0$ and $\Im_1(i) = 1$ and $\Im_2(i) = 0$ and $\Im_3(i) = 0$.

(31) $\Re(j) = 0$ and $\Im_1(j) = 0$ and $\Im_2(j) = 1$ and $\Im_3(j) = 0$ and $\Re(k) = 0$ and $\Im_1(k) = 0$ and $\Im_2(k) = 0$ and $\Im_3(k) = 1$.

(32) $\Re(z_1 + z_2 + z_3 + z_4) = \Re(z_1) + \Re(z_2) + \Re(z_3) + \Re(z_4)$ and $\Im_1(z_1 + z_2 + z_3 + z_4) = \Im_1(z_1) + \Im_1(z_2) + \Im_1(z_3) + \Im_1(z_4)$ and $\Im_2(z_1 + z_2 + z_3 + z_4) = \Im_2(z_1) + \Im_2(z_2) + \Im_2(z_3) + \Im_2(z_4)$ and $\Im_3(z_1 + z_2 + z_3 + z_4) = \Im_3(z_1) + \Im_3(z_2) + \Im_3(z_3) + \Im_3(z_4)$.

In the sequel x denotes a real number.

We now state three propositions:

(33) If $z_1 = x$, then $\Re(z_1 \cdot i) = 0$ and $\Im_1(z_1 \cdot i) = x$ and $\Im_2(z_1 \cdot i) = 0$ and $\Im_3(z_1 \cdot i) = 0$.

(34) If $z_1 = x$, then $\Re(z_1 \cdot j) = 0$ and $\Im_1(z_1 \cdot j) = 0$ and $\Im_2(z_1 \cdot j) = x$ and $\Im_3(z_1 \cdot j) = 0$.

(35) If $z_1 = x$, then $\Re(z_1 \cdot k) = 0$ and $\Im_1(z_1 \cdot k) = 0$ and $\Im_2(z_1 \cdot k) = 0$ and $\Im_3(z_1 \cdot k) = x$.

Let x be a real number and let y be a quaternion number. The functor $x+y$ is defined as follows:

(Def. 19) There exist elements y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $y = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ and $x + y = \langle x + y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$.

Let x be a real number and let y be a quaternion number. The functor $x-y$ is defined by:

(Def. 20) $x - y = x + -y$.

Let x be a real number and let y be a quaternion number. The functor $x \cdot y$ is defined as follows:

(Def. 21) There exist elements y_1, y_2, y_3, y_4 of $\mathbb{R}$ such that $y = \langle y_1, y_2, y_3, y_4 \rangle_{\mathbb{H}}$ and $x \cdot y = \langle x \cdot y_1, x \cdot y_2, x \cdot y_3, x \cdot y_4 \rangle_{\mathbb{H}}$.

Let x be a real number and let z' be a quaternion number. One can verify the following observations:

* $x + z'$ is quaternion,
* $x \cdot z'$ is quaternion, and
* $x - z'$ is quaternion.

Let z_1, z_2 be quaternion numbers. Then $z_1 + z_2$ is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 22) $z_1 + z_2 = \Re(z_1) + \Re(z_2) + (\Im_1(z_1) + \Im_1(z_2)) \cdot i + (\Im_2(z_1) + \Im_2(z_2)) \cdot j + (\Im_3(z_1) + \Im_3(z_2)) \cdot k$.

The following proposition is true

(36) $\Re(z_1 + z_2) = \Re(z_1) + \Re(z_2)$ and $\Im_1(z_1 + z_2) = \Im_1(z_1) + \Im_1(z_2)$ and $\Im_2(z_1 + z_2) = \Im_2(z_1) + \Im_2(z_2)$ and $\Im_3(z_1 + z_2) = \Im_3(z_1) + \Im_3(z_2)$.

Let z_1, z_2 be elements of $\mathbb{H}$. Then $z_1 \cdot z_2$ is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 23) $z_1 \cdot z_2 = (\Re(z_1) \cdot \Re(z_2) - \Im_1(z_1) \cdot \Im_1(z_2) - \Im_2(z_1) \cdot \Im_2(z_2) - \Im_3(z_1) \cdot \Im_3(z_2)) + ((\Re(z_1) \cdot \Im_1(z_2) + \Im_1(z_1) \cdot \Re(z_2) + \Im_2(z_1) \cdot \Im_3(z_2)) - \Im_3(z_1) \cdot \Im_2(z_2)) \cdot i + ((\Re(z_1) \cdot \Im_2(z_2) + \Im_2(z_1) \cdot \Re(z_2) + \Im_3(z_1) \cdot \Im_1(z_2)) - \Im_1(z_1) \cdot \Im_3(z_2)) \cdot j + ((\Re(z_1) \cdot \Im_3(z_2) + \Im_3(z_1) \cdot \Re(z_2) + \Im_1(z_1) \cdot \Im_2(z_2)) - \Im_2(z_1) \cdot \Im_1(z_2)) \cdot k$.

