

Pocklington's Theorem and Bertrand's Postulate

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Summary. The first four sections of this article include some auxiliary theorems related to number and finite sequence of numbers, in particular a primality test, the Pocklington's theorem (see [19]). The last section presents the formalization of Bertrand's postulate closely following the book [1], pp. 7–9.

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The articles [26], [4], [24], [28], [3], [2], [20], [17], [14], [16], [30], [10], [11], [6], [23], [13], [15], [5], [21], [8], [22], [27], [18], [29], [9], [7], [12], [25], and [31] provide the notation and terminology for this paper.

1. SOME THEOREMS ON REAL AND NATURAL NUMBERS

The following propositions are true:

- (1) For all real numbers r, s such that $0 \leq r$ and $s \cdot s < r \cdot r$ holds $s < r$.
- (2) For all real numbers r, s such that $1 < r$ and $r \cdot r \leq s$ holds $r < s$.
- (3) For all natural numbers a, n such that $a > 1$ holds $a^n > n$.
- (4) For all natural numbers n, k, m such that $k \leq n$ and $m = \lfloor \frac{n}{2} \rfloor$ holds $\binom{n}{m} \geq \binom{n}{k}$.
- (5) For all natural numbers n, m such that $m = \lfloor \frac{n}{2} \rfloor$ and $n \geq 2$ holds $\binom{n}{m} \geq \frac{2^n}{n}$.
- (6) For every natural number n holds $\binom{2 \cdot n}{n} \geq \frac{4^n}{2 \cdot n}$.
- (7) For all natural numbers n, p such that $p > 0$ and $n \mid p$ and $n \neq 1$ and $n \neq p$ holds $1 < n$ and $n < p$.

- (8) Let p be a natural number. Given a natural number n such that $n \mid p$ and $1 < n$ and $n < p$. Then there exists a natural number n such that $n \mid p$ and $1 < n$ and $n \cdot n \leq p$.
- (9) For all natural numbers i, j, k, l such that $i = j \cdot k + l$ and $l < j$ and $0 < l$ holds $j \nmid i$.
- (10) For all natural numbers n, q, b such that $\gcd(q, b) = 1$ and $q \neq 0$ and $b \neq 0$ holds $\gcd(q^n, b) = 1$.
- (11) For all natural numbers a, b, c holds $a^{2 \cdot b} \bmod c = (a^b \bmod c) \cdot (a^b \bmod c) \bmod c$.
- (12) Let p be a natural number. Then p is not prime if and only if one of the following conditions is satisfied:
- (i) $p \leq 1$, or
 - (ii) there exists a natural number n such that $n \mid p$ and $1 < n$ and $n < p$.
- (13) Let n, k be natural numbers. Suppose $n \mid k$ and $1 < n$. Then there exists a natural number p such that $p \mid k$ and $p \leq n$ and p is prime.
- (14) Let p be a natural number. Then p is prime if and only if the following conditions are satisfied:
- (i) $p > 1$, and
 - (ii) for every natural number n such that $1 < n$ and $n \cdot n \leq p$ and n is prime holds $n \nmid p$.
- (15) For all natural numbers a, p, k such that $a^k \bmod p = 1$ and $k \geq 1$ and p is prime holds a and p are relative prime.
- (16) Let p be a prime number, a be a natural number, and x be a set. Suppose $a \neq 0$ and $x = p^{p\text{-count}(a)}$. Then there exists a natural number b such that $b = x$ and $1 \leq b$ and $b \leq a$.
- (17) For all natural numbers k, q, n, d such that q is prime and $d \mid k \cdot q^{n+1}$ and $d \nmid k \cdot q^n$ holds $q^{n+1} \mid d$.
- (18) For all natural numbers q_1, q, n_1 such that $q_1 \mid q^{n_1}$ and q is prime and q_1 is prime and $n_1 > 0$ holds $q = q_1$.
- (19) For every prime number p and for every natural number n such that $n < p$ holds $p \nmid n!$.
- (20) Let a, b be non empty natural numbers. Suppose that for every natural number p such that p is prime holds $p\text{-count}(a) \leq p\text{-count}(b)$. Then there exists a natural number c such that $b = a \cdot c$.
- (21) Let a, b be non empty natural numbers. Suppose that for every natural number p such that p is prime holds $p\text{-count}(a) = p\text{-count}(b)$. Then $a = b$.
- (22) For all prime numbers p_1, p_2 and for every non empty natural number m such that $p_1^{p_1\text{-count}(m)} = p_2^{p_2\text{-count}(m)}$ and $p_1\text{-count}(m) > 0$ holds $p_1 = p_2$.

