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On Constructing Topological Spaces and Sorgenfrey Line[1]

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Summary. We continue Mizar formalization of General Topology according to the book [19] by Engelking. In the article the formalization of Section 1.2 is almost completed. Namely, we formalize theorems on introduction of topologies by bases, neighborhood systems, closed sets, closure operator, and interior operator. The Sorgenfrey line is defined by a basis. It is proved that the weight of it is continuum. Other techniques are used to demonstrate introduction of discrete and anti-discrete topologies.

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The terminology and notation used in this paper have been introduced in the following articles: [39], [17], [45], [30], [18], [38], [43], [46], [47], [15], [16], [10], [6], [7], [3], [5], [13], [20], [2], [8], [1], [14], [4], [42], [27], [44], [23], [37], [35], [11], [25], [24], [32], [33], [34], [29], [40], [26], [31], [48], [21], [22], [36], [12], [41], [28], and [9].

1. Introducing Topology

In this paper a is a set.

Let X be a set and let B be a family of subsets of X. We say that B is point-filtered if and only if:

(Def. 1) For all sets x, U_1, U_2 such that $U_1 \in B$ and $U_2 \in B$ and $x \in U_1 \cap U_2$ there exists a subset U of X such that $U \in B$ and $x \in U$ and $U \subseteq U_1 \cap U_2$.

We now state four propositions:

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(1) Let X be a set and B be a family of subsets of X. Then B is covering if and only if for every set x such that $x \in X$ there exists a subset U of X such that $U \in B$ and $x \in U$.

(2) Let X be a set and B be a family of subsets of X. Suppose B is point-filtered and covering. Let T be a topological structure. Suppose the carrier of $T = X$ and the topology of $T = \mathrm{UniCl}(B)$. Then T is a topological space and B is a basis of T.

(3) Let X be a set and B be a non-empty many sorted set indexed by X. Suppose that

(i) $\mathrm{rng}\, B \subseteq 2^{2^X}$,

(ii) for all sets x, U such that $x \in X$ and $U \in B(x)$ holds $x \in U$,

(iii) for all sets x, y, U such that $x \in U$ and $U \in B(y)$ and $y \in X$ there exists a set V such that $V \in B(x)$ and $V \subseteq U$, and

(iv) for all sets x, U_1, U_2 such that $x \in X$ and $U_1 \in B(x)$ and $U_2 \in B(x)$ there exists a set U such that $U \in B(x)$ and $U \subseteq U_1 \cap U_2$.

Then there exists a family P of subsets of X such that

(v) $P = \bigcup B$, and

(vi) for every topological structure T such that the carrier of $T = X$ and the topology of $T = \mathrm{UniCl}(P)$ holds T is a topological space and for every non empty topological space T' such that $T' = T$ holds B is a neighborhood system of T'.

(4) Let X be a set and F be a family of subsets of X. Suppose that

(i) $\emptyset \in F$,

(ii) $X \in F$,

(iii) for all sets A, B such that $A \in F$ and $B \in F$ holds $A \cup B \in F$, and

(iv) for every family G of subsets of X such that $G \subseteq F$ holds $\mathrm{Intersect}(G) \in F$.

Let T be a topological structure. Suppose the carrier of $T = X$ and the topology of $T = F^{\mathrm{c}}$. Then T is a topological space and for every subset A of T holds A is closed iff $A \in F$.

The scheme *TopDefByClosedPred* deals with a set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a strict topological space T such that the carrier of $T = \mathcal{A}$ and for every subset A of T holds A is closed iff $\mathcal{P}[A]$

provided the following conditions are satisfied:

- $\mathcal{P}[\emptyset]$ and $\mathcal{P}[\mathcal{A}]$,
- For all sets A, B such that $\mathcal{P}[A]$ and $\mathcal{P}[B]$ holds $\mathcal{P}[A \cup B]$, and
- For every family G of subsets of $\mathcal{A}$ such that for every set A such that $A \in G$ holds $\mathcal{P}[A]$ holds $\mathcal{P}[\mathrm{Intersect}(G)]$.

We now state two propositions:

(5) Let T_1, T_2 be topological spaces. Suppose that for every set A holds A is an open subset of T_1 iff A is an open subset of T_2. Then the topological structure of T_1 = the topological structure of T_2.

