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Some Differentiable Formulas of Special Functions

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Summary. This article contains some differentiable formulas of special functions.

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The terminology and notation used in this paper are introduced in the following papers: [13], [15], [16], [2], [4], [10], [12], [3], [1], [6], [9], [7], [8], [11], [17], [5], and [14].

For simplicity, we use the following convention: x, a, b are real numbers, n is a natural number, Z is an open subset of $\mathbb{R}$, and f, f_1, f_2, g are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Next we state a number of propositions:

(1) Suppose $Z \subseteq \mathrm{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_2(x) = a - x$ and $f_2(x) \neq 0$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{2 \cdot a}{(a-x)^{\mathbf{2}}}$.

(2) Suppose $Z \subseteq \mathrm{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_2(x) = x + a$ and $f_2(x) \neq 0$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{2 \cdot a}{(x+a)^{\mathbf{2}}}$.

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(3) Suppose $Z \subseteq \mathrm{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_2(x) = x - b$ and $f_2(x) \neq 0$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{a-b}{(x-b)^{\mathbf{2}}}$.

(4) Suppose $Z \subseteq \mathrm{dom}\, f$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then $\frac{1}{f}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{x^{\mathbf{2}}}$.

(5) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin}) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) (the function sin) $\cdot \frac{1}{f}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function sin}) \cdot \frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{x^{\mathbf{2}}} \cdot (\text{the function cos})(\frac{1}{x})$.

(6) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos}) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) (the function cos) $\cdot \frac{1}{f}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function cos}) \cdot \frac{1}{f})'_{\restriction Z}(x) = \frac{1}{x^{\mathbf{2}}} \cdot (\text{the function sin})(\frac{1}{x})$.

(7) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z\, ((\text{the function sin}) \cdot \frac{1}{f}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\mathrm{id}_Z\, ((\text{the function sin}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z\, ((\text{the function sin}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = (\text{the function sin})(\frac{1}{x}) - \frac{1}{x} \cdot (\text{the function cos})(\frac{1}{x})$.

(8) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z\, ((\text{the function cos}) \cdot \frac{1}{f}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\mathrm{id}_Z\, ((\text{the function cos}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z\, ((\text{the function cos}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = (\text{the function cos})(\frac{1}{x}) + \frac{1}{x} \cdot (\text{the function sin})(\frac{1}{x})$.

(9) Suppose $Z \subseteq \mathrm{dom}(((\text{the function sin}) \cdot \frac{1}{f})\, ((\text{the function cos}) \cdot \frac{1}{f}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $((\text{the function sin}) \cdot \frac{1}{f})\, ((\text{the function cos}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function sin}) \cdot \frac{1}{f})\, ((\text{the function cos}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = \frac{1}{x^{\mathbf{2}}} \cdot ((\text{the function sin})(\frac{1}{x})^{\mathbf{2}} - (\text{the function cos})(\frac{1}{x})^{\mathbf{2}})$.

(10) Suppose $Z \subseteq \mathrm{dom}(((\text{the function sin}) \cdot f)\, ((^{n}_{\mathbb{Z}}) \cdot (\text{the function sin})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function sin}) \cdot f)\, ((^{n}_{\mathbb{Z}}) \cdot (\text{the function sin}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function sin}) \cdot f)\, ((^{n}_{\mathbb{Z}}) \cdot (\text{the function sin})))'_{\restriction Z}(x) = n \cdot (\text{the function sin})(x)^{n-1}_{\mathbb{Z}} \cdot (\text{the function sin})((n + 1) \cdot x)$.

(11) Suppose $Z \subseteq \mathrm{dom}(((\text{the function cos}) \cdot f)\,((\frac{n}{\mathbb{Z}}) \cdot (\text{the function sin})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function cos}) \cdot f)\,((\frac{n}{\mathbb{Z}}) \cdot (\text{the function sin}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function cos}) \cdot f)\,((\frac{n}{\mathbb{Z}}) \cdot (\text{the function sin})))'_{\restriction Z}(x) = n \cdot (\text{the function sin})(x)^{n-1}_{\mathbb{Z}} \cdot (\text{the function cos})((n+1) \cdot x)$.

(12) Suppose $Z \subseteq \mathrm{dom}(((\text{the function cos}) \cdot f)\,((\frac{n}{\mathbb{Z}}) \cdot (\text{the function cos})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function cos}) \cdot f)\,((\frac{n}{\mathbb{Z}}) \cdot (\text{the function cos}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function cos}) \cdot f)\,((\frac{n}{\mathbb{Z}}) \cdot (\text{the function cos})))'_{\restriction Z}(x) = -n \cdot (\text{the function cos})(x)^{n-1}_{\mathbb{Z}} \cdot (\text{the function sin})((n+1) \cdot x)$.