2. POCKLINGTON'S THEOREM

One can prove the following propositions:

- (23) Let n, k, q, p, n_1, p, a be natural numbers. Suppose $n - 1 = k \cdot q^{n_1}$ and $k > 0$ and $n_1 > 0$ and q is prime and $a^{n-1} \bmod n = 1$ and p is prime and $p \mid n$. Then $p \mid a^{(n-1) \div q} - 1$ or $p \bmod q^{n_1} = 1$.
- (24) Let n, f, c be natural numbers. Suppose that
- (i) $n - 1 = f \cdot c$,
 - (ii) $f > c$,
 - (iii) $c > 0$,
 - (iv) $\gcd(f, c) = 1$, and
 - (v) for every natural number q such that $q \mid f$ and q is prime there exists a natural number a such that $a^{n-1} \bmod n = 1$ and $\gcd(a^{(n-1) \div q} - 1, n) = 1$. Then n is prime.
- (25) Let n, f, d, n_1, a, q be natural numbers. Suppose $n - 1 = q^{n_1} \cdot d$ and $q^{n_1} > d$ and $d > 0$ and $\gcd(q, d) = 1$ and q is prime and $a^{n-1} \bmod n = 1$ and $\gcd(a^{(n-1) \div q} - 1, n) = 1$. Then n is prime.

3. SOME PRIME NUMBERS

The following propositions are true:

- (26) 7 is prime.
- (27) 11 is prime.
- (28) 13 is prime.
- (29) 19 is prime.
- (30) 23 is prime.
- (31) 37 is prime.
- (32) 43 is prime.
- (33) 83 is prime.
- (34) 139 is prime.
- (35) 163 is prime.
- (36) 317 is prime.
- (37) 631 is prime.
- (38) 1259 is prime.
- (39) 2503 is prime.
- (40) 4001 is prime.

4. SOME THEOREMS ON FINITE SEQUENCE OF NUMBERS

One can prove the following propositions:

- (41) For all finite sequences f, f_0, f_1 of elements of \mathbb{R} such that $f = f_0 + f_1$ holds $\text{dom } f = \text{dom } f_0 \cap \text{dom } f_1$.
- (42) Let F be a finite sequence of elements of \mathbb{R} . If for every natural number k such that $k \in \text{dom } F$ holds $F(k) > 0$, then $\prod F > 0$.
- (43) For every set X_1 and for every finite set X_2 such that $X_1 \subseteq X_2$ and $X_2 \subseteq \mathbb{N}$ and $\emptyset \notin X_2$ holds $\prod \text{Sgm } X_1 \leq \prod \text{Sgm } X_2$.
- (44) Let a, k be natural numbers, X be a set, F be a finite sequence of elements of Prime, and p be a prime number such that $X \subseteq \text{Prime}$ and $X \subseteq \text{Seg } k$ and $F = \text{Sgm } X$ and $a = \prod F$. Then
 - (i) if $p \in \text{rng } F$, then $p\text{-count}(a) = 1$, and
 - (ii) if $p \notin \text{rng } F$, then $p\text{-count}(a) = 0$.
- (45) For every natural number n holds $\prod \text{Sgm}\{p; p \text{ ranges over prime numbers: } p \leq n + 1\} \leq 4^n$.
- (46) For every real number x such that $x \geq 2$ holds $\prod \text{Sgm}\{p; p \text{ ranges over prime numbers: } p \leq x\} \leq 4^{x-1}$.
- (47) Let n be a natural number and p be a prime number. Suppose $n \neq 0$. Then there exists a finite sequence f of elements of \mathbb{N} such that
 - (i) $\text{len } f = n$,
 - (ii) for every natural number k such that $k \in \text{dom } f$ holds $f(k) = 1$ iff $p^k \mid n$ and $f(k) = 0$ iff $p^k \nmid n$, and
 - (iii) $p\text{-count}(n) = \sum f$.
- (48) Let n be a natural number and p be a prime number. Then there exists a finite sequence f of elements of \mathbb{N} such that $\text{len } f = n$ and for every natural number k such that $k \in \text{dom } f$ holds $f(k) = \lfloor \frac{n}{p^k} \rfloor$ and $p\text{-count}(n!) = \sum f$.
- (49) Let n be a natural number and p be a prime number. Then there exists a finite sequence f of elements of \mathbb{R} such that $\text{len } f = 2 \cdot n$ and for every natural number k such that $k \in \text{dom } f$ holds $f(k) = \lfloor \frac{2 \cdot n}{p^k} \rfloor - 2 \cdot \lfloor \frac{n}{p^k} \rfloor$ and $p\text{-count}(\binom{2 \cdot n}{n}) = \sum f$.

