FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Inverse Trigonometric Functions Arcsin and Arccos[1]

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Summary. Notions of inverse sine and inverse cosine have been introduced. Their basic properties have been proved.

MML Identifier: SIN_COS6.

The papers [11], [14], [1], [10], [3], [13], [12], [9], [15], [2], [16], [6], [4], [5], [7], [8], and [17] provide the terminology and notation for this paper.

1. Preliminaries

In this paper r, s are real numbers and i is an integer number.

We now state two propositions:

(1) If $0 \leq r$ and $r < s$, then $\lfloor \frac{r}{s} \rfloor = 0$.

(2) For every function f and for all sets X, Y such that $f{\restriction}X$ is one-to-one and $Y \subseteq X$ holds $f{\restriction}Y$ is one-to-one.

2. Functions sine and cosine

We now state four propositions:

(3) $-1 \leq \sin r$.

(4) $\sin r \leq 1$.

[1]The paper was written during the first author's post-doctoral fellowship granted by the Shinshu University, Japan.

ISSN 1426–2630

(5) $-1 \leq \cos r$.

(6) $\cos r \leq 1$.

One can check that π is positive.

The following propositions are true:

(7) $\sin(-\frac{\pi}{2}) = -1$ and (the function sin)$(-\frac{\pi}{2}) = -1$.

(8) (The function sin)$(r) =$ (the function sin)$(r + 2 \cdot \pi \cdot i)$.

(9) $\cos(-\frac{\pi}{2}) = 0$ and (the function cos)$(-\frac{\pi}{2}) = 0$.

(10) (The function cos)$(r) =$ (the function cos)$(r + 2 \cdot \pi \cdot i)$.

(11) If $2 \cdot \pi \cdot i < r$ and $r < \pi + 2 \cdot \pi \cdot i$, then $\sin r > 0$.

(12) If $\pi + 2 \cdot \pi \cdot i < r$ and $r < 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\sin r < 0$.

(13) If $-\frac{\pi}{2} + 2 \cdot \pi \cdot i < r$ and $r < \frac{\pi}{2} + 2 \cdot \pi \cdot i$, then $\cos r > 0$.

(14) If $\frac{\pi}{2} + 2 \cdot \pi \cdot i < r$ and $r < \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i$, then $\cos r < 0$.

(15) If $\frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i < r$ and $r < 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\cos r > 0$.

(16) If $2 \cdot \pi \cdot i \leq r$ and $r \leq \pi + 2 \cdot \pi \cdot i$, then $\sin r \geq 0$.

(17) If $\pi + 2 \cdot \pi \cdot i \leq r$ and $r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\sin r \leq 0$.

(18) If $-\frac{\pi}{2} + 2 \cdot \pi \cdot i \leq r$ and $r \leq \frac{\pi}{2} + 2 \cdot \pi \cdot i$, then $\cos r \geq 0$.

(19) If $\frac{\pi}{2} + 2 \cdot \pi \cdot i \leq r$ and $r \leq \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i$, then $\cos r \leq 0$.

(20) If $\frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i \leq r$ and $r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\cos r \geq 0$.

(21) If $2 \cdot \pi \cdot i \leq r$ and $r < 2 \cdot \pi + 2 \cdot \pi \cdot i$ and $\sin r = 0$, then $r = 2 \cdot \pi \cdot i$ or $r = \pi + 2 \cdot \pi \cdot i$.

(22) If $2 \cdot \pi \cdot i \leq r$ and $r < 2 \cdot \pi + 2 \cdot \pi \cdot i$ and $\cos r = 0$, then $r = \frac{\pi}{2} + 2 \cdot \pi \cdot i$ or $r = \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i$.

(23) If $\sin r = -1$, then $r = \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$.

(24) If $\sin r = 1$, then $r = \frac{\pi}{2} + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$.

(25) If $\cos r = -1$, then $r = \pi + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$.

(26) If $\cos r = 1$, then $r = 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$.

(27) If $0 \leq r$ and $r \leq 2 \cdot \pi$ and $\sin r = -1$, then $r = \frac{3}{2} \cdot \pi$.

(28) If $0 \leq r$ and $r \leq 2 \cdot \pi$ and $\sin r = 1$, then $r = \frac{\pi}{2}$.

(29) If $0 \leq r$ and $r \leq 2 \cdot \pi$ and $\cos r = -1$, then $r = \pi$.

(30) If $0 \leq r$ and $r < \frac{\pi}{2}$, then $\sin r < 1$.

(31) If $0 \leq r$ and $r < \frac{3}{2} \cdot \pi$, then $\sin r > -1$.