We now state four propositions:

(37) $z = \Re(z) + \Im_1(z) \cdot i + \Im_2(z) \cdot j + \Im_3(z) \cdot k$.

(38) Suppose $\Im_1(z_1) = 0$ and $\Im_1(z_2) = 0$ and $\Im_2(z_1) = 0$ and $\Im_2(z_2) = 0$ and $\Im_3(z_1) = 0$ and $\Im_3(z_2) = 0$. Then $\Re(z_1 \cdot z_2) = \Re(z_1) \cdot \Re(z_2)$ and $\Im_1(z_1 \cdot z_2) = \Im_2(z_1) \cdot \Im_3(z_2) - \Im_3(z_1) \cdot \Im_2(z_2)$ and $\Im_2(z_1 \cdot z_2) = \Im_3(z_1) \cdot \Im_1(z_2) - \Im_1(z_1) \cdot \Im_3(z_2)$ and $\Im_3(z_1 \cdot z_2) = \Im_1(z_1) \cdot \Im_2(z_2) - \Im_2(z_1) \cdot \Im_1(z_2)$.

(39) Suppose $\Re(z_1) = 0$ and $\Re(z_2) = 0$. Then $\Re(z_1 \cdot z_2) = -\Im_1(z_1) \cdot \Im_1(z_2) - \Im_2(z_1) \cdot \Im_2(z_2) - \Im_3(z_1) \cdot \Im_3(z_2)$ and $\Im_1(z_1 \cdot z_2) = \Im_2(z_1) \cdot \Im_3(z_2) - \Im_3(z_1) \cdot \Im_2(z_2)$ and $\Im_2(z_1 \cdot z_2) = \Im_3(z_1) \cdot \Im_1(z_2) - \Im_1(z_1) \cdot \Im_3(z_2)$ and $\Im_3(z_1 \cdot z_2) = \Im_1(z_1) \cdot \Im_2(z_2) - \Im_2(z_1) \cdot \Im_1(z_2)$.

(40) $\Re(z \cdot z) = (\Re(z))^{\mathbf{2}} - (\Im_1(z))^{\mathbf{2}} - (\Im_2(z))^{\mathbf{2}} - (\Im_3(z))^{\mathbf{2}}$ and $\Im_1(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im_1(z))$ and $\Im_2(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im_2(z))$ and $\Im_3(z \cdot z) = 2 \cdot (\Re(z) \cdot \Im_3(z))$.

Let z be a quaternion number. Then $-z$ is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 24) $-z = -\Re(z) + (-\Im_1(z)) \cdot i + (-\Im_2(z)) \cdot j + (-\Im_3(z)) \cdot k$.

The following proposition is true

(41) $\Re(-z) = -\Re(z)$ and $\Im_1(-z) = -\Im_1(z)$ and $\Im_2(-z) = -\Im_2(z)$ and $\Im_3(-z) = -\Im_3(z)$.

Let z_1, z_2 be quaternion numbers. Then $z_1 - z_2$ is an element of $\mathbb{H}$ and it can be characterized by the condition:

(Def. 25) $z_1 - z_2 = (\Re(z_1) - \Re(z_2)) + (\Im_1(z_1) - \Im_1(z_2)) \cdot i + (\Im_2(z_1) - \Im_2(z_2)) \cdot j + (\Im_3(z_1) - \Im_3(z_2)) \cdot k$.

One can prove the following proposition

(42) $\Re(z_1 - z_2) = \Re(z_1) - \Re(z_2)$ and $\Im_1(z_1 - z_2) = \Im_1(z_1) - \Im_1(z_2)$ and $\Im_2(z_1 - z_2) = \Im_2(z_1) - \Im_2(z_2)$ and $\Im_3(z_1 - z_2) = \Im_3(z_1) - \Im_3(z_2)$.

Let z be a quaternion number. The functor $\overline{z}$ yielding a quaternion number is defined by:

(Def. 26) $\overline{z} = \Re(z) + (-\Im_1(z)) \cdot i + (-\Im_2(z)) \cdot j + (-\Im_3(z)) \cdot k$.

Let z be a quaternion number. Then $\overline{z}$ is an element of $\mathbb{H}$.

We now state a number of propositions:

(43) $\overline{z} = \langle \Re(z), -\Im_1(z), -\Im_2(z), -\Im_3(z) \rangle_{\mathbb{H}}$.

(44) $\Re(\overline{z}) = \Re(z)$ and $\Im_1(\overline{z}) = -\Im_1(z)$ and $\Im_2(\overline{z}) = -\Im_2(z)$ and $\Im_3(\overline{z}) = -\Im_3(z)$.

(45) If $z = 0$, then $\overline{z} = 0$.

(46) If $\overline{z} = 0$, then $z = 0$.

(47) $\overline{1_{\mathbb{H}}} = 1_{\mathbb{H}}$.

