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Set Sequences and Monotone Class

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Summary. In this paper we first defined the partial-union sequence, the partial-intersection sequence, and the partial-difference-union sequence of given sequence of subsets, and then proved the additive theorem of infinite sequences and sub-additive theorem of finite sequences for probability. Further, we defined the monotone class of families of subsets, and discussed the relations between the monotone class and the σ-field which are generated by the field of subsets of a given set.

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The articles [4], [3], [2], [20], [23], [19], [9], [21], [22], [18], [16], [6], [1], [13], [11], [24], [7], [8], [15], [14], [10], [12], [26], [25], [17], and [5] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: n, m, k are natural numbers, g is a real number, x, X, Y, Z are sets, A_1 is a sequence of subsets of X, F_1 is a finite sequence of elements of 2^X, R_1 is a finite sequence of elements of $\mathbb{R}$, S_1 is a σ-field of subsets of X, O_1 is a non empty set, S_2 is a σ-field of subsets of O_1, A_2, B_1 are sequences of subsets of S_2, and P is a probability on S_2.

One can prove the following propositions:

(1) For every finite sequence f holds $0 \notin \operatorname{dom} f$.

(2) For every finite sequence f holds $n \in \operatorname{dom} f$ iff $n \neq 0$ and $n \leq \operatorname{len} f$.

(3) Let f be a sequence of real numbers. Given k such that let given n. If $k \leq n$, then $f(n) = g$. Then f is convergent and $\lim f = g$.

(4) $(P \cdot A_2)(n) \geq 0$.

(5) If $A_2(n) \subseteq B_1(n)$, then $(P \cdot A_2)(n) \leq (P \cdot B_1)(n)$.

(6) If A_2 is non-decreasing, then $P \cdot A_2$ is non-decreasing.

(7) If A_2 is non-increasing, then $P \cdot A_2$ is non-increasing.

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Let A_1 be a function. The partial intersections of A_1 constitute a function defined by the conditions (Def. 1).

(Def. 1)(i) dom (the partial intersections of A_1) $= \mathbb{N}$,

(ii) (the partial intersections of A_1)$(0) = A_1(0)$, and

(iii) for every natural number n holds (the partial intersections of A_1)$(n+1) =$ (the partial intersections of A_1)$(n) \cap A_1(n+1)$.

Let X be a set and let A_1 be a sequence of subsets of X. Then the partial intersections of A_1 is a sequence of subsets of X.

Let A_1 be a function. The partial unions of A_1 constitute a function defined by the conditions (Def. 2).

(Def. 2)(i) dom (the partial unions of A_1) $= \mathbb{N}$,

(ii) (the partial unions of A_1)$(0) = A_1(0)$, and

(iii) for every natural number n holds (the partial unions of A_1)$(n+1) =$ (the partial unions of A_1)$(n) \cup A_1(n+1)$.

Let X be a set and let A_1 be a sequence of subsets of X. Then the partial unions of A_1 is a sequence of subsets of X.

The following propositions are true:

(8) (The partial intersections of A_1)$(n) \subseteq A_1(n)$.

(9) $A_1(n) \subseteq$ (the partial unions of A_1)(n).

(10) The partial intersections of A_1 are non-increasing.

(11) The partial unions of A_1 are non-decreasing.

(12) $x \in$ (the partial intersections of A_1)(n) iff for every k such that $k \leq n$ holds $x \in A_1(k)$.

(13) $x \in$ (the partial unions of A_1)(n) iff there exists k such that $k \leq n$ and $x \in A_1(k)$.

(14) Intersection (the partial intersections of A_1) $=$ Intersection A_1.

(15) $\bigcup$ (the partial unions of A_1) $= \bigcup A_1$.

Let A_1 be a function. The partial diff-unions of A_1 constitute a function defined by the conditions (Def. 3).

(Def. 3)(i) dom (the partial diff-unions of A_1) $= \mathbb{N}$,

(ii) (the partial diff-unions of A_1)$(0) = A_1(0)$, and

(iii) for every natural number n holds (the partial diff-unions of A_1)$(n+1) = A_1(n+1) \setminus$ (the partial unions of A_1)(n).

Let X be a set and let A_1 be a sequence of subsets of X. Then the partial diff-unions of A_1 is a sequence of subsets of X.

One can prove the following propositions:

(16) $x \in$ (the partial diff-unions of A_1)(n) iff $x \in A_1(n)$ and for every k such that $k < n$ holds $x \notin A_1(k)$.

(17) (The partial diff-unions of A_1)$(n) \subseteq A_1(n)$.

(18) (The partial diff-unions of $A_1)(n) \subseteq$ (the partial unions of $A_1)(n)$.

(19) The partial unions of the partial diff-unions of $A_1 =$ the partial unions of A_1.

(20) $\bigcup$(the partial diff-unions of $A_1) = \bigcup A_1$.

Let us consider X, A_1. Let us observe that A_1 is disjoint valued if and only if:

(Def. 4) For all m, n such that $m \neq n$ holds $A_1(m)$ misses $A_1(n)$.

We now state the proposition

(21) The partial diff-unions of A_1 are disjoint valued.

Let X be a set, let S_1 be a σ-field of subsets of X, and let X_1 be a sequence of subsets of S_1. Then the partial intersections of X_1 is a sequence of subsets of S_1.

Let X be a set, let S_1 be a σ-field of subsets of X, and let X_1 be a sequence of subsets of S_1. Then the partial unions of X_1 is a sequence of subsets of S_1.

Let X be a set, let S_1 be a σ-field of subsets of X, and let X_1 be a sequence of subsets of S_1. Then the partial diff-unions of X_1 is a sequence of subsets of S_1.

Next we state a number of propositions:

(22) $P \cdot$ the partial unions of A_2 is non-decreasing.

(23) $P \cdot$ the partial intersections of A_2 is non-increasing.

(24) $(\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa \in \mathbb{N}}$ is non-decreasing.

(25) $(P \cdot$ the partial unions of $A_2)(0) = (\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa \in \mathbb{N}}(0)$.

(26)(i) $P \cdot$ the partial unions of A_2 is convergent,

(ii) $\lim(P \cdot$ the partial unions of $A_2) = \sup(P \cdot$ the partial unions of $A_2)$, and

(iii) $\lim(P \cdot$ the partial unions of $A_2) = P(\bigcup A_2)$.

(27) If A_2 is disjoint valued, then for all n, m such that $n < m$ holds (the partial unions of $A_2)(n)$ misses $A_2(m)$.

(28) If A_2 is disjoint valued, then $(P \cdot$the partial unions of $A_2)(n) = (\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(29) If A_2 is disjoint valued, then $P \cdot$ the partial unions of $A_2 = (\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa \in \mathbb{N}}$.

(30) If A_2 is disjoint valued, then $(\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa \in \mathbb{N}}$ is convergent and $\lim((\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa \in \mathbb{N}}) = \sup((\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa \in \mathbb{N}})$ and $\lim((\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa \in \mathbb{N}}) = P(\bigcup A_2)$.

(31) If A_2 is disjoint valued, then $P(\bigcup A_2) = \sum(P \cdot A_2)$.

Let us consider X, F_1, n. Then $F_1(n)$ is a subset of X.

One can prove the following two propositions:

(32) There exists a finite sequence F_1 of elements of 2^X such that for every k such that $k \in \operatorname{dom} F_1$ holds $F_1(k) = X$.

(33) For every finite sequence F_1 of elements of 2^X holds $\bigcup \operatorname{rng} F_1$ is a subset of X.

Let X be a set and let F_1 be a finite sequence of elements of 2^X. Then $\bigcup F_1$ is a subset of X.

We now state the proposition

(34) $x \in \bigcup F_1$ iff there exists k such that $k \in \operatorname{dom} F_1$ and $x \in F_1(k)$.

Let us consider X, F_1. The functor Complement F_1 yields a finite sequence of elements of 2^X and is defined by:

(Def. 5) $\operatorname{len} \operatorname{Complement} F_1 = \operatorname{len} F_1$ and for every n such that $n \in \operatorname{dom} \operatorname{Complement} F_1$ holds $(\operatorname{Complement} F_1)(n) = F_1(n)^{\mathrm{c}}$.

Let us consider X, F_1. The functor Intersection F_1 yields a subset of X and is defined by:

(Def. 6) $\operatorname{Intersection} F_1 = \begin{cases} (\bigcup \operatorname{Complement} F_1)^{\mathrm{c}}, & \text{if } F_1 \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$

Next we state several propositions:

(35) $\operatorname{dom} \operatorname{Complement} F_1 = \operatorname{dom} F_1$.

(36) If $F_1 \neq \emptyset$, then $x \in \operatorname{Intersection} F_1$ iff for every k such that $k \in \operatorname{dom} F_1$ holds $x \in F_1(k)$.

(37) If $F_1 \neq \emptyset$, then $x \in \bigcap \operatorname{rng} F_1$ iff for every n such that $n \in \operatorname{dom} F_1$ holds $x \in F_1(n)$.

(38) $\operatorname{Intersection} F_1 = \bigcap \operatorname{rng} F_1$.

(39) Let F_1 be a finite sequence of elements of 2^X. Then there exists a sequence A_1 of subsets of X such that for every k such that $k \in \operatorname{dom} F_1$ holds $A_1(k) = F_1(k)$ and for every k such that $k \notin \operatorname{dom} F_1$ holds $A_1(k) = \emptyset$.

(40) Let F_1 be a finite sequence of elements of 2^X and A_1 be a sequence of subsets of X. Suppose for every k such that $k \in \operatorname{dom} F_1$ holds $A_1(k) = F_1(k)$ and for every k such that $k \notin \operatorname{dom} F_1$ holds $A_1(k) = \emptyset$. Then $A_1(0) = \emptyset$ and $\bigcup A_1 = \bigcup F_1$.

Let X be a set and let S_1 be a σ-field of subsets of X. A finite sequence of elements of 2^X is said to be a finite sequence of elements of S_1 if:

(Def. 7) For every k such that $k \in \operatorname{dom} \mathrm{it}$ holds $\mathrm{it}(k) \in S_1$.