Let f be a finite sequence of elements of \mathbb{N} and let p be a prime number.

The functor $p\text{-count}(f)$ yielding a finite sequence of elements of \mathbb{N} is defined by:

- (Def. 1) $\text{len}(p\text{-count}(f)) = \text{len } f$ and for every set i such that $i \in \text{dom}(p\text{-count}(f))$ holds $(p\text{-count}(f))(i) = p\text{-count}(f(i))$.

One can prove the following propositions:

- (50) For every prime number p and for every finite sequence f of elements of \mathbb{N} such that $f = \emptyset$ holds $p\text{-count}(f) = \emptyset$.
- (51) For every prime number p and for all finite sequences f_1, f_2 of elements of \mathbb{N} holds $p\text{-count}(f_1 \hat{\ } f_2) = (p\text{-count}(f_1)) \hat{\ } (p\text{-count}(f_2))$.

- (52) For every prime number p and for every non empty natural number n holds $p\text{-count}(\langle n \rangle) = \langle p\text{-count}(n) \rangle$.
- (53) For every finite sequence f of elements of \mathbb{N} and for every prime number p such that $\prod f \neq 0$ holds $p\text{-count}(\prod f) = \sum(p\text{-count}(f))$.
- (54) Let f_1, f_2 be finite sequences of elements of \mathbb{R} . Suppose $\text{len } f_1 = \text{len } f_2$ and for every natural number k such that $k \in \text{dom } f_1$ holds $f_1(k) \leq f_2(k)$ and $f_1(k) > 0$. Then $\prod f_1 \leq \prod f_2$.
- (55) For every natural number n and for every real number r such that $r > 0$ holds $\prod(n \mapsto r) = r^n$.

In this article we present several logical schemes. The scheme *scheme1* concerns a ternary predicate \mathcal{P} , and states that:

Let p be a prime number, n be a natural number, m be a non empty natural number, and X be a set. If $X = \{p^{p'\text{-count}(m)}; p'$ ranges over prime numbers: $\mathcal{P}[n, m, p']\}$, then $\prod \text{Sgm } X > 0$

for all values of the parameters.

The scheme *scheme2* concerns a ternary predicate \mathcal{P} , and states that:

Let p be a prime number, n be a natural number, m be a non empty natural number, and X be a set. If $X = \{p^{p'\text{-count}(m)}; p'$ ranges over prime numbers: $\mathcal{P}[n, m, p']\}$ and $p^{p\text{-count}(m)} \notin X$, then $p\text{-count}(\prod \text{Sgm } X) = 0$

for all values of the parameters.

The scheme *scheme3* concerns a ternary predicate \mathcal{P} , and states that:

Let p be a prime number, n be a natural number, m be a non empty natural number, and X be a set. If $X = \{p^{p'\text{-count}(m)}; p'$ ranges over prime numbers: $\mathcal{P}[n, m, p']\}$ and $p^{p\text{-count}(m)} \in X$, then $p\text{-count}(\prod \text{Sgm } X) = p\text{-count}(m)$

for all values of the parameters.

The scheme *scheme4* deals with a binary functor \mathcal{F} yielding a set and a binary predicate \mathcal{P} , and states that:

Let n, m be natural numbers, r be a real number, and X be a finite set. If $X = \{\mathcal{F}(p, m); p$ ranges over prime numbers: $p \leq r \wedge \mathcal{P}[p, m]\}$ and $r \geq 0$, then $\text{card } X \leq \lfloor r \rfloor$

for all values of the parameters.

5. BERTRAND'S POSTULATE

The following proposition is true

- (56) For every natural number n such that $n \geq 1$ there exists a prime number p such that $n < p$ and $p \leq 2 \cdot n$.

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