(32) If $\frac{3}{2} \cdot \pi < r$ and $r \leq 2 \cdot \pi$, then $\sin r > -1$.

(33) If $\frac{\pi}{2} < r$ and $r \leq 2 \cdot \pi$, then $\sin r < 1$.

(34) If $0 < r$ and $r < 2 \cdot \pi$, then $\cos r < 1$.

(35) If $0 \leq r$ and $r < \pi$, then $\cos r > -1$.

(36) If $\pi < r$ and $r \leq 2 \cdot \pi$, then $\cos r > -1$.

(37) If $2 \cdot \pi \cdot i \leq r$ and $r < \frac{\pi}{2} + 2 \cdot \pi \cdot i$, then $\sin r < 1$.

(38) If $2 \cdot \pi \cdot i \leq r$ and $r < \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i$, then $\sin r > -1$.

(39) If $\frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i < r$ and $r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\sin r > -1$.

(40) If $\frac{\pi}{2} + 2 \cdot \pi \cdot i < r$ and $r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\sin r < 1$.

(41) If $2 \cdot \pi \cdot i < r$ and $r < 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\cos r < 1$.

(42) If $2 \cdot \pi \cdot i \leq r$ and $r < \pi + 2 \cdot \pi \cdot i$, then $\cos r > -1$.

(43) If $\pi + 2 \cdot \pi \cdot i < r$ and $r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$, then $\cos r > -1$.

(44) If $\cos(2 \cdot \pi \cdot r) = 1$, then $r \in \mathbb{Z}$.

(45) (The function sin) $^\circ[-\frac{\pi}{2}, \frac{\pi}{2}] = [-1, 1]$.

(46) (The function sin) $^\circ]-\frac{\pi}{2}, \frac{\pi}{2}[\, = \,]-1, 1[$.

(47) (The function sin) $^\circ[\frac{\pi}{2}, \frac{3}{2} \cdot \pi] = [-1, 1]$.

(48) (The function sin) $^\circ]\frac{\pi}{2}, \frac{3}{2} \cdot \pi[\, = \,]-1, 1[$.

(49) (The function cos) $^\circ[0, \pi] = [-1, 1]$.

(50) (The function cos) $^\circ]0, \pi[\, = \,]-1, 1[$.

(51) (The function cos) $^\circ[\pi, 2 \cdot \pi] = [-1, 1]$.

(52) (The function cos) $^\circ]\pi, 2 \cdot \pi[\, = \,]-1, 1[$.

(53) The function sin is increasing on $[-\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{\pi}{2} + 2 \cdot \pi \cdot i]$.

(54) The function sin is decreasing on $[\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i]$.

(55) The function cos is decreasing on $[2 \cdot \pi \cdot i, \pi + 2 \cdot \pi \cdot i]$.

(56) The function cos is increasing on $[\pi + 2 \cdot \pi \cdot i, 2 \cdot \pi + 2 \cdot \pi \cdot i]$.

(57) (The function sin)$\restriction[-\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{\pi}{2} + 2 \cdot \pi \cdot i]$ is one-to-one.

(58) (The function sin)$\restriction[\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{3}{2} \cdot \pi + 2 \cdot \pi \cdot i]$ is one-to-one.

One can check that (the function sin)$\restriction[-\frac{\pi}{2}, \frac{\pi}{2}]$ is one-to-one and (the function sin)$\restriction[\frac{\pi}{2}, \frac{3}{2} \cdot \pi]$ is one-to-one.

One can check the following observations:

* (the function sin)$\restriction[-\frac{\pi}{2}, 0]$ is one-to-one,
* (the function sin)$\restriction[0, \frac{\pi}{2}]$ is one-to-one,
* (the function sin)$\restriction[\frac{\pi}{2}, \pi]$ is one-to-one,
* (the function sin)$\restriction[\pi, \frac{3}{2} \cdot \pi]$ is one-to-one, and
* (the function sin)$\restriction[\frac{3}{2} \cdot \pi, 2 \cdot \pi]$ is one-to-one.

One can verify the following observations:

* (the function sin)$\restriction]-\frac{\pi}{2}, \frac{\pi}{2}[$ is one-to-one,
* (the function sin)$\restriction]\frac{\pi}{2}, \frac{3}{2} \cdot \pi[$ is one-to-one,
* (the function sin)$\restriction]-\frac{\pi}{2}, 0[$ is one-to-one,
* (the function sin)$\restriction]0, \frac{\pi}{2}[$ is one-to-one,
* (the function sin)$\restriction]\frac{\pi}{2}, \pi[$ is one-to-one,
* (the function sin)$\restriction]\pi, \frac{3}{2} \cdot \pi[$ is one-to-one, and
* (the function sin)$\restriction]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$ is one-to-one.