Let X be a set, let S_1 be a σ-field of subsets of X, let F_2 be a finite sequence of elements of S_1, and let us consider n. Then $F_2(n)$ is an event of S_1.

We now state two propositions:

(41) Let F_2 be a finite sequence of elements of S_1. Then there exists a sequence A_2 of subsets of S_1 such that for every k such that $k \in \operatorname{dom} F_2$ holds $A_2(k) = F_2(k)$ and for every k such that $k \notin \operatorname{dom} F_2$ holds $A_2(k) = \emptyset$.

(42) For every finite sequence F_2 of elements of S_1 holds $\bigcup F_2 \in S_1$.

Let X be a set, let S be a σ-field of subsets of X, and let F be a finite sequence of elements of S. The functor $F^{\mathbf{c}}$ yielding a finite sequence of elements of S is defined as follows:

(Def. 8) $F^{\mathbf{c}} = \mathrm{Complement}\, F$.

We now state the proposition

(43) For every finite sequence F_2 of elements of S_1 holds $\mathrm{Intersection}\, F_2 \in S_1$.

In the sequel F_3 denotes a finite sequence of elements of S_2.

The following two propositions are true:

(44) $\mathrm{dom}(P \cdot F_3) = \mathrm{dom}\, F_3$.

(45) $P \cdot F_3$ is a finite sequence of elements of $\mathbb{R}$.

Let us consider O_1, S_2, F_3, P. Then $P \cdot F_3$ is a finite sequence of elements of $\mathbb{R}$.

Next we state several propositions:

(46) $\mathrm{len}(P \cdot F_3) = \mathrm{len}\, F_3$.

(47) If $\mathrm{len}\, R_1 = 0$, then $\sum R_1 = 0$.

(48) Suppose $\mathrm{len}\, R_1 \geq 1$. Then there exists a sequence f of real numbers such that $f(1) = R_1(1)$ and for every n such that $0 \neq n$ and $n < \mathrm{len}\, R_1$ holds $f(n+1) = f(n) + R_1(n+1)$ and $\sum R_1 = f(\mathrm{len}\, R_1)$.

(49) Let F_3 be a finite sequence of elements of S_2 and A_2 be a sequence of subsets of S_2. Suppose for every k such that $k \in \mathrm{dom}\, F_3$ holds $A_2(k) = F_3(k)$ and for every k such that $k \notin \mathrm{dom}\, F_3$ holds $A_2(k) = \emptyset$. Then $(\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa \in \mathbb{N}}$ is convergent and $\sum(P \cdot A_2) = (\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa \in \mathbb{N}}(\mathrm{len}\, F_3)$ and $P(\bigcup A_2) \leq \sum(P \cdot A_2)$ and $\sum(P \cdot F_3) = \sum(P \cdot A_2)$.

(50) $P(\bigcup F_3) \leq \sum(P \cdot F_3)$ and if F_3 is disjoint valued, then $P(\bigcup F_3) = \sum(P \cdot F_3)$.

Let us consider X and let I_1 be a family of subsets of X. We say that I_1 is non-decreasing-union-closed if and only if:

(Def. 9) For every sequence A_1 of subsets of X such that A_1 is non-decreasing and for every n holds $A_1(n) \in I_1$ holds $\bigcup A_1 \in I_1$.

We say that I_1 is non-increasing-intersection-closed if and only if:

(Def. 10) For every sequence A_1 of subsets of X such that A_1 is non-increasing and for every n holds $A_1(n) \in I_1$ holds $\mathrm{Intersection}\, A_1 \in I_1$.

We now state three propositions:

(51) Let I_1 be a family of subsets of X. Then I_1 is non-decreasing-union-closed if and only if for every sequence A_1 of subsets of X such that A_1 is non-decreasing and for every n holds $A_1(n) \in I_1$ holds $\lim A_1 \in I_1$.

(52) Let I_1 be a family of subsets of X. Then I_1 is non-increasing-intersection-closed if and only if for every sequence A_1 of subsets of X such that A_1 is

non-increasing and for every n holds $A_1(n) \in I_1$ holds $\lim A_1 \in I_1$.

(53) 2^X is non-decreasing-union-closed and 2^X is non-increasing-intersection-closed.

Let us consider X. A family of subsets of X is said to be a monotone class of X if:

(Def. 11) It is non-decreasing-union-closed and it is non-increasing-intersection-closed.

Next we state four propositions:

(54) Z is a monotone class of X if and only if the following conditions are satisfied:

(i) $Z \subseteq 2^X$, and

(ii) for every sequence A_1 of subsets of X such that A_1 is monotone and for every n holds $A_1(n) \in Z$ holds $\lim A_1 \in Z$.

(55) Let F be a field of subsets of X. Then F is a σ-field of subsets of X if and only if F is a monotone class of X.

(56) 2^{O_1} is a monotone class of O_1.

(57) Let X be a family of subsets of O_1. Then there exists a monotone class Y of O_1 such that $X \subseteq Y$ and for every Z such that $X \subseteq Z$ and Z is a monotone class of O_1 holds $Y \subseteq Z$.

Let us consider O_1 and let X be a family of subsets of O_1. The functor monotone-class(X) yielding a monotone class of O_1 is defined as follows:

(Def. 12) $X \subseteq$ monotone-class(X) and for every Z such that $X \subseteq Z$ and Z is a monotone class of O_1 holds monotone-class(X) $\subseteq Z$.

We now state two propositions:

(58) For every field Z of subsets of O_1 holds monotone-class(Z) is a field of subsets of O_1.

(59) For every field Z of subsets of O_1 holds $\sigma(Z) =$ monotone-class(Z).

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A Theory of Sequential Files

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Summary. This article is a continuation of [6]. We present the notion of files and records. These are two finite sequences. One is a record and another is a separator for the carriage return and/or line feed. So, we define the record. The sequential text file contains records and separators. Generally, a record and a separator are paired in the file. And in a special situation, the separator does not exist in the file, for that the record is only one record or record is nothing. And the record does not exist in the file, for that some separator is in the file. In this article, we present a theory for files and records.

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The terminology and notation used here are introduced in the following articles: [11], [12], [7], [1], [10], [13], [8], [2], [3], [4], [9], [5], and [6].

In this paper a, b, c denote sets.

The following propositions are true:

(1) Let D be a non empty set and p, q, r, s be finite sequences of elements of D. Then $p \frown q \frown r \frown s = p \frown (q \frown r) \frown s$ and $(p \frown q \frown r) \frown s = p \frown q \frown (r \frown s)$ and $(p \frown (q \frown r)) \frown s = p \frown q \frown (r \frown s)$.

(2) For every set D and for every finite sequence f of elements of D holds $f \upharpoonright \operatorname{len} f = f$.

(3) For every non empty set D and for all finite sequences p, q of elements of D such that $\operatorname{len} p = 0$ holds $q = p \frown q$.

(4) Let D be a non empty set, f be a finite sequence of elements of D, and n, m be natural numbers. If $n \leq m$, then $\operatorname{len}(f_{\lfloor m}) \leq \operatorname{len}(f_{\lfloor n})$.

(5) For every non empty set D and for all finite sequences f, g of elements of D such that $\operatorname{len} g \geq 1$ holds $\operatorname{mid}(f \frown g, \operatorname{len} f + 1, \operatorname{len} f + \operatorname{len} g) = g$.

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(6) Let D be a non empty set, f, g be finite sequences of elements of D, and i, j be natural numbers. If $1 \leq i$ and $i \leq j$ and $j \leq \operatorname{len} f$, then $\operatorname{mid}(f \frown g, i, j) = \operatorname{mid}(f, i, j)$.

(7) Let D be a non empty set, f be a finite sequence of elements of D, and i, j, n be natural numbers. If $1 \leq i$ and $i \leq j$ and $i \leq \operatorname{len}(f \restriction n)$ and $j \leq \operatorname{len}(f \restriction n)$, then $\operatorname{mid}(f, i, j) = \operatorname{mid}(f \restriction n, i, j)$.

(8) For every non empty set D and for every finite sequence f of elements of D such that $f = \langle a \rangle$ holds $a \in D$.

(9) For every non empty set D and for every finite sequence f of elements of D such that $f = \langle a, b \rangle$ holds $a \in D$ and $b \in D$.

(10) Let D be a non empty set and f be a finite sequence of elements of D. If $f = \langle a, b, c \rangle$, then $a \in D$ and $b \in D$ and $c \in D$.

(11) For every non empty set D and for every finite sequence f of elements of D such that $f = \langle a \rangle$ holds $f \restriction 1 = \langle a \rangle$.

(12) For every non empty set D and for every finite sequence f of elements of D such that $f = \langle a, b \rangle$ holds $f_{\downharpoonright 1} = \langle b \rangle$.

(13) For every non empty set D and for every finite sequence f of elements of D such that $f = \langle a, b, c \rangle$ holds $f \restriction 1 = \langle a \rangle$.

(14) For every non empty set D and for every finite sequence f of elements of D such that $f = \langle a, b, c \rangle$ holds $f \restriction 2 = \langle a, b \rangle$.

(15) For every non empty set D and for every finite sequence f of elements of D such that $f = \langle a, b, c \rangle$ holds $f_{\downharpoonright 1} = \langle b, c \rangle$.

(16) For every non empty set D and for every finite sequence f of elements of D such that $f = \langle a, b, c \rangle$ holds $f_{\downharpoonright 2} = \langle c \rangle$.

(17) For every non empty set D and for every finite sequence f of elements of D such that $\operatorname{len} f = 0$ holds $\operatorname{Rev}(f) = f$.

(18) Let D be a non empty set, r be a finite sequence of elements of D, and i be a natural number. If $i \leq \operatorname{len} r$, then $\operatorname{Rev}(r_{\downharpoonright i}) = \operatorname{Rev}(r) \restriction (\operatorname{len} r -' i)$.

(19) Let D be a non empty set and f, C_1 be finite sequences of elements of D. If C_1 is not a substring of f and C_1 separates uniquely, then $\operatorname{instr}(1, f \frown C_1) = \operatorname{len} f + 1$.

(20) For every non empty set D and for every finite sequence f of elements of D holds every finite sequence f, g of elements of D is a preposition of $(f \frown g)_{\downharpoonright \operatorname{len} f}$.

