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The Fundamental Group of the Circle

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Summary. The article formalizes a proof of the theorem counting the fundamental group of a circle taken from [18]. The last result describes an isomorphism between the additive group of integers and the fundamental group of a simple closed curve.

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The notation and terminology used in this paper have been introduced in the following articles: [38], [10], [44], [2], [45], [33], [7], [1], [46], [9], [27], [8], [6], [40], [12], [3], [37], [19], [41], [26], [4], [34], [28], [32], [42], [36], [43], [20], [35], [39], [11], [30], [31], [29], [22], [21], [14], [13], [5], [15], [47], [16], [17], [25], [23], and [24].

1. Preliminaries

Let us observe that every element of $\mathbb{Z}^+$ is integer.

Let us note that $\mathbb{Z}^+$ is infinite.

Let S be an infinite 1-sorted structure. Note that the carrier of S is infinite.

In the sequel a, r, s denote real numbers.

One can prove the following propositions:

(1) If $r \leq s$ and $0 < a$, then for every point p of $[r, s]_{\rm M}$ holds $\mathrm{Ball}(p, a) = [r, s]$ or $\mathrm{Ball}(p, a) = [r, p+a[$ or $\mathrm{Ball}(p, a) =]p-a, s]$ or $\mathrm{Ball}(p, a) =]p-a, p+a[$.

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(2) Suppose $r \leq s$. Then there exists a basis B of $[r, s]_{\mathrm{T}}$ such that

(i) there exists a many sorted set f indexed by $[r, s]_{\mathrm{T}}$ such that for every point y of $[r, s]_{\mathrm{M}}$ holds $f(y) = \{\mathrm{Ball}(y, \frac{1}{n}); n$ ranges over natural numbers: $n \neq 0\}$ and $B = \bigcup f$, and

(ii) for every subset X of $[r, s]_{\mathrm{T}}$ such that $X \in B$ holds X is connected.

(3) For every topological structure T and for every subset A of T and for every point t of T such that $t \in A$ holds $\mathrm{skl}(t, A) \subseteq A$.

Let T be a topological space and let A be an open subset of T. Observe that $T \restriction A$ is open.

Next we state several propositions:

(4) Let T be a topological space, S be a subspace of T, A be a subset of T, and B be a subset of S. If $A = B$, then $T \restriction A = S \restriction B$.

(5) Let S, T be topological spaces, A, B be subsets of T, and C, D be subsets of S. Suppose that

(i) the topological structure of $S =$ the topological structure of T,

(ii) $A = C$,

(iii) $B = D$, and

(iv) A and B are separated.

Then C and D are separated.

(6) Let S, T be topological spaces. Suppose the topological structure of $S =$ the topological structure of T and S is connected. Then T is connected.

(7) Let S, T be topological spaces, A be a subset of S, and B be a subset of T. Suppose the topological structure of $S =$ the topological structure of T and $A = B$ and A is connected. Then B is connected.

(8) Let S, T be non empty topological spaces, s be a point of S, t be a point of T, and A be a neighbourhood of s. Suppose the topological structure of $S =$ the topological structure of T and $s = t$. Then A is a neighbourhood of t.

(9) Let S, T be non empty topological spaces, A be a subset of S, B be a subset of T, and N be a neighbourhood of A. Suppose the topological structure of $S =$ the topological structure of T and $A = B$. Then N is a neighbourhood of B.

(10) Let S, T be non empty topological spaces, A, B be subsets of T, and f be a map from S into T. Suppose f is a homeomorphism and A is a component of B. Then $f^{-1}(A)$ is a component of $f^{-1}(B)$.

2. Local Connectedness

The following propositions are true:

(11) Let T be a non empty topological space, S be a non empty subspace of T, A be a non empty subset of T, and B be a non empty subset of S. If $A = B$ and A is locally connected, then B is locally connected.

(12) Let S, T be non empty topological spaces. Suppose the topological structure of $S =$ the topological structure of T and S is locally connected. Then T is locally connected.

(13) For every non empty topological space T holds T is locally connected iff Ω_T is locally connected.