Next we state two propositions:

(59) (The function cos)$\restriction[2 \cdot \pi \cdot i, \pi + 2 \cdot \pi \cdot i]$ is one-to-one.

(60) (The function cos)$\restriction[\pi + 2 \cdot \pi \cdot i, 2 \cdot \pi + 2 \cdot \pi \cdot i]$ is one-to-one.

Let us note that (the function cos)$\restriction[0, \pi]$ is one-to-one and (the function cos)$\restriction[\pi, 2 \cdot \pi]$ is one-to-one.

One can check the following observations:

* (the function cos)$\restriction[0, \frac{\pi}{2}]$ is one-to-one,
* (the function cos)$\restriction[\frac{\pi}{2}, \pi]$ is one-to-one,
* (the function cos)$\restriction[\pi, \frac{3}{2} \cdot \pi]$ is one-to-one, and
* (the function cos)$\restriction[\frac{3}{2} \cdot \pi, 2 \cdot \pi]$ is one-to-one.

One can check the following observations:

* (the function cos)$\restriction]0, \pi[$ is one-to-one,
* (the function cos)$\restriction]\pi, 2 \cdot \pi[$ is one-to-one,
* (the function cos)$\restriction]0, \frac{\pi}{2}[$ is one-to-one,
* (the function cos)$\restriction]\frac{\pi}{2}, \pi[$ is one-to-one,
* (the function cos)$\restriction]\pi, \frac{3}{2} \cdot \pi[$ is one-to-one, and
* (the function cos)$\restriction]\frac{3}{2} \cdot \pi, 2 \cdot \pi[$ is one-to-one.

The following proposition is true

(61) If $2 \cdot \pi \cdot i \leq r$ and $r < 2 \cdot \pi + 2 \cdot \pi \cdot i$ and $2 \cdot \pi \cdot i \leq s$ and $s < 2 \cdot \pi + 2 \cdot \pi \cdot i$ and $\sin r = \sin s$ and $\cos r = \cos s$, then $r = s$.

3. Function arcsin

The function arcsin is a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined by:

(Def. 1) The function arcsin = ((the function sin)$\restriction[-\frac{\pi}{2}, \frac{\pi}{2}])^{-1}$.

Let r be a set. The functor $\arcsin r$ is defined by:

(Def. 2) $\arcsin r$ = (the function arcsin)(r).

Let r be a set. Then $\arcsin r$ is a real number.

Next we state two propositions:

(62) (The function arcsin) $^{-1}$ = (the function sin)$\restriction[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(63) rng (the function arcsin) = $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Let us note that the function arcsin is one-to-one.

The following propositions are true:

(64) dom (the function arcsin) = $[-1, 1]$.

(65) ((The function sin)$\restriction[-\frac{\pi}{2}, \frac{\pi}{2}]$ **qua** function) $\cdot$(the function arcsin) = $\mathrm{id}_{[-1,1]}$.

(66) (The function arcsin) $\cdot$((the function sin)$\restriction[-\frac{\pi}{2}, \frac{\pi}{2}]) = \mathrm{id}_{[-1,1]}$.

(67) ((The function sin)$\restriction[-\frac{\pi}{2},\frac{\pi}{2}]$) $\cdot$ (the function arcsin) $= \mathrm{id}_{[-\frac{\pi}{2},\frac{\pi}{2}]}$.

(68) (The function arcsin **qua** function) $\cdot$((the function sin)$\restriction[-\frac{\pi}{2},\frac{\pi}{2}]$) $= \mathrm{id}_{[-\frac{\pi}{2},\frac{\pi}{2}]}$.

(69) If $-1 \leq r$ and $r \leq 1$, then $\sin \arcsin r = r$.

(70) If $-\frac{\pi}{2} \leq r$ and $r \leq \frac{\pi}{2}$, then $\arcsin \sin r = r$.

(71) $\arcsin(-1) = -\frac{\pi}{2}$.

(72) $\arcsin 0 = 0$.

(73) $\arcsin 1 = \frac{\pi}{2}$.

(74) If $-1 \leq r$ and $r \leq 1$ and $\arcsin r = -\frac{\pi}{2}$, then $r = -1$.

(75) If $-1 \leq r$ and $r \leq 1$ and $\arcsin r = 0$, then $r = 0$.

(76) If $-1 \leq r$ and $r \leq 1$ and $\arcsin r = \frac{\pi}{2}$, then $r = 1$.

(77) If $-1 \leq r$ and $r \leq 1$, then $-\frac{\pi}{2} \leq \arcsin r$ and $\arcsin r \leq \frac{\pi}{2}$.