(21) Let D be a non empty set and f, C_1 be finite sequences of elements of D. Suppose C_1 is not a substring of f and C_1 separates uniquely. Then $f \frown C_1$ is terminated by C_1.

Let D be a set. We introduce file of D as a synonym of finite sequence of elements of D.

Let D be a non empty set and let r, f, C_1 be files of D. We say that r is a record of f and C_1 if and only if:

(Def. 1) $C_1 \frown r$ is a substring of $\operatorname{addcr}(f, C_1)$ or r is a preposition of $\operatorname{addcr}(f, C_1)$ but r is terminated by C_1.

The following propositions are true:

(22) For every non empty set D and for every finite sequence r of elements of D holds $\operatorname{ovlpart}(\varepsilon_D, r) = \varepsilon_D$ and $\operatorname{ovlpart}(r, \varepsilon_D) = \varepsilon_D$.

(23) For every non empty set D holds every finite sequence C_1 of elements of D is a record of ε_D and C_1.

(24) Let D be a non empty set, a, b be sets, and f, r, C_1 be files of D. Suppose $a \neq b$ and $D = \{a, b\}$ and $C_1 = \langle b \rangle$ and $f = \langle b, a, b \rangle$ and $r = \langle a, b \rangle$. Then C_1 is a record of f and C_1 and r is a record of f and C_1.

(25) For every non empty set D and for all files f, C_1 of D holds f is a preposition of $f \frown C_1$.

(26) For every non empty set D and for all files f, C_1 of D holds f is a preposition of $\operatorname{addcr}(f, C_1)$.

(27) For every non empty set D and for all files r, C_1 of D such that C_1 is a postposition of r holds $0 \leq \operatorname{len} r - \operatorname{len} C_1$.

(28) For every non empty set D and for all files C_1, r of D such that C_1 is a postposition of r holds $r = \operatorname{addcr}(r, C_1)$.

(29) For every non empty set D and for all files C_1, r of D such that r is terminated by C_1 holds $r = \operatorname{addcr}(r, C_1)$.

(30) For every non empty set D and for all files f, g of D such that f is terminated by g holds $\operatorname{len} g \leq \operatorname{len} f$.

(31) For every non empty set D and for all files f, C_1 of D holds $\operatorname{len} \operatorname{addcr}(f, C_1) \geq \operatorname{len} f$ and $\operatorname{len} \operatorname{addcr}(f, C_1) \geq \operatorname{len} C_1$.

(32) For every non empty set D and for all finite sequences f, g of elements of D holds $g = (\operatorname{ovlpart}(f, g)) \frown \operatorname{ovlrdiff}(f, g)$.

(33) For every non empty set D and for all finite sequences f, g of elements of D holds $\operatorname{ovlcon}(f, g) = (\operatorname{ovlldiff}(f, g)) \frown g$.

(34) For every non empty set D and for all files C_1, r of D holds $\operatorname{addcr}(r, C_1) = (\operatorname{ovlldiff}(r, C_1)) \frown C_1$.

(35) Let D be a non empty set and r_1, r_2, f be files of D. If $f = r_1 \frown r_2$, then r_1 is a substring of f and r_2 is a substring of f.

(36) Let D be a non empty set and r_1, r_2, r_3, f be files of D. Suppose $f = r_1 \frown r_2 \frown r_3$. Then r_1 is a substring of f and r_2 is a substring of f and r_3 is a substring of f.

(37) Let D be a non empty set and C_1, r_1, r_2 be files of D. Suppose r_1 is terminated by C_1 and r_2 is terminated by C_1. Then $C_1 \frown r_2$ is a substring

of addcr($r_1 \frown r_2, C_1$).

(38) Let D be a non empty set, f, g be files of D, and n be a natural number. If $0 < n$ and $g = \emptyset$, then $\mathrm{instr}(n, f) = n$.

(39) Let D be a non empty set, f, g be files of D, and n be a natural number. If $0 < n$ and $n \leq \mathrm{len}\, f$, then $\mathrm{instr}(n, f) \leq \mathrm{len}\, f$.

(40) For every non empty set D and for every file f of D holds every file f, C_1 of D is a substring of $\mathrm{ovlcon}(f, C_1)$.

(41) For every non empty set D and for every file f of D holds every file f, C_1 of D is a substring of $\mathrm{addcr}(f, C_1)$.

(42) Let D be a non empty set, f, g be finite sequences of elements of D, and n be a natural number. If g is a substring of $f \restriction n$ and $\mathrm{len}\, g > 0$ and $\mathrm{len}\, g \leq n$, then g is a substring of f.

(43) For every non empty set D and for all files f, C_1 of D holds there exists a file of D which is a record of f and C_1.

(44) For every non empty set D and for all files f, C_1, r of D such that r is a record of f and C_1 holds r is a record of r and C_1.

(45) Let D be a non empty set and C_1, r_1, r_2, f be files of D. Suppose r_1 is terminated by C_1 and r_2 is terminated by C_1 and $f = r_1 \frown r_2$. Then r_1 is a record of f and C_1 and r_2 is a record of f and C_1.

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Circled Sets, Circled Hull, and Circled Family

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Summary. In this article, we prove some basic properties of the circled sets. We also define the circled hull, and give the definition of a circled family.

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The articles [15], [19], [14], [3], [4], [12], [5], [11], [13], [18], [9], [8], [2], [17], [16], [6], [1], [7], and [10] provide the terminology and notation for this paper.

1. Circled Sets

One can prove the following proposition

(1) For every real linear space V and for all circled subsets A, B of V holds $A - B$ is circled.

Let V be a real linear space and let M, N be circled subsets of V. Note that $M - N$ is circled.

Next we state the proposition

(2) Let V be a non empty RLS structure and M be a subset of V. Then M is circled if and only if for every vector u of V and for every real number r such that $|r| \leq 1$ and $u \in M$ holds $r \cdot u \in M$.

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Let V be a non empty RLS structure and let M be a subset of V. Let us observe that M is circled if and only if:

(Def. 1) For every vector u of V and for every real number r such that $|r| \leq 1$ and $u \in M$ holds $r \cdot u \in M$.

The following propositions are true:

(3) Let V be a real linear space, M be a subset of V, and r be a real number. If M is circled, then $r \cdot M$ is circled.

(4) Let V be a real linear space, M_1, M_2 be subsets of V, and r_1, r_2 be real numbers. If M_1 is circled and M_2 is circled, then $r_1 \cdot M_1 + r_2 \cdot M_2$ is circled.

(5) Let V be a real linear space, M_1, M_2, M_3 be subsets of V, and r_1, r_2, r_3 be real numbers. Suppose M_1 is circled and M_2 is circled and M_3 is circled. Then $r_1 \cdot M_1 + r_2 \cdot M_2 + r_3 \cdot M_3$ is circled.

(6) For every real linear space V holds $\mathrm{Up}(\mathbf{0}_V)$ is circled.

(7) For every real linear space V holds $\mathrm{Up}(\Omega_V)$ is circled.

(8) For every real linear space V and for all circled subsets M, N of V holds $M \cap N$ is circled.

(9) For every real linear space V and for all circled subsets M, N of V holds $M \cup N$ is circled.

2. Circled Hull and Circled Family

Let V be a non empty RLS structure and let M be a subset of V. The functor Circled-Family M yields a family of subsets of V and is defined as follows:

(Def. 2) For every subset N of V holds $N \in$ Circled-Family M iff N is circled and $M \subseteq N$.

Let V be a real linear space and let M be a subset of V. The functor Cir M yielding a circled subset of V is defined by:

(Def. 3) $\mathrm{Cir}\, M = \bigcap \text{Circled-Family}\, M$.

Let V be a real linear space and let M be a subset of V. Note that Circled-Family M is non empty.

We now state several propositions:

(10) For every real linear space V and for all subsets M_1, M_2 of V such that $M_1 \subseteq M_2$ holds Circled-Family $M_2 \subseteq$ Circled-Family M_1.

(11) For every real linear space V and for all subsets M_1, M_2 of V such that $M_1 \subseteq M_2$ holds $\mathrm{Cir}\, M_1 \subseteq \mathrm{Cir}\, M_2$.

(12) For every real linear space V and for every subset M of V holds $M \subseteq \mathrm{Cir}\, M$.

(13) Let V be a real linear space, M be a subset of V, and N be a circled subset of V. If $M \subseteq N$, then $\mathrm{Cir}\, M \subseteq N$.

(14) For every real linear space V and for every circled subset M of V holds $\mathrm{Cir}\,M = M$.

(15) For every real linear space V holds $\mathrm{Cir}(\emptyset_V) = \emptyset$.

(16) For every real linear space V and for every subset M of V and for every real number r holds $r \cdot \mathrm{Cir}\,M = \mathrm{Cir}(r \cdot M)$.

3. Basic Properties of Combination

Let V be a real linear space and let L be a linear combination of V. We say that L is circled if and only if the condition (Def. 4) is satisfied.

(Def. 4) There exists a finite sequence F of elements of the carrier of V such that

(i) F is one-to-one,

(ii) $\mathrm{rng}\,F =$ the support of L, and

(iii) there exists a finite sequence f of elements of $\mathbb{R}$ such that $\mathrm{len}\,f = \mathrm{len}\,F$ and $\sum f = 1$ and for every natural number n such that $n \in \mathrm{dom}\,f$ holds $f(n) = L(F(n))$ and $f(n) \geq 0$.

The following propositions are true:

(17) Let V be a real linear space and L be a linear combination of V. If L is circled, then the support of $L \neq \emptyset$.

(18) Let V be a real linear space, L be a linear combination of V, and v be a vector of V. If L is circled and $L(v) \leq 0$, then $v \notin$ the support of L.

(19) For every real linear space V and for every linear combination L of V such that L is circled holds $L \neq \mathbf{0}_{\mathrm{LC}_V}$.

(20) For every real linear space V holds there exists a linear combination of V which is circled.

Let V be a real linear space. One can check that there exists a linear combination of V which is circled.

Let V be a real linear space. A circled combination of V is a circled linear combination of V.

We now state the proposition

(21) For every real linear space V and for every non empty subset M of V holds there exists a linear combination of M which is circled.

