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# A Theory of Matrices of Complex Elements

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**Summary.** A concept of "Matrix of Complex" is defined here. Addition, subtraction, scalar multiplication and product are introduced using correspondent definitions of "Matrix of Field". Many equations for such operations consist of a case of "Matrix of Field". A calculation method of product of matrices is shown using a finite sequence of Complex in the last theorem.

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The articles [11], [14], [1], [4], [2], [15], [6], [10], [9], [3], [8], [7], [13], [12], and [5] provide the terminology and notation for this paper.

The following two propositions are true:

(1) $1 = 1_{\mathbb{C}_{\rm F}}$.

(2) $0_{\mathbb{C}_{\rm F}} = 0$.

Let $A$ be a matrix over $\mathbb{C}$. The functor $A_{\mathbb{C}_{\rm F}}$ yields a matrix over $\mathbb{C}_{\rm F}$ and is defined by:

(Def. 1) $A_{\mathbb{C}_{\rm F}} = A$.

Let $A$ be a matrix over $\mathbb{C}_{\rm F}$. The functor $A_{\mathbb{C}}$ yielding a matrix over $\mathbb{C}$ is defined by:

(Def. 2) $A_{\mathbb{C}} = A$.

We now state four propositions:

(3) For all matrices $A$, $B$ over $\mathbb{C}$ such that $A_{\mathbb{C}_{\rm F}} = B_{\mathbb{C}_{\rm F}}$ holds $A = B$.

(4) For all matrices $A$, $B$ over $\mathbb{C}_{\rm F}$ such that $A_{\mathbb{C}} = B_{\mathbb{C}}$ holds $A = B$.

(5) For every matrix $A$ over $\mathbb{C}$ holds $A = (A_{\mathbb{C}_{\rm F}})_{\mathbb{C}}$.

(6) For every matrix $A$ over $\mathbb{C}_{\rm F}$ holds $A = (A_{\mathbb{C}})_{\mathbb{C}_{\rm F}}$.

Let $A$, $B$ be matrices over $\mathbb{C}$. The functor $A + B$ yielding a matrix over $\mathbb{C}$ is defined as follows:




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(Def. 3) $A + B = (A_{\mathbb{C}_{\rm F}} + B_{\mathbb{C}_{\rm F}})_{\mathbb{C}}$.

Let $A$ be a matrix over $\mathbb{C}$. The functor $-A$ yielding a matrix over $\mathbb{C}$ is defined as follows:

(Def. 4) $-A = (-A_{\mathbb{C}_{\rm F}})_{\mathbb{C}}$.

Let $A$, $B$ be matrices over $\mathbb{C}$. The functor $A - B$ yields a matrix over $\mathbb{C}$ and is defined as follows:

(Def. 5) $A - B = (A_{\mathbb{C}_{\rm F}} - B_{\mathbb{C}_{\rm F}})_{\mathbb{C}}$.

Let $A$, $B$ be matrices over $\mathbb{C}$. The functor $A \cdot B$ yielding a matrix over $\mathbb{C}$ is defined as follows:

(Def. 6) $A \cdot B = (A_{\mathbb{C}_{\rm F}} \cdot B_{\mathbb{C}_{\rm F}})_{\mathbb{C}}$.

Let $x$ be a complex number and let $A$ be a matrix over $\mathbb{C}$. The functor $x \cdot A$ yielding a matrix over $\mathbb{C}$ is defined as follows:

(Def. 7) For every element $e_1$ of $\mathbb{C}_{\rm F}$ such that $e_1 = x$ holds $x \cdot A = (e_1 \cdot A_{\mathbb{C}_{\rm F}})_{\mathbb{C}}$.

One can prove the following propositions:

(7) For every matrix $A$ over $\mathbb{C}$ holds $\operatorname{len} A = \operatorname{len}(A_{\mathbb{C}_{\rm F}})$ and $\operatorname{width} A = \operatorname{width}(A_{\mathbb{C}_{\rm F}})$.

(8) For every matrix $A$ over $\mathbb{C}_{\rm F}$ holds $\operatorname{len} A = \operatorname{len}(A_{\mathbb{C}})$ and $\operatorname{width} A = \operatorname{width}(A_{\mathbb{C}})$.