(14) Let T be a non empty topological space and S be a non empty open subspace of T. If T is locally connected, then S is locally connected.

(15) Let S, T be non empty topological spaces. Suppose S and T are homeomorphic and S is locally connected. Then T is locally connected.

(16) Let T be a non empty topological space. Given a basis B of T such that let X be a subset of T. If $X \in B$, then X is connected. Then T is locally connected.

(17) If $r \leq s$, then $[r, s]_{\mathrm{T}}$ is locally connected.

Let us mention that $\mathbb{I}$ is locally connected.

Let A be a non empty open subset of $\mathbb{I}$. Observe that $\mathbb{I} \restriction A$ is locally connected.

3. Some Useful Functions

Let r be a real number. The functor ExtendInt r yielding a map from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ is defined as follows:

(Def. 1) For every point x of $\mathbb{I}$ holds $(\mathrm{ExtendInt}\, r)(x) = r \cdot x$.

Let r be a real number. One can check that ExtendInt r is continuous.

Let r be a real number. Then ExtendInt r is a path from $R^1 0$ to $R^1 r$.

Let S, T, Y be non empty topological spaces, let H be a map from $[: S, T :]$ into Y, and let t be a point of T. The functor $\mathrm{Prj1}(t, H)$ yields a map from S into Y and is defined by:

(Def. 2) For every point s of S holds $(\mathrm{Prj1}(t, H))(s) = H(s, t)$.

Let S, T, Y be non empty topological spaces, let H be a map from $[: S, T :]$ into Y, and let s be a point of S. The functor $\mathrm{Prj2}(s, H)$ yields a map from T into Y and is defined as follows:

(Def. 3) For every point t of T holds $(\mathrm{Prj2}(s, H))(t) = H(s, t)$.

Let S, T, Y be non empty topological spaces, let H be a continuous map from $[: S, T :]$ into Y, and let t be a point of T. Note that $\mathrm{Prj1}(t, H)$ is continuous.

Let S, T, Y be non empty topological spaces, let H be a continuous map from $[: S, T :]$ into Y, and let s be a point of S. One can check that $\mathrm{Prj2}(s, H)$ is continuous.

One can prove the following two propositions:

(18) Let T be a non empty topological space, a, b be points of T, P, Q be paths from a to b, H be a homotopy between P and Q, and t be a point of $\mathbb{I}$. If H is continuous, then $\mathrm{Prj1}(t, H)$ is continuous.

(19) Let T be a non empty topological space, a, b be points of T, P, Q be paths from a to b, H be a homotopy between P and Q, and s be a point of $\mathbb{I}$. If H is continuous, then $\mathrm{Prj2}(s, H)$ is continuous.

Let r be a real number. The functor $\mathrm{cLoop}\, r$ yielding a map from $\mathbb{I}$ into TopUnitCircle 2 is defined as follows:

(Def. 4) For every point x of $\mathbb{I}$ holds $(\mathrm{cLoop}\, r)(x) = [\cos(2 \cdot \pi \cdot r \cdot x), \sin(2 \cdot \pi \cdot r \cdot x)]$.

The following proposition is true

(20) $\mathrm{cLoop}\, r = \mathrm{CircleMap} \cdot \mathrm{ExtendInt}\, r$.

Let n be an integer. Then $\mathrm{cLoop}\, n$ is a loop of $c[10]$.

4. Main Theorems

Next we state four propositions:

(21) Let U_1 be a family of subsets of TopUnitCircle 2. Suppose U_1 is a cover of TopUnitCircle 2 and open. Let Y be a non empty topological space, F be a continuous map from $[: Y, \mathbb{I} :]$ into TopUnitCircle 2, and y be a point of Y. Then there exists a non empty finite sequence T of elements of $\mathbb{R}$ such that

(i) $T(1) = 0$,

(ii) $T(\mathrm{len}\, T) = 1$,

(iii) T is increasing, and

(iv) there exists an open subset N of Y such that $y \in N$ and for every natural number i such that $i \in \mathrm{dom}\, T$ and $i + 1 \in \mathrm{dom}\, T$ there exists a non empty subset U_2 of TopUnitCircle 2 such that $U_2 \in U_1$ and $F^{\circ}[: N, [T(i), T(i+1)] :] \subseteq U_2$.

