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Completeness of the Real Euclidean Space

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The terminology and notation used here are introduced in the following articles: [21], [8], [24], [25], [6], [26], [7], [3], [14], [2], [5], [1], [20], [22], [4], [23], [15], [16], [13], [12], [11], [9], [18], [10], [19], and [17].

1. The Real Euclidean Space as a Real Linear Space

In this paper n is a natural number.

Let n be a natural number. The functor $\langle \mathcal{E}^n, \|\cdot\|\rangle$ yields a strict non empty normed structure and is defined by the conditions (Def. 1).

(Def. 1)(i) The carrier of $\langle \mathcal{E}^n, \|\cdot\|\rangle = \mathcal{R}^n$,

(ii) the zero of $\langle \mathcal{E}^n, \|\cdot\|\rangle = \langle\underbrace{0, \ldots, 0}_{n}\rangle$,

(iii) for all elements a, b of $\mathcal{R}^n$ holds (the addition of $\langle \mathcal{E}^n, \|\cdot\|\rangle$)$(a, b) = a+b$,

(iv) for every element r of $\mathbb{R}$ and for every element x of $\mathcal{R}^n$ holds (the external multiplication of $\langle \mathcal{E}^n, \|\cdot\|\rangle$)$(r, x) = r \cdot x$, and

(v) for every element x of $\mathcal{R}^n$ holds (the norm of $\langle \mathcal{E}^n, \|\cdot\|\rangle$)$(x) = |x|$.

Let n be a natural number. Note that the addition of $\langle \mathcal{E}^n, \|\cdot\|\rangle$ is commutative and associative.

Let n be a non empty natural number. Note that $\langle \mathcal{E}^n, \|\cdot\|\rangle$ is non trivial.

One can prove the following propositions:

(1) For every vector x of $\langle \mathcal{E}^n, \|\cdot\|\rangle$ and for every element y of $\mathcal{R}^n$ such that $x = y$ holds $\|x\| = |y|$.

(2) Let n be a natural number, x, y be vectors of $\langle \mathcal{E}^n, \|\cdot\|\rangle$, and a, b be elements of $\mathcal{R}^n$. If $x = a$ and $y = b$, then $x + y = a + b$.

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(3) For every vector x of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and for every element y of $\mathcal{R}^n$ and for every real number a such that $x = y$ holds $a \cdot x = a \cdot y$.

Let n be a natural number. Note that $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

One can prove the following propositions:

(4) For every vector x of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and for every element a of $\mathcal{R}^n$ such that $x = a$ holds $-x = -a$.

(5) For all vectors x, y of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and for all elements a, b of $\mathcal{R}^n$ such that $x = a$ and $y = b$ holds $x - y = a - b$.

(6) For every finite sequence f of elements of $\mathbb{R}$ such that $\operatorname{dom} f = \operatorname{Seg} n$ holds f is an element of $\mathcal{R}^n$.

(7) Let n be a natural number and x be an element of $\mathcal{R}^n$. Suppose that for every natural number i such that $i \in \operatorname{Seg} n$ holds $0 \leq x(i)$. Then $0 \leq \sum x$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $x(i) \leq \sum x$.

(8) For every element x of $\mathcal{R}^n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $|x(i)| \leq |x|$.

(9) Let x be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and y be an element of $\mathcal{R}^n$. If $x = y$, then for every natural number i such that $i \in \operatorname{Seg} n$ holds $|y(i)| \leq \|x\|$.

(10) For every element x of $\mathcal{R}^{n+1}$ holds $|x|^{\mathbf{2}} = |x{\restriction}n|^{\mathbf{2}} + x(n+1)^{\mathbf{2}}$.

Let n be a natural number, let f be a function from $\mathbb{N}$ into $\mathcal{R}^n$, and let k be a natural number. Then $f(k)$ is an element of $\mathcal{R}^n$.

We now state two propositions:

(11) Let n be a natural number, x be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x_2 be an element of $\mathcal{R}^n$, s_1 be a sequence of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and x_1 be a function from $\mathbb{N}$ into $\mathcal{R}^n$. Suppose $x_2 = x$ and $x_1 = s_1$. Then s_1 is convergent and $\lim s_1 = x$ if and only if for every natural number i such that $i \in \operatorname{Seg} n$ there exists a sequence r_1 of real numbers such that for every natural number k holds $r_1(k) = x_1(k)(i)$ and r_1 is convergent and $x_2(i) = \lim r_1$.

(12) For every sequence f of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that f is Cauchy sequence by norm holds f is convergent.

Let us consider n. Note that $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is complete.

2. The Real Euclidean Space as a Real Normed Space

Let n be a natural number. The functor $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ yields a strict non empty unitary space structure and is defined by the conditions (Def. 2).

(Def. 2)(i) The RLS structure of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ = the RLS structure of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and

(ii) for all elements x, y of $\mathcal{R}^n$ holds (the scalar product of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$)($x$, y) $= \sum (x \bullet y)$.

Let n be a non empty natural number. One can verify that $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is non trivial.

Let n be a natural number. Observe that $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is real unitary space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

The following propositions are true:

(13) Let n be a natural number, a be a real number, x_3, y_1 be points of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and x_4, y_2 be points of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $x_3 = x_4$ and $y_1 = y_2$, then $x_3 + y_1 = x_4 + y_2$ and $-x_3 = -x_4$ and $a \cdot x_3 = a \cdot x_4$.

(14) For every natural number n and for every point x_3 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ and for every point x_4 of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ such that $x_3 = x_4$ holds $\|x_3\|^{\mathbf{2}} = (x_4|x_4)$.

(15) Let n be a natural number and f be a set. Then f is a sequence of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ if and only if f is a sequence of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$.

(16) Let n be a natural number, s_2 be a sequence of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and s_3 be a sequence of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ such that $s_2 = s_3$. Then
   (i) if s_2 is convergent, then s_3 is convergent and $\lim s_2 = \lim s_3$, and
   (ii) if s_3 is convergent, then s_2 is convergent and $\lim s_2 = \lim s_3$.

(17) Let n be a natural number, s_2 be a sequence of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and s_3 be a sequence of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $s_2 = s_3$ and s_2 is Cauchy sequence by norm, then s_3 is Cauchy.

(18) Let n be a natural number, s_2 be a sequence of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and s_3 be a sequence of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $s_2 = s_3$ and s_3 is Cauchy, then s_2 is Cauchy sequence by norm.

Let us consider n. Note that $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is Hilbert.

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