Let V be a real linear space and let M be a non empty subset of V. Note that there exists a linear combination of M which is circled.

Let V be a real linear space and let M be a non empty subset of V. A circled combination of M is a circled linear combination of M.

Let V be a real linear space. The functor $\mathrm{circledComb}\,V$ is defined as follows:

(Def. 5) For every set L holds $L \in \mathrm{circledComb}\,V$ iff L is a circled combination of V.

Let V be a real linear space and let M be a non empty subset of V. The functor circledComb M is defined by:

(Def. 6) For every set L holds $L \in$ circledComb M iff L is a circled combination of M.

The following propositions are true:

(22) Let V be a real linear space and v be a vector of V. Then there exists a circled combination L of V such that $\sum L = v$ and for every non empty subset A of V such that $v \in A$ holds L is a circled combination of A.

(23) Let V be a real linear space and v_1, v_2 be vectors of V. Suppose $v_1 \neq v_2$. Then there exists a circled combination L of V such that for every non empty subset A of V if $\{v_1, v_2\} \subseteq A$, then L is a circled combination of A.

(24) Let V be a real linear space, L_1, L_2 be circled combinations of V, and a, b be real numbers. Suppose $a \cdot b > 0$. Then the support of $a \cdot L_1 + b \cdot L_2 =$ (the support of $a \cdot L_1$) $\cup$ (the support of $b \cdot L_2$).

(25) Let V be a real linear space, v be a vector of V, and L be a linear combination of V. If L is circled and the support of $L = \{v\}$, then $L(v) = 1$ and $\sum L = L(v) \cdot v$.

(26) Let V be a real linear space, v_1, v_2 be vectors of V, and L be a linear combination of V. Suppose L is circled and the support of $L = \{v_1, v_2\}$ and $v_1 \neq v_2$. Then $L(v_1) + L(v_2) = 1$ and $L(v_1) \geq 0$ and $L(v_2) \geq 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

(27) Let V be a real linear space, v be a vector of V, and L be a linear combination of $\{v\}$. If L is circled, then $L(v) = 1$ and $\sum L = L(v) \cdot v$.

(28) Let V be a real linear space, v_1, v_2 be vectors of V, and L be a linear combination of $\{v_1, v_2\}$. Suppose $v_1 \neq v_2$ and L is circled. Then $L(v_1) + L(v_2) = 1$ and $L(v_1) \geq 0$ and $L(v_2) \geq 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

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On the Borel Families of Subsets of Topological Spaces[1]

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Summary. This is the next Mizar article in a series aiming at complete formalization of "General Topology" [14] by Engelking. We cover the second part of Section 1.3.

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The papers [27], [30], [31], [9], [1], [2], [26], [3], [28], [10], [12], [21], [29], [22], [5], [16], [6], [23], [32], [11], [20], [17], [18], [19], [7], [13], [25], [24], [15], [4], and [8] provide the terminology and notation for this paper.

1. Preliminaries

Let T be a 1-sorted structure. The functor $\mathrm{TotFam}\,T$ yielding a family of subsets of T is defined by:

(Def. 1) $\mathrm{TotFam}\,T = 2^{\text{the carrier of } T}$.

The following proposition is true

(1) For every set T and for every family F of subsets of T holds F is countable iff $F^{\rm c}$ is countable.

Let us note that $\mathbb{Q}$ is countable.

The scheme *FraenCoun11* concerns a unary predicate $\mathcal{P}$, and states that:

$\{\{n\}; n$ ranges over elements of $\mathbb{Q}$: $\mathcal{P}[n]\}$ is countable

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for all values of the parameters.

One can prove the following proposition

(2) For every non empty topological space T and for every subset A of T holds $\operatorname{Der} A = \{x; x \text{ ranges over points of } T\colon x \in \overline{A \setminus \{x\}}\}$.

Let us note that every topological structure which is finite is also second-countable.

One can verify that $\mathbb{R}$ is non countable.

One can verify the following observations:

* every set which is non countable is also non finite,
* every set which is non finite is also non trivial, and
* there exists a set which is non countable and non empty.

We adopt the following rules: T is a non empty topological space, A, B are subsets of T, and F, G are families of subsets of T.

One can prove the following propositions:

(3) A is closed iff $\operatorname{Der} A \subseteq A$.

(4) Let T be a non empty topological structure, B be a basis of T, and V be a subset of T. Suppose V is open and $V \neq \emptyset$. Then there exists a subset W of T such that $W \in B$ and $W \subseteq V$ and $W \neq \emptyset$.

2. Regular Formalization: Separable Spaces

The following propositions are true:

(5) $\operatorname{density} T \leq \operatorname{weight} T$.

(6) T is separable iff there exists a subset of T which is dense and countable.

(7) If T is second-countable, then T is separable.

One can check that every non empty topological space which is second-countable is also separable.

The following four propositions are true:

(8) Let T be a non empty topological space and A, B be subsets of T. If A and B are separated, then $\operatorname{Fr}(A \cup B) = \operatorname{Fr} A \cup \operatorname{Fr} B$.

(9) If F is locally finite, then $\operatorname{Fr} \bigcup F \subseteq \bigcup \operatorname{Fr} F$.

(10) For every discrete non empty topological space T holds T is separable iff $\overline{\overline{\Omega_T}} \leq \aleph_0$.

(11) For every discrete non empty topological space T holds T is separable iff T is countable.

3. Families of Subsets Closed for Countable Unions and Complement

Let us consider T, F. We say that F is all-open-containing if and only if:

(Def. 2) For every subset A of T such that A is open holds $A \in F$.

Let us consider T, F. We say that F is all-closed-containing if and only if:

(Def. 3) For every subset A of T such that A is closed holds $A \in F$.

Let T be a set and let F be a family of subsets of T. We say that F is closed for countable unions if and only if:

(Def. 4) For every countable family G of subsets of T such that $G \subseteq F$ holds $\bigcup G \in F$.

Let T be a set. Note that every σ-field of subsets of T is closed for countable unions.

One can prove the following proposition

(12) For every set T and for every family F of subsets of T such that F is closed for countable unions holds $\emptyset \in F$.

Let T be a set. One can verify that every family of subsets of T which is closed for countable unions is also non empty.

Next we state the proposition

(13) Let T be a set and F be a family of subsets of T. Then F is a σ-field of subsets of T if and only if F is closed for complement operator and closed for countable unions.

Let T be a set and let F be a family of subsets of T. We say that F is closed for countable meets if and only if:

(Def. 5) For every countable family G of subsets of T such that $G \subseteq F$ holds $\bigcap G \in F$.

Next we state four propositions:

(14) Let F be a family of subsets of T. Then the following statements are equivalent

(i) F is all-closed-containing and closed for complement operator,

(ii) F is all-open-containing and closed for complement operator.

(15) For every set T and for every family F of subsets of T such that F is closed for complement operator holds $F = F^{c}$.

(16) Let T be a set and F, G be families of subsets of T. If $F \subseteq G$ and G is closed for complement operator, then $F^{c} \subseteq G$.

(17) Let T be a set and F be a family of subsets of T. Then the following statements are equivalent

(i) F is closed for countable meets and closed for complement operator,

(ii) F is closed for countable unions and closed for complement operator.

Let us consider T. One can verify that every family of subsets of T which is all-open-containing, closed for complement operator, and closed for countable unions is also all-closed-containing and closed for countable meets and every family of subsets of T which is all-closed-containing, closed for complement operator, and closed for countable meets is also all-open-containing and closed for countable unions.

4. On the Families of Subsets

Let T be a set and let F be a countable family of subsets of T. Note that F^{c} is countable.

Let us consider T. Note that every family of subsets of T which is empty is also open and closed.

Let us consider T. One can check that there exists a family of subsets of T which is countable, open, and closed.

We now state the proposition

(18) For every set T holds $\emptyset$ is an empty family of subsets of T.

Let us observe that every set which is empty is also countable.

5. Collective Properties of Families

One can prove the following two propositions:

(19) If $F = \{A\}$, then A is open iff F is open.

(20) If $F = \{A\}$, then A is closed iff F is closed.

Let T be a set and let F, G be families of subsets of T. Then $F \Cap G$ is a family of subsets of T. Then $F \Cup G$ is a family of subsets of T.

Next we state a number of propositions:

(21) If F is closed and G is closed, then $F \Cap G$ is closed.

(22) If F is closed and G is closed, then $F \Cup G$ is closed.

(23) If F is open and G is open, then $F \Cap G$ is open.

(24) If F is open and G is open, then $F \Cup G$ is open.

(25) For every set T and for all families F, G of subsets of T holds $\overline{\overline{F \Cap G}} \leq \overline{\overline{[\!: F, G :\!]}}$.

(26) For every set T and for all families F, G of subsets of T holds $\overline{\overline{F \Cup G}} \leq \overline{\overline{[\!: F, G :\!]}}$.

(27) For all sets F, G holds $\bigcup(F \Cup G) \subseteq \bigcup F \cup \bigcup G$.

(28) For all sets F, G such that $F \neq \emptyset$ and $G \neq \emptyset$ holds $\bigcup F \cup \bigcup G = \bigcup(F \Cup G)$.

(29) For every set F holds $\emptyset \Cup F = \emptyset$.

(30) For all sets F, G such that $F \Cup G = \emptyset$ holds $F = \emptyset$ or $G = \emptyset$.

(31) For all sets F, G such that $F \Cap G = \emptyset$ holds $F = \emptyset$ or $G = \emptyset$.

(32) For all sets F, G holds $\bigcap(F \Cup G) \subseteq \bigcap F \cup \bigcap G$.

(33) For all sets F, G such that $F \neq \emptyset$ and $G \neq \emptyset$ holds $\bigcap(F \Cup G) = \bigcap F \cup \bigcap G$.

(34) For all sets F, G such that $F \neq \emptyset$ and $G \neq \emptyset$ holds $\bigcap F \cap \bigcap G = \bigcap(F \Cap G)$.

6. F_σ and G_δ Types of Subsets

Let us consider T, A. We say that A is F_σ if and only if:

(Def. 6) There exists a closed countable family F of subsets of T such that $A = \bigcup F$.