(6) Let T_1, T_2 be topological spaces. Suppose that for every set A holds A is a closed subset of T_1 iff A is a closed subset of T_2. Then the topological structure of T_1 = the topological structure of T_2.

Let X, Y be sets and let r be a subset of $[\!: X, 2^Y :\!]$. Then $\operatorname{rng} r$ is a family of subsets of Y.

We now state the proposition

(7) Let X be a set and c be a function from 2^X into 2^X. Suppose that

(i) $c(\emptyset) = \emptyset$,

(ii) for every subset A of X holds $A \subseteq c(A)$,

(iii) for all subsets A, B of X holds $c(A \cup B) = c(A) \cup c(B)$, and

(iv) for every subset A of X holds $c(c(A)) = c(A)$.

Let T be a topological structure. Suppose the carrier of $T = X$ and the topology of $T = (\operatorname{rng} c)^{\mathrm{c}}$. Then T is a topological space and for every subset A of T holds $\overline{A} = c(A)$.

The scheme *TopDefByClosureOp* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a strict topological space T such that the carrier of $T = \mathcal{A}$ and for every subset A of T holds $\overline{A} = \mathcal{F}(A)$

provided the parameters satisfy the following conditions:

- $\mathcal{F}(\emptyset) = \emptyset$,
- For every subset A of $\mathcal{A}$ holds $A \subseteq \mathcal{F}(A)$ and $\mathcal{F}(A) \subseteq \mathcal{A}$,
- For all subsets A, B of $\mathcal{A}$ holds $\mathcal{F}(A \cup B) = \mathcal{F}(A) \cup \mathcal{F}(B)$, and
- For every subset A of $\mathcal{A}$ holds $\mathcal{F}(\mathcal{F}(A)) = \mathcal{F}(A)$.

We now state two propositions:

(8) Let T_1, T_2 be topological spaces. Suppose that

(i) the carrier of T_1 = the carrier of T_2, and

(ii) for every subset A_1 of T_1 and for every subset A_2 of T_2 such that $A_1 = A_2$ holds $\overline{A_1} = \overline{A_2}$.

Then the topology of T_1 = the topology of T_2.

(9) Let X be a set and i be a function from 2^X into 2^X. Suppose that

(i) $i(X) = X$,

(ii) for every subset A of X holds $i(A) \subseteq A$,

(iii) for all subsets A, B of X holds $i(A \cap B) = i(A) \cap i(B)$, and

(iv) for every subset A of X holds $i(i(A)) = i(A)$.

Let T be a topological structure. Suppose the carrier of $T = X$ and the topology of $T = \operatorname{rng} i$. Then T is a topological space and for every subset A of T holds $\operatorname{Int} A = i(A)$.

The scheme *TopDefByInteriorOp* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a strict topological space T such that the carrier of $T = \mathcal{A}$ and for every subset A of T holds $\operatorname{Int} A = \mathcal{F}(A)$

provided the following conditions are met:

- $\mathcal{F}(\mathcal{A}) = \mathcal{A}$,
- For every subset A of $\mathcal{A}$ holds $\mathcal{F}(A) \subseteq A$,
- For all subsets A, B of $\mathcal{A}$ holds $\mathcal{F}(A \cap B) = \mathcal{F}(A) \cap \mathcal{F}(B)$, and
- For every subset A of $\mathcal{A}$ holds $\mathcal{F}(\mathcal{F}(A)) = \mathcal{F}(A)$.

Next we state the proposition

(10) Let T_1, T_2 be topological spaces. Suppose that
 (i) the carrier of T_1 = the carrier of T_2, and
 (ii) for every subset A_1 of T_1 and for every subset A_2 of T_2 such that $A_1 = A_2$ holds $\operatorname{Int} A_1 = \operatorname{Int} A_2$.
 Then the topology of T_1 = the topology of T_2.

2. Sorgenfrey Line

In the sequel x, q denote elements of $\mathbb{R}$.

The strict non empty topological space Sorgenfrey line is defined by the conditions (Def. 2).

