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Hölder's Inequality and Minkowski's Inequality

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Summary. In this article, Hölder's inequality and Minkowski's inequality are proved. These equalities are basic ones of functional analysis.

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The papers [12], [13], [14], [3], [1], [11], [4], [2], [7], [5], [6], [10], [8], and [9] provide the notation and terminology for this paper.

1. Hölder's Inequality

In this paper a, b, p, q are real numbers.

Let x be a real number. One can verify that $[x, +\infty[$ is non empty.

Next we state several propositions:

(1) For all real numbers p, q such that $0 < p$ and $0 < q$ and for every real number a such that $0 \leq a$ holds $a^p \cdot a^q = a^{p+q}$.

(2) For all real numbers p, q such that $0 < p$ and $0 < q$ and for every real number a such that $0 \leq a$ holds $(a^p)^q = a^{p \cdot q}$.

(3) For every real number p such that $0 < p$ and for all real numbers a, b such that $0 \leq a$ and $a \leq b$ holds $a^p \leq b^p$.

(4) If $1 < p$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $0 < a$ and $0 < b$, then $a \cdot b \leq \frac{a_{\mathbb{R}}^p}{p} + \frac{b_{\mathbb{R}}^q}{q}$ and $a \cdot b = \frac{a_{\mathbb{R}}^p}{p} + \frac{b_{\mathbb{R}}^q}{q}$ iff $a_{\mathbb{R}}^p = b_{\mathbb{R}}^q$.

(5) If $1 < p$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $0 \leq a$ and $0 \leq b$, then $a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}$ and $a \cdot b = \frac{a^p}{p} + \frac{b^q}{q}$ iff $a^p = b^q$.

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2. Minkowski's Inequality

Next we state several propositions:

(6) Let p, q be real numbers. Suppose $1 < p$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let a, b, a_1, b_1, a_2 be sequences of real numbers. Suppose that for every natural number n holds $a_1(n) = |a(n)|^p$ and $b_1(n) = |b(n)|^q$ and $a_2(n) = |a(n) \cdot b(n)|$. Let n be a natural number. Then $(\sum_{\alpha=0}^{\kappa}(a_2)(\alpha))_{\kappa\in\mathbb{N}}(n) \leq (\sum_{\alpha=0}^{\kappa}(a_1)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{p}} \cdot (\sum_{\alpha=0}^{\kappa}(b_1)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{q}}$.

(7) Let p be a real number. Suppose $1 < p$. Let a, b, a_1, b_2, a_2 be sequences of real numbers. Suppose that for every natural number n holds $a_1(n) = |a(n)|^p$ and $b_2(n) = |b(n)|^p$ and $a_2(n) = |a(n) + b(n)|^p$. Let n be a natural number. Then $(\sum_{\alpha=0}^{\kappa}(a_2)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{p}} \leq (\sum_{\alpha=0}^{\kappa}(a_1)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{p}} + (\sum_{\alpha=0}^{\kappa}(b_2)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{p}}$.

(8) Let a, b be sequences of real numbers. Suppose for every natural number n holds $a(n) \leq b(n)$ and b is convergent and a is non-decreasing. Then a is convergent and $\lim a \leq \lim b$.

(9) Let a, b, c be sequences of real numbers. Suppose for every natural number n holds $a(n) \leq b(n) + c(n)$ and b is convergent and c is convergent and a is non-decreasing. Then a is convergent and $\lim a \leq \lim b + \lim c$.

(10) Let p be a real number. Suppose $0 < p$. Let a, a_1 be sequences of real numbers. Suppose a is convergent and for every natural number n holds $0 \leq a(n)$ and for every natural number n holds $a_1(n) = a(n)^p$. Then a_1 is convergent and $\lim a_1 = (\lim a)^p$.

(11) Let p be a real number. Suppose $0 < p$. Let a, a_1 be sequences of real numbers. Suppose a is summable and for every natural number n holds $0 \leq a(n)$ and for every natural number n holds $a_1(n) = (\sum_{\alpha=0}^{\kappa} a(\alpha))_{\kappa\in\mathbb{N}}(n)^p$. Then a_1 is convergent and $\lim a_1 = (\sum a)^p$ and a_1 is non-decreasing and for every natural number n holds $a_1(n) \leq (\sum a)^p$.

(12) Let p, q be real numbers. Suppose $1 < p$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let a, b, a_1, b_1, a_2 be sequences of real numbers. Suppose for every natural number n holds $a_1(n) = |a(n)|^p$ and $b_1(n) = |b(n)|^q$ and $a_2(n) = |a(n) \cdot b(n)|$ and a_1 is summable and b_1 is summable. Then a_2 is summable and $\sum a_2 \leq (\sum a_1)^{\frac{1}{p}} \cdot (\sum b_1)^{\frac{1}{q}}$.

(13) Let p be a real number. Suppose $1 < p$. Let a, b, a_1, b_2, a_2 be sequences of real numbers. Suppose that

(i) for every natural number n holds $a_1(n) = |a(n)|^p$ and $b_2(n) = |b(n)|^p$ and $a_2(n) = |a(n) + b(n)|^p$,

(ii) a_1 is summable, and

(iii) b_2 is summable.

Then a_2 is summable and $(\sum a_2)^{\frac{1}{p}} \leq (\sum a_1)^{\frac{1}{p}} + (\sum b_2)^{\frac{1}{p}}$.

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