(9) For every matrix $M$ over $\mathbb{C}$ such that $\operatorname{len} M > 0$ holds $--M = M$.

(10) For every field $K$ and for every matrix $M$ over $K$ holds $1_K \cdot M = M$.

(11) For every matrix $M$ over $\mathbb{C}$ holds $1 \cdot M = M$.

(12) For every field $K$ and for all elements $a$, $b$ of $K$ and for every matrix $M$ over $K$ holds $a \cdot (b \cdot M) = (a \cdot b) \cdot M$.

(13) For every field $K$ and for all elements $a$, $b$ of $K$ and for every matrix $M$ over $K$ holds $(a + b) \cdot M = a \cdot M + b \cdot M$.

(14) For every matrix $M$ over $\mathbb{C}$ holds $M + M = 2 \cdot M$.

(15) For every matrix $M$ over $\mathbb{C}$ holds $M + M + M = 3 \cdot M$.

Let $n$, $m$ be natural numbers. The functor $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}^{n \times m}_{\mathbb{C}}$ yields a matrix over $\mathbb{C}$ and is defined by:

(Def. 8) $$\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}^{n \times m}_{\mathbb{C}} = \left(\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}^{n \times m}_{\mathbb{C}_{\rm F}}\right)_{\mathbb{C}}.$$

One can prove the following propositions:

(16) For all natural numbers $n$, $m$ holds $\left(\begin{array}{ccc}0 & \ldots & 0\\ \vdots & \ddots & \vdots\\ 0 & \ldots & 0\end{array}\right)^{n\times m}_{\mathbb{C}\ \ \mathbb{C}_{\rm F}} = \left(\begin{array}{ccc}0 & \ldots & 0\\ \vdots & \ddots & \vdots\\ 0 & \ldots & 0\end{array}\right)^{n\times m}_{\mathbb{C}_{\rm F}}$.

(17) For every matrix $M$ over $\mathbb{C}$ such that $\operatorname{len} M > 0$ holds $M + -M = \left(\begin{array}{ccc}0 & \ldots & 0\\ \vdots & \ddots & \vdots\\ 0 & \ldots & 0\end{array}\right)^{(\operatorname{len} M)\times(\operatorname{width} M)}_{\mathbb{C}}$.

(18) For every matrix $M$ over $\mathbb{C}$ such that $\operatorname{len} M > 0$ holds $M - M = \left(\begin{array}{ccc}0 & \ldots & 0\\ \vdots & \ddots & \vdots\\ 0 & \ldots & 0\end{array}\right)^{(\operatorname{len} M)\times(\operatorname{width} M)}_{\mathbb{C}}$.

(19) For all matrices $M_1$, $M_2$, $M_3$ over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$ and $M_1 + M_3 = M_2 + M_3$ holds $M_1 = M_2$.

(20) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\operatorname{len} M_2 > 0$ holds $M_1 - -M_2 = M_1 + M_2$.

(21) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ and $M_1 = M_1 + M_2$ holds $M_2 = \left(\begin{array}{ccc}0 & \ldots & 0\\ \vdots & \ddots & \vdots\\ 0 & \ldots & 0\end{array}\right)^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)}_{\mathbb{C}}$.

(22) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ and $M_1 - M_2 = \left(\begin{array}{ccc}0 & \ldots & 0\\ \vdots & \ddots & \vdots\\ 0 & \ldots & 0\end{array}\right)^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)}_{\mathbb{C}}$ holds $M_1 = M_2$.

(23) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ and $M_1 + M_2 = \left(\begin{array}{ccc}0 & \ldots & 0\\ \vdots & \ddots & \vdots\\ 0 & \ldots & 0\end{array}\right)^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)}_{\mathbb{C}}$ holds $M_2 = -M_1$.

(24) For all natural numbers $n$, $m$ such that $n > 0$ holds

$$-\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{\mathbb{C}}^{n\times m} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{\mathbb{C}}^{n\times m}.$$

(25) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ and $M_2 - M_1 = M_2$ holds $M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{\mathbb{C}}^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)}$.