Let us consider T, A. We say that A is G_δ if and only if:

(Def. 7) There exists an open countable family F of subsets of T such that $A = \bigcap F$.

The following propositions are true:

(35) $\emptyset_T$ is F_σ.

(36) $\emptyset_T$ is G_δ.

Let us consider T. Note that $\emptyset_T$ is F_σ and G_δ.

Next we state two propositions:

(37) Ω_T is F_σ.

(38) Ω_T is G_δ.

Let us consider T. One can verify that Ω_T is F_σ and G_δ.

One can prove the following propositions:

(39) If A is F_σ, then A^{c} is G_δ.

(40) If A is G_δ, then A^{c} is F_σ.

(41) If A is F_σ and B is F_σ, then $A \cap B$ is F_σ.

(42) If A is F_σ and B is F_σ, then $A \cup B$ is F_σ.

(43) If A is G_δ and B is G_δ, then $A \cup B$ is G_δ.

(44) If A is G_δ and B is G_δ, then $A \cap B$ is G_δ.

(45) For every subset A of T such that A is closed holds A is F_σ.

(46) For every subset A of T such that A is open holds A is G_δ.

(47) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{Q}$ holds A is F_σ.

7. $T_{1/2}$ Topological Spaces

Let T be a topological space. We say that T is $T_{1/2}$ if and only if:

(Def. 8) For every subset A of T holds $\operatorname{Der} A$ is closed.

We now state three propositions:

(48) For every topological space T such that T is T_1 holds T is $T_{1/2}$.

(49) For every non empty topological space T such that T is $T_{1/2}$ holds T is T_0.

(50) For every non empty topological space T holds every point p of T is isolated in Ω_T or an accumulation point of Ω_T.

Let us note that every topological space which is $T_{1/2}$ is also T_0 and every topological space which is T_1 is also $T_{1/2}$.

8. Condensation Points

Let us consider T, A and let x be a point of T. We say that x is a condensation point of A if and only if:

(Def. 9) For every neighbourhood N of x holds $N \cap A$ is not countable.

In the sequel x denotes a point of T.

One can prove the following proposition

(51) If x is a condensation point of A and $A \subseteq B$, then x is a condensation point of B.

Let us consider T, A. The functor A^0 yielding a subset of T is defined as follows:

(Def. 10) For every point x of T holds $x \in A^0$ iff x is a condensation point of A.

The following propositions are true:

(52) For every point p of T such that p is a condensation point of A holds p is an accumulation point of A.

(53) $A^0 \subseteq \operatorname{Der} A$.

(54) $A^0 = \overline{A^0}$.

(55) If $A \subseteq B$, then $A^0 \subseteq B^0$.

(56) If x is a condensation point of $A \cup B$, then x is a condensation point of A or a condensation point of B.

(57) $A \cup B^0 = A^0 \cup B^0$.

(58) If A is countable, then there exists no point of T which is a condensation point of A.

(59) If A is countable, then $A^0 = \emptyset$.

Let us consider T and let A be a countable subset of T. Note that A^0 is empty.

The following proposition is true

(60) If T is second-countable, then there exists a basis of T which is countable.

Let us mention that there exists a topological space which is second-countable and non empty.

9. Borel Families of Subsets

Let us consider T. Observe that $\mathrm{TotFam}\, T$ is non empty, all-open-containing, closed for complement operator, and closed for countable unions.

We now state four propositions:

(61) For every set T and for every sequence A of subsets of T holds $\mathrm{rng}\, A$ is a countable non empty family of subsets of T.

(62) Let T, F be sets. Then F is a σ-field of subsets of T if and only if F is a closed for complement operator σ-field of subsets-like non empty family of subsets of T.

(63) For all families F, G of subsets of T such that F is all-open-containing and $F \subseteq G$ holds G is all-open-containing.

(64) Let F, G be families of subsets of T. Suppose F is all-closed-containing and $F \subseteq G$. Then G is all-closed-containing.

Let T be a 1-sorted structure. A σ-field of subsets of T is a σ-field of subsets of the carrier of T.

Let T be a non empty topological space. Note that there exists a family of subsets of T which is closed for complement operator, closed for countable unions, closed for countable meets, all-closed-containing, and all-open-containing.

We now state the proposition

(65) $\sigma(\mathrm{TotFam}\, T)$ is all-open-containing, closed for complement operator, and closed for countable unions.

Let us consider T. One can verify that $\sigma(\mathrm{TotFam}\, T)$ is all-open-containing, closed for complement operator, and closed for countable unions.

Let T be a non empty 1-sorted structure. Note that there exists a family of subsets of T which is σ-field of subsets-like, closed for complement operator, closed for countable unions, and non empty.

Let T be a non empty topological space. One can verify that every σ-field of subsets of T is closed for countable unions.

We now state the proposition

(66) Let T be a non empty topological space and F be a family of subsets of T. Suppose F is closed for complement operator and closed for countable unions. Then F is a σ-field of subsets of T.

Let T be a non empty topological space. Note that there exists a σ-field of subsets of T which is all-open-containing.

Let T be a non empty topological space. Note that Topology(T) is open and all-open-containing.

We now state the proposition

(67) Let X be a family of subsets of T. Then there exists an all-open-containing closed for complement operator closed for countable unions family Y of subsets of T such that

(i) $X \subseteq Y$, and

(ii) for every all-open-containing closed for complement operator closed for countable unions family Z of subsets of T such that $X \subseteq Z$ holds $Y \subseteq Z$.

Let us consider T. The functor BorelSets T yields an all-open-containing closed for complement operator closed for countable unions family of subsets of T and is defined by the condition (Def. 11).

(Def. 11) Let G be an all-open-containing closed for complement operator closed for countable unions family of subsets of T. Then BorelSets $T \subseteq G$.

Next we state three propositions:

(68) For every closed family F of subsets of T holds $F \subseteq$ BorelSets T.

(69) For every open family F of subsets of T holds $F \subseteq$ BorelSets T.

(70) BorelSets $T = \sigma(\mathrm{Topology}(T))$.

Let us consider T, A. We say that A is Borel if and only if:

(Def. 12) $A \in$ BorelSets T.

Let us consider T. Note that every subset of T which is F_σ is also Borel.

Let us consider T. Note that every subset of T which is G_δ is also Borel.

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Linearity of Lebesgue Integral of Simple Valued Function[1]

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Summary. In this article the authors prove linearity of the Lebesgue integral of simple valued function.

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The notation and terminology used here are introduced in the following papers: [16], [17], [1], [15], [2], [18], [7], [9], [8], [3], [4], [5], [6], [10], [11], [12], [14], and [13].

One can prove the following propositions:

(1) Let F, G, H be finite sequences of elements of $\overline{\mathbb{R}}$. Suppose that

(i) for every natural number i such that $i \in \operatorname{dom} F$ holds $0_{\overline{\mathbb{R}}} \leq F(i)$,

(ii) for every natural number i such that $i \in \operatorname{dom} G$ holds $0_{\overline{\mathbb{R}}} \leq G(i)$,

(iii) $\operatorname{dom} F = \operatorname{dom} G$, and

(iv) $H = F + G$.

Then $\sum H = \sum F + \sum G$.

(2) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, n be a natural number, f be a partial function from X to $\overline{\mathbb{R}}$, F be a finite sequence of separated subsets of S, and a, x be finite sequences of elements of $\overline{\mathbb{R}}$. Suppose that f is simple function in S and $\operatorname{dom} f \neq \emptyset$ and for every set x such that $x \in \operatorname{dom} f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$ and F and a are representation of f and $\operatorname{dom} x = \operatorname{dom} F$ and for every natural number i such that $i \in \operatorname{dom} x$ holds $x(i) = a(i) \cdot (M \cdot F)(i)$ and $\operatorname{len} F = n$. Then $\int_X f \, \mathrm{d}M = \sum x$.

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(3) Let X be a non empty set, S be a σ-field of subsets of X, f be a partial function from X to $\overline{\mathbb{R}}$, M be a σ-measure on S, F be a finite sequence of separated subsets of S, and a, x be finite sequences of elements of $\overline{\mathbb{R}}$. Suppose that

(i) f is simple function in S,
(ii) $\mathrm{dom}\, f \neq \emptyset$,
(iii) for every set x such that $x \in \mathrm{dom}\, f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$,
(iv) F and a are representation of f,
(v) $\mathrm{dom}\, x = \mathrm{dom}\, F$, and
(vi) for every natural number n such that $n \in \mathrm{dom}\, x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$.

Then $\int\limits_X f \,\mathrm{d}M = \sum x$.

(4) Let X be a non empty set, S be a σ-field of subsets of X, f be a partial function from X to $\overline{\mathbb{R}}$, and M be a σ-measure on S. Suppose f is simple function in S and $\mathrm{dom}\, f \neq \emptyset$ and for every set x such that $x \in \mathrm{dom}\, f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$. Then there exists a finite sequence F of separated subsets of S and there exist finite sequences a, x of elements of $\overline{\mathbb{R}}$ such that

(i) F and a are representation of f,
(ii) $\mathrm{dom}\, x = \mathrm{dom}\, F$,
(iii) for every natural number n such that $n \in \mathrm{dom}\, x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$, and
(iv) $\int\limits_X f \,\mathrm{d}M = \sum x$.

(5) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that

(i) f is simple function in S,
(ii) $\mathrm{dom}\, f \neq \emptyset$,
(iii) for every set x such that $x \in \mathrm{dom}\, f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$,
(iv) g is simple function in S,
(v) $\mathrm{dom}\, g = \mathrm{dom}\, f$, and
(vi) for every set x such that $x \in \mathrm{dom}\, g$ holds $0_{\overline{\mathbb{R}}} \leq g(x)$.

Then

(vii) $f + g$ is simple function in S,
(viii) $\mathrm{dom}(f + g) \neq \emptyset$,
(ix) for every set x such that $x \in \mathrm{dom}(f + g)$ holds $0_{\overline{\mathbb{R}}} \leq (f + g)(x)$, and
(x) $\int\limits_X f + g \,\mathrm{d}M = \int\limits_X f \,\mathrm{d}M + \int\limits_X g \,\mathrm{d}M$.