(22) Let Y be a non empty topological space, F be a map from $[: Y, \mathbb{I} :]$ into TopUnitCircle 2, and F_1 be a map from $[: Y, \mathrm{Sspace}(0_{\mathbb{I}}) :]$ into $\mathbb{R}^{\mathbf{1}}$. Suppose F is continuous and F_1 is continuous and $F{\restriction}[: \text{the carrier of } Y, \{0\} :] = \mathrm{CircleMap} \cdot F_1$. Then there exists a map G from $[: Y, \mathbb{I} :]$ into $\mathbb{R}^{\mathbf{1}}$ such that

(i) G is continuous,

(ii) $F = \mathrm{CircleMap} \cdot G$,

(iii) $G{\restriction}[: \text{the carrier of } Y, \{0\} :] = F_1$, and

(iv) for every map H from $[: Y, \mathbb{I} :]$ into $\mathbb{R}^{\mathbf{1}}$ such that H is continuous and $F = \mathrm{CircleMap} \cdot H$ and $H{\restriction}[: \text{the carrier of } Y, \{0\} :] = F_1$ holds $G = H$.

(23) Let x_0, y_0 be points of TopUnitCircle 2, x_1 be a point of $\mathbb{R}^{\mathbf{1}}$, and f be a path from x_0 to y_0. Suppose $x_1 \in \mathrm{CircleMap}^{-1}(\{x_0\})$. Then there exists a map f_1 from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ such that

(i) $f_1(0) = x_1$,

(ii) $f = \mathrm{CircleMap} \cdot f_1$,

(iii) f_1 is continuous, and

(iv) for every map f_2 from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ such that f_2 is continuous and $f = \mathrm{CircleMap} \cdot f_2$ and $f_2(0) = x_1$ holds $f_1 = f_2$.

(24) Let x_0, y_0 be points of TopUnitCircle 2, P, Q be paths from x_0 to y_0, F be a homotopy between P and Q, and x_1 be a point of $\mathbb{R}^{\mathbf{1}}$. Suppose P, Q are homotopic and $x_1 \in \mathrm{CircleMap}^{-1}(\{x_0\})$. Then there exists a point y_1 of $\mathbb{R}^{\mathbf{1}}$ and there exist paths P_1, Q_1 from x_1 to y_1 and there exists a homotopy F_1 between P_1 and Q_1 such that P_1, Q_1 are homotopic and $F = \mathrm{CircleMap} \cdot F_1$ and $y_1 \in \mathrm{CircleMap}^{-1}(\{y_0\})$ and for every homotopy F_2 between P_1 and Q_1 such that $F = \mathrm{CircleMap} \cdot F_2$ holds $F_1 = F_2$.

The map Ciso from $\mathbb{Z}^+$ into $\pi_1(\mathrm{TopUnitCircle}\,2, c[10])$ is defined by:

(Def. 5) For every integer n holds $(\mathrm{Ciso})(n) = [\mathrm{cLoop}\,n]_{\mathrm{EqRel}(\mathrm{TopUnitCircle}\,2, c[10])}$.

One can prove the following proposition

(25) For every integer i and for every path f from $R^1 0$ to $R^1 i$ holds $(\mathrm{Ciso})(i) = [\mathrm{CircleMap} \cdot f]_{\mathrm{EqRel}(\mathrm{TopUnitCircle}\,2, c[10])}$.

Ciso is a homomorphism from $\mathbb{Z}^+$ to $\pi_1(\mathrm{TopUnitCircle}\,2, c[10])$.

Let us mention that Ciso is one-to-one and onto.

We now state two propositions:

(26) Ciso is isomorphism.

(27) Let S be a subspace of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve and x be a point of S. Then $\mathbb{Z}^+$ and $\pi_1(S, x)$ are isomorphic.

Let S be a subspace of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve and let x be a point of S. Note that $\pi_1(S, x)$ is infinite.

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