(26) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ holds $M_1 = M_1 - (M_2 - M_2)$.

(27) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ holds $-(M_1 + M_2) = -M_1 + -M_2$.

(28) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ holds $M_1 - (M_1 - M_2) = M_2$.

(29) For all matrices $M_1$, $M_2$, $M_3$ over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$ and $M_1 - M_3 = M_2 - M_3$ holds $M_1 = M_2$.

(30) For all matrices $M_1$, $M_2$, $M_3$ over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$ and $M_3 - M_1 = M_3 - M_2$ holds $M_1 = M_2$.

(31) For all matrices $M_1$, $M_2$, $M_3$ over $\mathbb{C}$ such that $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{width} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_1 > 0$ and $\operatorname{len} M_2 > 0$ holds $M_1 \cdot (M_2 + M_3) = M_1 \cdot M_2 + M_1 \cdot M_3$.

(32) For all matrices $M_1$, $M_2$, $M_3$ over $\mathbb{C}$ such that $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_2 > 0$ and $\operatorname{len} M_1 > 0$ holds $(M_2 + M_3) \cdot M_1 = M_2 \cdot M_1 + M_3 \cdot M_1$.

(33) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ holds $M_1 + M_2 = M_2 + M_1$.

(34) For all matrices $M_1$, $M_2$, $M_3$ over $\mathbb{C}$ such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_1 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_3$ holds $(M_1 + M_2) + M_3 = M_1 + (M_2 + M_3)$.

(35) For every matrix $M$ over $\mathbb{C}$ such that $\operatorname{len} M > 0$ holds $M + \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{\mathbb{C}}^{(\operatorname{len} M)\times(\operatorname{width} M)} = M$.

(36) Let $K$ be a field, $b$ be an element of $K$, and $M_1$, $M_2$ be matrices over $K$. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$, then $b \cdot (M_1 + M_2) = b \cdot M_1 + b \cdot M_2$.

(37) Let $M_1$, $M_2$ be matrices over $\mathbb{C}$ and $a$ be a complex number. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$, then $a \cdot (M_1 + M_2) = a \cdot M_1 + a \cdot M_2$.

(38) For every field $K$ and for all matrices $M_1$, $M_2$ over $K$ such that $\operatorname{width} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_1 > 0$ and $\operatorname{len} M_2 > 0$ holds $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)} \cdot M_2 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} M_1)\times(\operatorname{width} M_2)}$.

(39) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\operatorname{width} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_1 > 0$ and $\operatorname{len} M_2 > 0$ holds $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{C}}^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)} \cdot M_2 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{C}}^{(\operatorname{len} M_1)\times(\operatorname{width} M_2)}$.

(40) For every field $K$ and for every matrix $M_1$ over $K$ such that $\operatorname{len} M_1 > 0$ holds $0_K \cdot M_1 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)}$.

(41) For every matrix $M_1$ over $\mathbb{C}$ such that $\operatorname{len} M_1 > 0$ holds $0 \cdot M_1 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_{\mathbb{C}}^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)}$.

Let $s$ be a finite sequence of elements of $\mathbb{C}$ and let $k$ be a natural number. Then $s(k)$ is an element of $\mathbb{C}$.

We now state the proposition

(42) Let $i$, $j$ be natural numbers and $M_1$, $M_2$ be matrices over $\mathbb{C}$. Suppose $\operatorname{len} M_1 > 0$ and $\operatorname{len} M_2 > 0$ and $\operatorname{width} M_1 = \operatorname{len} M_2$ and $1 \le i$ and $i \le \operatorname{len} M_1$ and $1 \le j$ and $j \le \operatorname{width} M_2$. Then there exists a finite sequence $s$ of elements of $\mathbb{C}$ such that $\operatorname{len} s = \operatorname{len} M_2$ and $s(1) = (M_1 \circ (i,1)) \cdot (M_2 \circ (1,j))$ and for every natural number $k$ such that $1 \le k$ and $k < \operatorname{len} M_2$ holds $s(k+1) = s(k) + (M_1 \circ (i,k+1)) \cdot (M_2 \circ (k+1,j))$.

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