(6) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f, g be partial functions from X to $\overline{\mathbb{R}}$, and c be an extended real number. Suppose that f is simple function in S and $\mathrm{dom}\, f \neq \emptyset$ and for every set x such that $x \in \mathrm{dom}\, f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$ and $0_{\overline{\mathbb{R}}} \leq c$ and

$c < +\infty$ and $\operatorname{dom} g = \operatorname{dom} f$ and for every set x such that $x \in \operatorname{dom} g$ holds $g(x) = c \cdot f(x)$. Then $\int_X g \, \mathrm{d}M = c \cdot \int_X f \, \mathrm{d}M$.

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The Fashoda Meet Theorem for Continuous Mappings

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The articles [21], [25], [2], [20], [26], [5], [27], [6], [3], [1], [24], [10], [18], [16], [9], [4], [13], [11], [19], [23], [17], [7], [8], [22], [12], [15], and [14] provide the terminology and notation for this paper.

We use the following convention: n is a natural number, p_1, p_2 are points of $\mathcal{E}^n_{\rm T}$, and a, b, c, d are real numbers.

Let us consider a, b, c, d. One can verify that ClosedInsideOfRectangle(a, b, c, d) is convex.

Let us consider a, b, c, d. Observe that Trectangle(a, b, c, d) is convex.

The following propositions are true:

(1) Let e be a positive real number and g be a continuous map from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$. Then there exists a finite sequence h of elements of $\mathbb{R}$ such that

(i) $h(1) = 0$,

(ii) $h(\operatorname{len} h) = 1$,

(iii) $5 \leq \operatorname{len} h$,

(iv) $\operatorname{rng} h \subseteq$ the carrier of $\mathbb{I}$,

(v) h is increasing, and

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(vi) for every natural number i and for every subset Q of $\mathbb{I}$ and for every subset W of $\mathcal{E}^n$ such that $1 \leq i$ and $i < \operatorname{len} h$ and $Q = [h_i, h_{i+1}]$ and $W = g^\circ Q$ holds $\varnothing W < e$.

(2) For every subset P of $\mathcal{E}^n_{\mathrm{T}}$ such that $P \subseteq \mathcal{L}(p_1, p_2)$ and $p_1 \in P$ and $p_2 \in P$ and P is connected holds $P = \mathcal{L}(p_1, p_2)$.

(3) For every path g from p_1 to p_2 such that $\operatorname{rng} g \subseteq \mathcal{L}(p_1, p_2)$ holds $\operatorname{rng} g = \mathcal{L}(p_1, p_2)$.

(4) Let P, Q be non empty subsets of $\mathcal{E}^2_{\mathrm{T}}$, p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$, f be a path from p_1 to p_2, and g be a path from q_1 to q_2. Suppose that

(i) $\operatorname{rng} f = P$,

(ii) $\operatorname{rng} g = Q$,

(iii) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in P$ holds $(p_1)_{\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$,

(iv) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in Q$ holds $(p_1)_{\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$,

(v) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in P$ holds $(q_1)_{\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq (q_2)_{\mathbf{2}}$, and

(vi) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in Q$ holds $(q_1)_{\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq (q_2)_{\mathbf{2}}$.

Then P meets Q.

(5) Let f, g be continuous maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$ and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and $f(O)_{\mathbf{1}} = a$ and $f(I)_{\mathbf{1}} = b$ and $g(O)_{\mathbf{2}} = c$ and $g(I)_{\mathbf{2}} = d$ and for every point r of $\mathbb{I}$ holds $a \leq f(r)_{\mathbf{1}}$ and $f(r)_{\mathbf{1}} \leq b$ and $a \leq g(r)_{\mathbf{1}}$ and $g(r)_{\mathbf{1}} \leq b$ and $c \leq f(r)_{\mathbf{2}}$ and $f(r)_{\mathbf{2}} \leq d$ and $c \leq g(r)_{\mathbf{2}}$ and $g(r)_{\mathbf{2}} \leq d$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(6) Let a_1, b_1, c_1, d_1 be points of $\operatorname{Trectangle}(a, b, c, d)$, h be a path from a_1 to b_1, v be a path from d_1 to c_1, and A_1, B_1, C_1, D_1 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $(A_1)_{\mathbf{1}} = a$ and $(B_1)_{\mathbf{1}} = b$ and $(C_1)_{\mathbf{2}} = c$ and $(D_1)_{\mathbf{2}} = d$ and $a_1 = A_1$ and $b_1 = B_1$ and $c_1 = C_1$ and $d_1 = D_1$. Then there exist points s, t of $\mathbb{I}$ such that $h(s) = v(t)$.

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Tietze Extension Theorem

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Summary. In this paper we formalize the Tietze extension theorem using as a basis the proof presented at the PlanetMath web server[1].

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The articles [24], [26], [1], [2], [23], [11], [4], [21], [27], [5], [28], [7], [6], [17], [16], [22], [18], [20], [19], [25], [9], [10], [13], [14], [8], [12], [3], and [15] provide the notation and terminology for this paper.

We adopt the following rules: r, s denote real numbers, X denotes a set, and f, g, h denote real-yielding functions.

The following propositions are true:

(1) For all real numbers a, b, c such that $|a-b| \leq c$ holds $b-c \leq a$ and $a \leq b+c$.

(2) If $r < s$, then $]-\infty, r]$ misses $[s, +\infty[$.

(3) If $r \leq s$, then $]-\infty, r[$ misses $]s, +\infty[$.

(4) If $f \subseteq g$, then $h-f \subseteq h-g$.

(5) If $f \subseteq g$, then $f-h \subseteq g-h$.

[1] `http://planetmath.org/encyclopedia/ProofOfTietzeExtensionTheorem2.html`

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Let f be a real-yielding function, let r be a real number, and let X be a set. We say that f is absolutely bounded by r in X if and only if:

(Def. 1) For every set x such that $x \in X \cap \operatorname{dom} f$ holds $|f(x)| \leq r$.

Let us mention that there exists a sequence of real numbers which is summable, constant, and convergent.

We now state the proposition

(6) For every empty topological space T_1 and for every topological space T_2 holds every map from T_1 into T_2 is continuous.

Let T_1 be a topological space and let T_2 be a non empty topological space. Observe that there exists a map from T_1 into T_2 which is continuous.

We now state several propositions:

(7) For all summable sequences f, g of real numbers such that for every natural number n holds $f(n) \leq g(n)$ holds $\sum f \leq \sum g$.

(8) For every sequence f of real numbers such that f is absolutely summable holds $|\sum f| \leq \sum |f|$.

(9) Let f be a sequence of real numbers and a, r be positive real numbers. Suppose $r < 1$ and for every natural number n holds $|f(n) - f(n+1)| \leq a \cdot r^n$. Then f is convergent and for every natural number n holds $|\lim f - f(n)| \leq \frac{a \cdot r^n}{1-r}$.

(10) Let f be a sequence of real numbers and a, r be positive real numbers. Suppose $r < 1$ and for every natural number n holds $|f(n) - f(n+1)| \leq a \cdot r^n$. Then $\lim f \geq f(0) - \frac{a}{1-r}$ and $\lim f \leq f(0) + \frac{a}{1-r}$.

(11) Let X, Z be non empty sets and F be a sequence of partial functions from X into $\mathbb{R}$. Suppose Z is common for elements of F. Let a, r be positive real numbers. Suppose $r < 1$ and for every natural number n holds $F(n) - F(n+1)$ is absolutely bounded by $a \cdot r^n$ in Z. Then F is uniform-convergent on Z and for every natural number n holds $\lim_Z F - F(n)$ is absolutely bounded by $\frac{a \cdot r^n}{1-r}$ in Z.

(12) Let X, Z be non empty sets and F be a sequence of partial functions from X into $\mathbb{R}$. Suppose Z is common for elements of F. Let a, r be positive real numbers. Suppose $r < 1$ and for every natural number n holds $F(n) - F(n+1)$ is absolutely bounded by $a \cdot r^n$ in Z. Let z be an element of Z. Then $(\lim_Z F)(z) \geq F(0)(z) - \frac{a}{1-r}$ and $(\lim_Z F)(z) \leq F(0)(z) + \frac{a}{1-r}$.

(13) Let X, Z be non empty sets and F be a sequence of partial functions from X into $\mathbb{R}$. Suppose Z is common for elements of F. Let a, r be positive real numbers and f be a function from Z into $\mathbb{R}$. Suppose $r < 1$ and for every natural number n holds $F(n) - f$ is absolutely bounded by $a \cdot r^n$ in Z. Then F is point-convergent on Z and $\lim_Z F = f$.

Let S, T be topological structures, let A be an empty subset of S, and let f be a map from S into T. Note that $f \restriction A$ is empty.

Let T be a topological space and let A be a closed subset of T. Note that $T \restriction A$ is closed.

The following propositions are true:

(14) Let X, Y be non empty topological spaces, X_1, X_2 be non empty subspaces of X, f_1 be a map from X_1 into Y, and f_2 be a map from X_2 into Y. Suppose X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$. Let x be a point of X. Then

(i) if $x \in$ the carrier of X_1, then $(f_1 \cup f_2)(x) = f_1(x)$, and

(ii) if $x \in$ the carrier of X_2, then $(f_1 \cup f_2)(x) = f_2(x)$.

(15) Let X, Y be non empty topological spaces, X_1, X_2 be non empty subspaces of X, f_1 be a map from X_1 into Y, and f_2 be a map from X_2 into Y. If X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$, then $\mathrm{rng}(f_1 \cup f_2) \subseteq \mathrm{rng}\, f_1 \cup \mathrm{rng}\, f_2$.

(16) Let X, Y be non empty topological spaces, X_1, X_2 be non empty subspaces of X, f_1 be a map from X_1 into Y, and f_2 be a map from X_2 into Y. Suppose X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$. Then for every subset A of X_1 holds $(f_1 \cup f_2)^\circ A = f_1{}^\circ A$ and for every subset A of X_2 holds $(f_1 \cup f_2)^\circ A = f_2{}^\circ A$.

(17) If $f \subseteq g$ and g is absolutely bounded by r in X, then f is absolutely bounded by r in X.