(13) Suppose $Z \subseteq \mathrm{dom}(((\text{the function sin}) \cdot f)\,((\frac{n}{\mathbb{Z}}) \cdot (\text{the function cos})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function sin}) \cdot f)\,((\frac{n}{\mathbb{Z}}) \cdot (\text{the function cos}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function sin}) \cdot f)\,((\frac{n}{\mathbb{Z}}) \cdot (\text{the function cos})))'_{\restriction Z}(x) = n \cdot (\text{the function cos})(x)^{n-1}_{\mathbb{Z}} \cdot (\text{the function cos})((n+1) \cdot x)$.

(14) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{f}\,(\text{the function sin}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\frac{1}{f}\,(\text{the function sin})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{f}\,(\text{the function sin}))'_{\restriction Z}(x) = \frac{1}{x} \cdot (\text{the function cos})(x) - \frac{1}{x^{\mathbf{2}}} \cdot (\text{the function sin})(x)$.

(15) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{f}\,(\text{the function cos}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\frac{1}{f}\,(\text{the function cos})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{f}\,(\text{the function cos}))'_{\restriction Z}(x) = -\frac{1}{x} \cdot (\text{the function sin})(x) - \frac{1}{x^{\mathbf{2}}} \cdot (\text{the function cos})(x)$.

(16) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin}) + (\frac{\frac{1}{2}}{\mathbb{R}}) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) > 0$. Then

(i) $(\text{the function sin}) + (\frac{\frac{1}{2}}{\mathbb{R}}) \cdot f$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function sin}) + (\frac{\frac{1}{2}}{\mathbb{R}}) \cdot f)'_{\restriction Z}(x) = (\text{the function cos})(x) + \frac{1}{2} \cdot x_{\mathbb{R}}^{-\frac{1}{2}}$.

(17) Suppose $Z \subseteq \mathrm{dom}(g\,((\text{the function sin}) \cdot \frac{1}{f}))$ and $g = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $g\,((\text{the function sin}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(g\,((\text{the function sin}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = 2 \cdot x \cdot (\text{the function sin})(\frac{1}{x}) - (\text{the function cos})(\frac{1}{x})$.

(18) Suppose $Z \subseteq \text{dom}(g\,((\text{the function cos}) \cdot \frac{1}{f}))$ and $g = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $g\,((\text{the function cos}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(g\,((\text{the function cos}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = 2 \cdot x \cdot (\text{the function cos})(\frac{1}{x}) + (\text{the function sin})(\frac{1}{x})$.

(19) Suppose $Z \subseteq \text{dom}(\log\_(e) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) > 0$. Then $\log\_(e) \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot f)'_{\restriction Z}(x) = \frac{1}{x}$.

(20) Suppose $Z \subseteq \text{dom}(\text{id}_Z\, f)$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$. Then $\text{id}_Z\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\text{id}_Z\, f)'_{\restriction Z}(x) = 1 + (\log\_(e))(x)$.

(21) Suppose $Z \subseteq \text{dom}(g\, f)$ and $g = \frac{2}{\mathbb{Z}}$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$. Then $g\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(g\, f)'_{\restriction Z}(x) = x + 2 \cdot x \cdot (\log\_(e))(x)$.

(22) Suppose $Z \subseteq \text{dom}(\frac{f_1+f_2}{f_1-f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = a$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $(f_1 - f_2)(x) > 0$. Then $\frac{f_1+f_2}{f_1-f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1+f_2}{f_1-f_2})'_{\restriction Z}(x) = \frac{4 \cdot a \cdot x}{(a-x^2)^2}$.

(23) Suppose that

(i) $Z \subseteq \text{dom}(\log\_(e) \cdot \frac{f_1+f_2}{f_1-f_2})$,

(ii) for every x such that $x \in Z$ holds $f_1(x) = a$,

(iii) $f_2 = \frac{2}{\mathbb{Z}}$,

(iv) for every x such that $x \in Z$ holds $(f_1 - f_2)(x) > 0$, and

(v) for every x such that $x \in Z$ holds $(f_1 + f_2)(x) > 0$.

Then $\log\_(e) \cdot \frac{f_1+f_2}{f_1-f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot \frac{f_1+f_2}{f_1-f_2})'_{\restriction Z}(x) = \frac{4 \cdot a \cdot x}{a^2-x^4}$.

(24) Suppose $Z \subseteq \text{dom}(\frac{1}{f}\, g)$ and for every x such that $x \in Z$ holds $f(x) = x$ and $g = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$. Then $\frac{1}{f}\, g$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f}\, g)'_{\restriction Z}(x) = \frac{1}{x^{\mathbf{2}}} \cdot (1 - (\log\_(e))(x))$.

(25) Suppose $Z \subseteq \text{dom}(\frac{1}{f})$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$ and for every x such that $x \in Z$ holds $f(x) \neq 0$. Then $\frac{1}{f}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{x \cdot (\log\_(e))(x)^{\mathbf{2}}}$.

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