(78) If $-1 < r$ and $r < 1$, then $-\frac{\pi}{2} < \arcsin r$ and $\arcsin r < \frac{\pi}{2}$.

(79) If $-1 \leq r$ and $r \leq 1$, then $\arcsin r = -\arcsin(-r)$.

(80) If $0 \leq s$ and $r^{\mathbf{2}} + s^{\mathbf{2}} = 1$, then $\cos \arcsin r = s$.

(81) If $s \leq 0$ and $r^{\mathbf{2}} + s^{\mathbf{2}} = 1$, then $\cos \arcsin r = -s$.

(82) If $-1 \leq r$ and $r \leq 1$, then $\cos \arcsin r = \sqrt{1 - r^{\mathbf{2}}}$.

(83) The function arcsin is increasing on $[-1, 1]$.

(84) The function arcsin is differentiable on $]-1, 1[$ and if $-1 < r$ and $r < 1$, then (the function arcsin)$'(r) = \frac{1}{\sqrt{1-r^{\mathbf{2}}}}$.

(85) The function arcsin is continuous on $[-1, 1]$.

4. Function arccos

The function arccos is a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined by:

(Def. 3) The function arccos = ((the function cos)$\restriction[0, \pi]$)$^{-1}$.

Let r be a set. The functor $\arccos r$ is defined by:

(Def. 4) $\arccos r$ = (the function arccos)(r).

Let r be a set. Then $\arccos r$ is a real number.

One can prove the following two propositions:

(86) (The function arccos) $^{-1}$ = (the function cos)$\restriction[0, \pi]$.

(87) rng (the function arccos) $= [0, \pi]$.

Let us note that the function arccos is one-to-one.

The following propositions are true:

(88) dom (the function arccos) $= [-1, 1]$.

(89) ((The function cos)$\restriction[0, \pi]$ **qua** function) $\cdot$(the function arccos) $= \mathrm{id}_{[-1,1]}$.

(90) (The function arccos) $\cdot$((the function cos)$\restriction[0, \pi]$) $= \mathrm{id}_{[-1,1]}$.

(91) ((The function cos)$\restriction[0,\pi]$) $\cdot$ (the function arccos) $= \mathrm{id}_{[0,\pi]}$.

(92) (The function arccos **qua** function) $\cdot$((the function cos)$\restriction[0,\pi]$) $= \mathrm{id}_{[0,\pi]}$.

(93) If $-1 \leq r$ and $r \leq 1$, then $\cos \arccos r = r$.

(94) If $0 \leq r$ and $r \leq \pi$, then $\arccos \cos r = r$.

(95) $\arccos(-1) = \pi$.

(96) $\arccos 0 = \frac{\pi}{2}$.

(97) $\arccos 1 = 0$.

(98) If $-1 \leq r$ and $r \leq 1$ and $\arccos r = 0$, then $r = 1$.

(99) If $-1 \leq r$ and $r \leq 1$ and $\arccos r = \frac{\pi}{2}$, then $r = 0$.

(100) If $-1 \leq r$ and $r \leq 1$ and $\arccos r = \pi$, then $r = -1$.

(101) If $-1 \leq r$ and $r \leq 1$, then $0 \leq \arccos r$ and $\arccos r \leq \pi$.

(102) If $-1 < r$ and $r < 1$, then $0 < \arccos r$ and $\arccos r < \pi$.

(103) If $-1 \leq r$ and $r \leq 1$, then $\arccos r = \pi - \arccos(-r)$.

(104) If $0 \leq s$ and $r^{\mathbf{2}} + s^{\mathbf{2}} = 1$, then $\sin \arccos r = s$.

(105) If $s \leq 0$ and $r^{\mathbf{2}} + s^{\mathbf{2}} = 1$, then $\sin \arccos r = -s$.

(106) If $-1 \leq r$ and $r \leq 1$, then $\sin \arccos r = \sqrt{1 - r^{\mathbf{2}}}$.

(107) The function arccos is decreasing on $[-1, 1]$.

(108) The function arccos is differentiable on $]-1, 1[$ and if $-1 < r$ and $r < 1$, then (the function arccos)$'(r) = -\frac{1}{\sqrt{1-r^{\mathbf{2}}}}$.

(109) The function arccos is continuous on $[-1, 1]$.

(110) If $-1 \leq r$ and $r \leq 1$, then $\arcsin r + \arccos r = \frac{\pi}{2}$.

(111) If $-1 \leq r$ and $r \leq 1$, then $\arccos(-r) - \arcsin r = \frac{\pi}{2}$.

(112) If $-1 \leq r$ and $r \leq 1$, then $\arccos r - \arcsin(-r) = \frac{\pi}{2}$.

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Received September 5, 2004