(18) If $X \subseteq \mathrm{dom}\, f$ or $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ and if $f \restriction X = g \restriction X$ and if f is absolutely bounded by r in X, then g is absolutely bounded by r in X.

In the sequel T is a non empty topological space and A is a closed subset of T.

One can prove the following propositions:

(19) Suppose $r > 0$ and T is T_4. Let f be a continuous map from $T \restriction A$ into $\mathbb{R}^{\mathbf{1}}$. Suppose f is absolutely bounded by r in A. Then there exists a continuous map g from T into $\mathbb{R}^{\mathbf{1}}$ such that g is absolutely bounded by $\frac{r}{3}$ in $\mathrm{dom}\, g$ and $f - g$ is absolutely bounded by $\frac{2 \cdot r}{3}$ in A.

(20) Suppose that for all non empty closed subsets A, B of T such that A misses B there exists a continuous map f from T into $\mathbb{R}^{\mathbf{1}}$ such that $f^\circ A = \{0\}$ and $f^\circ B = \{1\}$. Then T is a T_4 space.

(21) Let f be a map from T into $\mathbb{R}^{\mathbf{1}}$ and x be a point of T. Then f is continuous at x if and only if for every real number e such that $e > 0$ there exists a subset H of T such that H is open and $x \in H$ and for every point y of T such that $y \in H$ holds $|f(y) - f(x)| < e$.

(22) Let F be a sequence of partial functions from the carrier of T into $\mathbb{R}$. Suppose that

(i) F is uniform-convergent on the carrier of T, and

(ii) for every natural number i holds $F(i)$ is a continuous map from T into

$\mathbb{R}^{\mathbf{1}}$.

Then $\lim_{\text{the carrier of } T} F$ is a continuous map from T into $\mathbb{R}^{\mathbf{1}}$.

(23) Let T be a non empty topological space, f be a map from T into $\mathbb{R}^{\mathbf{1}}$, and r be a positive real number. Then f is absolutely bounded by r in the carrier of T if and only if f is a map from T into $[-r, r]_{\mathrm{T}}$.

(24) If $f-g$ is absolutely bounded by r in X, then $g-f$ is absolutely bounded by r in X.

(25) Suppose T is T_4. Let given A and f be a map from $T \restriction A$ into $[-1, 1]_{\mathrm{T}}$. Suppose f is continuous. Then there exists a continuous map g from T into $[-1, 1]_{\mathrm{T}}$ such that $g \restriction A = f$.

(26) Suppose that for every non empty closed subset A of T and for every continuous map f from $T \restriction A$ into $[-1, 1]_{\mathrm{T}}$ there exists a continuous map g from T into $[-1, 1]_{\mathrm{T}}$ such that $g \restriction A = f$. Then T is T_4.

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Homeomorphisms of Jordan Curves

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Summary. In this paper we prove that simple closed curves can be homeomorphically framed into a given rectangle. We also show that homeomorphisms preserve the Jordan property.

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The notation and terminology used in this paper are introduced in the following articles: [20], [21], [1], [3], [22], [4], [5], [19], [10], [18], [7], [17], [11], [2], [8], [9], [16], [13], [14], [15], [6], [23], and [12].

In this paper p_1, p_2 are points of $\mathcal{E}^2_{\mathrm{T}}$, C is a simple closed curve, and P is a subset of $\mathcal{E}^2_{\mathrm{T}}$.

Let n be a natural number, let A be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and let a, b be points of $\mathcal{E}^n_{\mathrm{T}}$. We say that a and b realize maximal distance in A if and only if:

(Def. 1) $a \in A$ and $b \in A$ and for all points x, y of $\mathcal{E}^n_{\mathrm{T}}$ such that $x \in A$ and $y \in A$ holds $\rho(a, b) \geq \rho(x, y)$.

Next we state the proposition

(1) There exist p_1, p_2 such that p_1 and p_2 realize maximal distance in C.

Let M be a non empty metric structure and let f be a map from M_{top} into M_{top}. We say that f is isometric if and only if:

(Def. 2) There exists an isometric map g from M into M such that $g = f$.

Let M be a non empty metric structure. Note that there exists a map from M_{top} into M_{top} which is isometric.

Let M be a non empty metric space. Observe that every map from M_{top} into M_{top} which is isometric is also continuous.

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Let M be a non empty metric space. Note that every map from M_{top} into M_{top} which is isometric is also homeomorphism.

Let a be a real number. The functor Rotate a yields a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 3) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\mathrm{Rotate}\, a)(p) = [\Re(\mathbf{p_1} + \mathbf{p_2} \cdot i \circlearrowleft a), \Im(\mathbf{p_1} + \mathbf{p_2} \cdot i \circlearrowleft a)]$, where $a = [r_1, 0]$ and $r_1 = -1$.

The following propositions are true:

(2) Let a be a real number. Suppose $0 \leq a$ and $a < 2 \cdot \pi$. Let f be a map from $(\mathcal{E}^2)_{\mathrm{top}}$ into $(\mathcal{E}^2)_{\mathrm{top}}$. If $f = \mathrm{Rotate}\, a$, then f is isometric, where $a = [r_1, 0]$ and $r_1 = -1$.

(3) Let A, B, D be real numbers. Suppose p_1 and p_2 realize maximal distance in P. Then $(\mathrm{AffineMap}(A, B, A, D))(p_1)$ and $(\mathrm{AffineMap}(A, B, A, D))(p_2)$ realize maximal distance in $(\mathrm{AffineMap}(A, B, A, D))^\circ P$.

(4) Let A be a real number. Suppose $0 \leq A$ and $A < 2 \cdot \pi$ and p_1 and p_2 realize maximal distance in P. Then $(\mathrm{Rotate}\, A)(p_1)$ and $(\mathrm{Rotate}\, A)(p_2)$ realize maximal distance in $(\mathrm{Rotate}\, A)^\circ P$.

(5) For every complex number z and for every real number r holds $z \circlearrowleft -r = z \circlearrowleft 2 \cdot \pi - r$.

(6) For every real number r holds $\mathrm{Rotate}(-r) = \mathrm{Rotate}(2 \cdot \pi - r)$.

(7) There exists a homeomorphism f of $\mathcal{E}^2_{\mathrm{T}}$ such that $[-1, 0]$ and $[1, 0]$ realize maximal distance in $f^\circ C$.

Let T_1, T_2 be topological structures and let f be a map from T_1 into T_2. We say that f is closed if and only if:

(Def. 4) For every subset A of T_1 such that A is closed holds $f^\circ A$ is closed.

One can prove the following propositions:

(8) Let X, Y be non empty topological spaces and f be a continuous map from X into Y. Suppose f is one-to-one and onto. Then f is a homeomorphism if and only if f is closed.

(9) For every set X and for every subset A of X holds $A^{\mathrm{c}} = \emptyset$ iff $A = X$.

(10) Let T_1, T_2 be non empty topological spaces and f be a map from T_1 into T_2. Suppose f is a homeomorphism. Let A be a subset of T_1. If A is connected, then $f^\circ A$ is connected.

(11) Let T_1, T_2 be non empty topological spaces and f be a map from T_1 into T_2. Suppose f is a homeomorphism. Let A be a subset of T_1. If A is a component of T_1, then $f^\circ A$ is a component of T_2.

(12) Let T_1, T_2 be non empty topological spaces, f be a map from T_1 into T_2, and A be a subset of T_1. Then $f \restriction A$ is a map from $T_1 \restriction A$ into $T_2 \restriction f^\circ A$.

(13) Let T_1, T_2 be non empty topological spaces and f be a map from T_1 into

T_2. Suppose f is continuous. Let A be a subset of T_1 and g be a map from $T_1 \restriction A$ into $T_2 \restriction f^\circ A$. If $g = f \restriction A$, then g is continuous.

(14) Let T_1, T_2 be non empty topological spaces and f be a map from T_1 into T_2. Suppose f is a homeomorphism. Let A be a subset of T_1 and g be a map from $T_1 \restriction A$ into $T_2 \restriction f^\circ A$. If $g = f \restriction A$, then g is a homeomorphism.

(15) Let T_1, T_2 be non empty topological spaces and f be a map from T_1 into T_2. Suppose f is a homeomorphism. Let A, B be subsets of T_1. If A is a component of B, then $f^\circ A$ is a component of $f^\circ B$.

(16) For every subset S of $\mathcal{E}^2_{\mathrm{T}}$ and for every homeomorphism f of $\mathcal{E}^2_{\mathrm{T}}$ such that S is Jordan holds $f^\circ S$ is Jordan.

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Jordan Curve Theorem

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Summary. This paper formalizes the Jordan curve theorem following [42] and [17].

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The articles [44], [47], [9], [1], [45], [48], [5], [8], [6], [4], [7], [10], [43], [21], [2], [40], [39], [49], [46], [12], [11], [37], [38], [33], [22], [3], [13], [18], [15], [16], [14], [31], [32], [35], [20], [34], [30], [25], [26], [19], [29], [24], [23], [36], [41], [28], and [27] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: a, b, c, d, r, s denote real numbers, n denotes a natural number, p, p_1, p_2 denote points of $\mathcal{E}^2_{\rm T}$, x, y denote points of $\mathcal{E}^n_{\rm T}$, C denotes a simple closed curve, A, B, P denote subsets of $\mathcal{E}^2_{\rm T}$, U, V denote subsets of $(\mathcal{E}^2_{\rm T})\restriction C^{\rm c}$, and D denotes a compact middle-intersecting subset of $\mathcal{E}^2_{\rm T}$.

Let M be a symmetric triangle Reflexive metric structure and let x, y be points of M. One can verify that $\rho(x,y)$ is non negative.

Let n be a natural number and let x, y be points of $\mathcal{E}^n_{\rm T}$. Note that $\rho(x,y)$ is non negative.

Let n be a natural number and let x, y be points of $\mathcal{E}^n_{\rm T}$. Observe that $|x-y|$ is non negative.

We now state several propositions:

(1) For all points p_1, p_2 of $\mathcal{E}^n_{\rm T}$ such that $p_1 \neq p_2$ holds $\frac{1}{2} \cdot (p_1 + p_2) \neq p_1$.