(Def. 2)(i) The carrier of Sorgenfrey line = $\mathbb{R}$, and
 (ii) there exists a family B of subsets of $\mathbb{R}$ such that the topology of Sorgenfrey line = UniCl(B) and $B = \{[x, q[: x < q \ \wedge \ q \text{ is rational}\}$.

We now state several propositions:

(11) For all real numbers x, y holds $[x, y[$ is an open subset of Sorgenfrey line.

(12) For all real numbers x, y holds $]x, y[$ is an open subset of Sorgenfrey line.

(13) For every real number x holds $]-\infty, x[$ is an open subset of Sorgenfrey line.

(14) For every real number x holds $]x, +\infty[$ is an open subset of Sorgenfrey line.

(15) For every real number x holds $[x, +\infty[$ is an open subset of Sorgenfrey line.

(16) $\overline{\overline{\mathbb{Z}}} = \aleph_0$.

(17) $\overline{\overline{\mathbb{Q}}} = \aleph_0$.

(18) Let A be a set. Suppose A is mutually-disjoint and for every a such that $a \in A$ there exist real numbers x, y such that $x < y$ and $]x, y[\subseteq a$. Then A is countable.

Let X be a set and let x be a real number. We say that x is local minimum of X if and only if:

(Def. 3) $x \in X$ and there exists a real number y such that $y < x$ and $]y, x[$ misses X.

In the sequel x is an element of $\mathbb{R}$.

One can prove the following proposition

(19) For every subset U of $\mathbb{R}$ holds $\{x : x$ is local minimum of $U\}$ is countable.

One can check the following observations:

* $\mathbb{Z}$ is infinite,
* $\mathbb{Q}$ is infinite, and
* $\mathbb{R}$ is infinite.

Let X be an infinite set. Note that 2^X is infinite.

Let M be an aleph. Observe that 2^M is infinite.

The infinite cardinal number $\mathfrak{c}$ is defined by:

(Def. 4) $\mathfrak{c} = \overline{\overline{\mathbb{R}}}$.

In the sequel x, q are elements of $\mathbb{R}$.

One can prove the following proposition

(20) $\overline{\overline{\{[x, q[: x < q \ \wedge \ q \text{ is rational}\}}} = \mathfrak{c}$.

Let X be an infinite set. Observe that there exists a subset of X which is infinite.

Let X be a set and let r be a real number. The functor X-powers(r) yields a function from $\mathbb{N}$ into $\mathbb{R}$ and is defined by:

(Def. 5) For every natural number i holds $i \in X$ and $(X\text{-powers}(r))(i) = r^i$ or $i \notin X$ and $(X\text{-powers}(r))(i) = 0$.

Next we state the proposition

(21) For every set X and for every real number r such that $0 < r$ and $r < 1$ holds X-powers(r) is summable.

In the sequel r denotes a real number, X denotes a set, and n denotes an element of $\mathbb{N}$.

The following propositions are true:

(22) If $0 < r$ and $r < 1$, then $\sum((r^\kappa)_{\kappa \in \mathbb{N}} \upharpoonright n) = \frac{r^n}{1-r}$.

(23) $\sum(((\frac{1}{2})^\kappa)_{\kappa \in \mathbb{N}} \upharpoonright (n+1)) = (\frac{1}{2})^n$.

(24) If $0 < r$ and $r < 1$, then $\sum(X\text{-powers}(r)) \leq \sum((r^\kappa)_{\kappa \in \mathbb{N}})$.

(25) $\sum((X\text{-powers}(\frac{1}{2})) \upharpoonright (n+1)) \leq (\frac{1}{2})^n$.

(26) For every infinite subset X of $\mathbb{N}$ and for every natural number i holds $(\sum_{\alpha=0}^{\kappa}(X\text{-powers}(\frac{1}{2}))(\alpha))_{\kappa \in \mathbb{N}}(i) < \sum(X\text{-powers}(\frac{1}{2}))$.

(27) For all infinite subsets X, Y of $\mathbb{N}$ such that $\sum(X\text{-powers}(\frac{1}{2})) = \sum(Y\text{-powers}(\frac{1}{2}))$ holds $X = Y$.

(28) If X is countable, then $\operatorname{Fin} X$ is countable.