(48) $\Re(\overline{i}) = 0$ and $\Im_1(\overline{i}) = -1$ and $\Im_2(\overline{i}) = 0$ and $\Im_3(\overline{i}) = 0$.

(49) $\Re(\overline{j}) = 0$ and $\Im_1(\overline{j}) = 0$ and $\Im_2(\overline{j}) = -1$ and $\Im_3(\overline{j}) = 0$.

(50) $\Re(\overline{k}) = 0$ and $\Im_1(\overline{k}) = 0$ and $\Im_2(\overline{k}) = 0$ and $\Im_3(\overline{k}) = -1$.

(51) $\overline{i} = -i$.

(52) $\overline{j} = -j$.

(53) $\overline{k} = -k$.

(54) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

(55) $\overline{-z} = -\overline{z}$.

(56) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.

(57) If $\Im_2(z_1) \cdot \Im_3(z_2) \neq \Im_3(z_1) \cdot \Im_2(z_2)$, then $\overline{z_1 \cdot z_2} \neq \overline{z_1} \cdot \overline{z_2}$.

(58) If $\Im_1(z) = 0$ and $\Im_2(z) = 0$ and $\Im_3(z) = 0$, then $\overline{z} = z$.

(59) If $\Re(z) = 0$, then $\overline{z} = -z$.

(60) $\Re(z \cdot \overline{z}) = (\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}}$ and $\Im_1(z \cdot \overline{z}) = 0$ and $\Im_2(z \cdot \overline{z}) = 0$ and $\Im_3(z \cdot \overline{z}) = 0$.

(61) $\Re(z+\overline{z}) = 2 \cdot \Re(z)$ and $\Im_1(z+\overline{z}) = 0$ and $\Im_2(z+\overline{z}) = 0$ and $\Im_3(z+\overline{z}) = 0$.

(62) $-z = \langle -\Re(z), -\Im_1(z), -\Im_2(z), -\Im_3(z) \rangle_{\mathbb{H}}$.

(63) $z_1 - z_2 = \langle \Re(z_1) - \Re(z_2), \Im_1(z_1) - \Im_1(z_2), \Im_2(z_1) - \Im_2(z_2), \Im_3(z_1) - \Im_3(z_2) \rangle_{\mathbb{H}}$.

(64) $\Re(z - \overline{z}) = 0$ and $\Im_1(z - \overline{z}) = 2 \cdot \Im_1(z)$ and $\Im_2(z - \overline{z}) = 2 \cdot \Im_2(z)$ and $\Im_3(z - \overline{z}) = 2 \cdot \Im_3(z)$.

Let us consider z. The functor $|z|$ yielding a real number is defined by:

(Def. 27) $|z| = \sqrt{(\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}}}$.

We now state a number of propositions:

(65) $|0_{\mathbb{H}}| = 0$.

(66) If $|z| = 0$, then $z = 0$.

(67) $0 \leq |z|$.

(68) $|1_{\mathbb{H}}| = 1$.

(69) $|i| = 1$.

(70) $|j| = 1$.

(71) $|k| = 1$.

(72) $|-z| = |z|$.

(73) $|\overline{z}| = |z|$.

(74) $0 \leq (\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}}$.

(75) $\Re(z) \leq |z|$.

(76) $\Im_1(z) \leq |z|$.

(77) $\Im_2(z) \leq |z|$.

(78) $\Im_3(z) \leq |z|$.

(79) $|z_1 + z_2| \leq |z_1| + |z_2|$.

(80) $|z_1 - z_2| \leq |z_1| + |z_2|$.

(81) $|z_1| - |z_2| \leq |z_1 + z_2|$.

(82) $|z_1| - |z_2| \leq |z_1 - z_2|$.

(83) $|z_1 - z_2| = |z_2 - z_1|$.

(84) $|z_1 - z_2| = 0$ iff $z_1 = z_2$.

(85) $|z_1 - z_2| \leq |z_1 - z| + |z - z_2|$.

(86) $||z_1| - |z_2|| \leq |z_1 - z_2|$.

(87) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

(88) $|z \cdot z| = (\Re(z))^{\mathbf{2}} + (\Im_1(z))^{\mathbf{2}} + (\Im_2(z))^{\mathbf{2}} + (\Im_3(z))^{\mathbf{2}}$.

(89) $|z \cdot z| = |z \cdot \overline{z}|$.

References

[1] Grzegorz Bancerek. Arithmetic of non-negative rational numbers. *To appear in Formalized Mathematics.*

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[9] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[10] Wojciech Leończuk and Krzysztof Prażmowski. Incidence projective spaces. *Formalized Mathematics*, 2(**2**):225–232, 1991.
[11] Andrzej Trybulec. Introduction to arithmetics. *To appear in Formalized Mathematics.*
[12] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[13] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[15] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[16] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[17] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[18] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received November 14, 2006