(29) $\mathfrak{c} = 2^{\aleph_0}$.

(30) $\aleph_0 < \mathfrak{c}$.

(31) For every family A of subsets of $\mathbb{R}$ such that $\overline{\overline{A}} < \mathfrak{c}$ holds $\overline{\overline{\{x : \bigvee_{U : \mathrm{set}} (U \in \mathrm{UniCl}(A) \ \wedge \ x \text{ is local minimum of } U)\}}} < \mathfrak{c}$.

(32) Let X be a family of subsets of $\mathbb{R}$. Suppose $\overline{\overline{X}} < \mathfrak{c}$. Then there exists a real number x and there exists a rational number q such that $x < q$ and $[x, q[\notin \mathrm{UniCl}(X)$.

(33) weight Sorgenfrey line $= \mathfrak{c}$.

3. Example: closed = finite

Let X be a set. The functor ClFinTop(X) yielding a strict topological space is defined by:

(Def. 6) The carrier of ClFinTop(X) $= X$ and for every subset F of ClFinTop(X) holds F is closed iff F is finite or $F = X$.

The following two propositions are true:

(34) For every set X and for every subset A of ClFinTop(X) holds A is open iff $A = \emptyset$ or A^{c} is finite.

(35) For every infinite set X and for every subset A of X such that A is finite holds A^{c} is infinite.

Let X be a non empty set. Note that ClFinTop(X) is non empty.

The following proposition is true

(36) For every infinite set X and for all non empty open subsets U, V of ClFinTop(X) holds U meets V.

4. Example: one point closure

Let X, x_0 be sets. The functor x_0-PointClTop(X) yielding a strict topological space is defined as follows:

(Def. 7) The carrier of x_0-PointClTop(X) $= X$ and for every subset A of x_0-PointClTop(X) holds $\overline{A} = (A = \emptyset \rightarrow A, A \cup \{x_0\} \cap X)$.

Let X be a non empty set and let x_0 be a set. One can check that x_0-PointClTop(X) is non empty.

We now state two propositions:

(37) For every non empty set X and for every element x_0 of X and for every non empty subset A of x_0-PointClTop(X) holds $\overline{A} = A \cup \{x_0\}$.

(38) Let X be a non empty set, x_0 be an element of X, and A be a non empty subset of x_0-PointClTop(X). Then A is closed if and only if $x_0 \in A$.

Let X be a non empty set and let A be a proper subset of X. Observe that A^{c} is non empty.

The following propositions are true:

(39) Let X be a non empty set, x_0 be an element of X, and A be a proper subset of x_0-PointClTop(X). Then A is open if and only if $x_0 \notin A$.

(40) For all sets X, x_0, x such that $x_0 \in X$ holds $\{x\}$ is a closed subset of x_0-PointClTop(X) iff $x = x_0$.

(41) For all sets X, x_0, x such that $\{x_0\} \subset X$ holds $\{x\}$ is an open subset of x_0-PointClTop(X) iff $x \in X$ and $x \neq x_0$.

5. Example: discrete on subset

Let X, X_0 be sets. The functor X_0-DiscreteTop(X) yielding a strict topological space is defined as follows:

(Def. 8) The carrier of X_0-DiscreteTop(X) $= X$ and for every subset A of X_0-DiscreteTop(X) holds $\operatorname{Int} A = (A = X \to A, A \cap X_0)$.

Let X be a non empty set and let X_0 be a set. One can check that X_0-DiscreteTop(X) is non empty.

We now state several propositions:

(42) For every non empty set X and for every set X_0 and for every proper subset A of X_0-DiscreteTop(X) holds $\operatorname{Int} A = A \cap X_0$.

(43) For every non empty set X and for every set X_0 and for every proper subset A of X_0-DiscreteTop(X) holds A is open iff $A \subseteq X_0$.

(44) For every set X and for every subset X_0 of X holds the topology of X_0-DiscreteTop(X) $= \{X\} \cup 2^{X_0}$.

(45) For every set X holds ADTS(X) $= \emptyset$-DiscreteTop(X).

(46) For every set X holds $\{X\}_{\mathrm{top}} = X$-DiscreteTop(X).

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