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(2) If $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$, then $(p_1)_{\mathbf{2}} < (\frac{1}{2} \cdot (p_1 + p_2))_{\mathbf{2}}$.

(3) If $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$, then $(\frac{1}{2} \cdot (p_1 + p_2))_{\mathbf{2}} < (p_2)_{\mathbf{2}}$.

(4) For every vertical subset A of $\mathcal{E}^2_{\mathrm{T}}$ holds $A \cap B$ is vertical.

(5) For every horizontal subset A of $\mathcal{E}^2_{\mathrm{T}}$ holds $A \cap B$ is horizontal.

(6) If $p \in \mathcal{L}(p_1, p_2)$ and $\mathcal{L}(p_1, p_2)$ is vertical, then $\mathcal{L}(p, p_2)$ is vertical.

(7) If $p \in \mathcal{L}(p_1, p_2)$ and $\mathcal{L}(p_1, p_2)$ is horizontal, then $\mathcal{L}(p, p_2)$ is horizontal.

Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. One can verify the following observations:

* $\mathcal{L}(\text{SW-corner}(P), \text{SE-corner}(P))$ is horizontal,
* $\mathcal{L}(\text{NW-corner}(P), \text{SW-corner}(P))$ is vertical, and
* $\mathcal{L}(\text{NE-corner}(P), \text{SE-corner}(P))$ is vertical.

Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. One can check the following observations:

* $\mathcal{L}(\text{SE-corner}(P), \text{SW-corner}(P))$ is horizontal,
* $\mathcal{L}(\text{SW-corner}(P), \text{NW-corner}(P))$ is vertical, and
* $\mathcal{L}(\text{SE-corner}(P), \text{NE-corner}(P))$ is vertical.

Let us note that every subset of $\mathcal{E}^2_{\mathrm{T}}$ which is vertical, non empty, and compact is also middle-intersecting.

The following propositions are true:

(8) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $X \subseteq Y$ but $\mathrm{W}_{\mathrm{min}}(Y) \in X$ or $\mathrm{W}_{\mathrm{max}}(Y) \in X$ holds W-bound(X) = W-bound(Y).

(9) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $X \subseteq Y$ but $\mathrm{E}_{\mathrm{min}}(Y) \in X$ or $\mathrm{E}_{\mathrm{max}}(Y) \in X$ holds E-bound(X) = E-bound(Y).

(10) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $X \subseteq Y$ but $\mathrm{N}_{\mathrm{min}}(Y) \in X$ or $\mathrm{N}_{\mathrm{max}}(Y) \in X$ holds N-bound(X) = N-bound(Y).

(11) For all non empty compact subsets X, Y of $\mathcal{E}^2_{\mathrm{T}}$ such that $X \subseteq Y$ but $\mathrm{S}_{\mathrm{min}}(Y) \in X$ or $\mathrm{S}_{\mathrm{max}}(Y) \in X$ holds S-bound(X) = S-bound(Y).

(12) W-bound(C) = W-bound(NorthArc(C)).

(13) E-bound(C) = E-bound(NorthArc(C)).

(14) W-bound(C) = W-bound(SouthArc(C)).

(15) E-bound(C) = E-bound(SouthArc(C)).

(16) If $(p_1)_{\mathbf{1}} \leq r$ and $r \leq (p_2)_{\mathbf{1}}$, then $\mathcal{L}(p_1, p_2)$ meets VerticalLine(r).

(17) If $(p_1)_{\mathbf{2}} \leq r$ and $r \leq (p_2)_{\mathbf{2}}$, then $\mathcal{L}(p_1, p_2)$ meets HorizontalLine(r).

Let us consider n. One can check that every subset of $\mathcal{E}^n_{\mathrm{T}}$ which is empty is also Bounded and every subset of $\mathcal{E}^n_{\mathrm{T}}$ which is non Bounded is also non empty.

Let n be a non empty natural number. Note that there exists a subset of $\mathcal{E}^n_{\mathrm{T}}$ which is open, closed, non Bounded, and convex.

Next we state several propositions:

(18) For every compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds NorthHalfline UMP $C \setminus \{\mathrm{UMP}\, C\}$ misses C.

(19) For every compact subset C of $\mathcal{E}^2_{\rm T}$ holds SouthHalfline LMP $C \setminus \{$LMP $C\}$ misses C.

(20) For every compact subset C of $\mathcal{E}^2_{\rm T}$ holds NorthHalfline UMP $C \setminus \{$UMP $C\} \subseteq$ UBD C.

(21) For every compact subset C of $\mathcal{E}^2_{\rm T}$ holds SouthHalfline LMP $C \setminus \{$LMP $C\} \subseteq$ UBD C.

(22) If A is an inside component of B, then UBD B misses A.

(23) If A is an outside component of B, then BDD B misses A.

One can prove the following propositions:

(24) For every positive real number r and for every point a of $\mathcal{E}^n_{\rm T}$ holds $a \in \mathrm{Ball}(a, r)$.

(25) For every non negative real number r holds every point p of $\mathcal{E}^n_{\rm T}$ is a point of $\mathrm{Tdisk}(p, r)$.

Let r be a positive real number, let n be a non empty natural number, and let p, q be points of $\mathcal{E}^n_{\rm T}$. Observe that $\overline{\mathrm{Ball}}(p, r) \setminus \{q\}$ is non empty.

We now state several propositions:

(26) If $r \leq s$, then $\mathrm{Ball}(x, r) \subseteq \mathrm{Ball}(x, s)$.

(27) $\overline{\mathrm{Ball}}(x, r) \setminus \mathrm{Ball}(x, r) = \mathrm{Sphere}(x, r)$.

(28) If $y \in \mathrm{Sphere}(x, r)$, then $\mathcal{L}(x, y) \setminus \{x, y\} \subseteq \mathrm{Ball}(x, r)$.

(29) If $r < s$, then $\overline{\mathrm{Ball}}(x, r) \subseteq \mathrm{Ball}(x, s)$.

(30) If $r < s$, then $\mathrm{Sphere}(x, r) \subseteq \mathrm{Ball}(x, s)$.

(31) For every non zero real number r holds $\overline{\mathrm{Ball}(x, r)} = \overline{\mathrm{Ball}}(x, r)$.

(32) For every non zero real number r holds $\mathrm{Fr}\,\mathrm{Ball}(x, r) = \mathrm{Sphere}(x, r)$.

Let n be a non empty natural number. Note that every subset of $\mathcal{E}^n_{\rm T}$ which is Bounded is also proper.

Let us consider n. Note that there exists a subset of $\mathcal{E}^n_{\rm T}$ which is non empty, closed, convex, and Bounded and there exists a subset of $\mathcal{E}^n_{\rm T}$ which is non empty, open, convex, and Bounded.

Let n be a natural number and let A be a Bounded subset of $\mathcal{E}^n_{\rm T}$. Observe that $\overline{A}$ is Bounded.

Let n be a natural number and let A be a Bounded subset of $\mathcal{E}^n_{\rm T}$. One can check that Fr A is Bounded.

The following propositions are true:

(33) Let A be a closed subset of $\mathcal{E}^n_{\rm T}$ and p be a point of $\mathcal{E}^n_{\rm T}$. If $p \notin A$, then there exists a positive real number r such that $\mathrm{Ball}(p, r)$ misses A.

(34) For every Bounded subset A of $\mathcal{E}^n_{\rm T}$ and for every point a of $\mathcal{E}^n_{\rm T}$ there exists a positive real number r such that $A \subseteq \mathrm{Ball}(a, r)$.

(35) For all topological structures S, T and for every map f from S into T such that f is a homeomorphism holds f is onto.

(36) Let T be a topological space, S be a subspace of T, A be a subset of T, and B be a subset of S. If $A = B$, then $T\restriction A = S\restriction B$.

Let T be a non empty T_2 topological space. Note that every non empty subspace of T is T_2.

Let us consider p, r. Observe that $\mathrm{Tdisk}(p, r)$ is closed.

Let us consider p, r. Observe that $\mathrm{Tdisk}(p, r)$ is compact.

2. Paths

Next we state a number of propositions:

(37) Let T be a non empty topological space, a, b be points of T, and f be a path from a to b. If a, b are connected, then $\mathrm{rng}\, f$ is connected.

(38) Let X be a non empty topological space, Y be a non empty subspace of X, x_1, x_2 be points of X, y_1, y_2 be points of Y, and f be a path from x_1 to x_2. Suppose $x_1 = y_1$ and $x_2 = y_2$ and x_1, x_2 are connected and $\mathrm{rng}\, f \subseteq$ the carrier of Y. Then y_1, y_2 are connected and f is a path from y_1 to y_2.

(39) Let X be an arcwise connected non empty topological space, Y be a non empty subspace of X, x_1, x_2 be points of X, y_1, y_2 be points of Y, and f be a path from x_1 to x_2. Suppose $x_1 = y_1$ and $x_2 = y_2$ and $\mathrm{rng}\, f \subseteq$ the carrier of Y. Then y_1, y_2 are connected and f is a path from y_1 to y_2.

(40) Let T be a non empty topological space, a, b be points of T, and f be a path from a to b. If a, b are connected, then $\mathrm{rng}\, f = \mathrm{rng}(-f)$.

(41) Let T be an arcwise connected non empty topological space, a, b be points of T, and f be a path from a to b. Then $\mathrm{rng}\, f = \mathrm{rng}(-f)$.

(42) Let T be a non empty topological space, a, b, c be points of T, f be a path from a to b, and g be a path from b to c. If a, b are connected and b, c are connected, then $\mathrm{rng}\, f \subseteq \mathrm{rng}(f + g)$.

(43) Let T be an arcwise connected non empty topological space, a, b, c be points of T, f be a path from a to b, and g be a path from b to c. Then $\mathrm{rng}\, f \subseteq \mathrm{rng}(f + g)$.

(44) Let T be a non empty topological space, a, b, c be points of T, f be a path from b to c, and g be a path from a to b. If a, b are connected and b, c are connected, then $\mathrm{rng}\, f \subseteq \mathrm{rng}(g + f)$.

(45) Let T be an arcwise connected non empty topological space, a, b, c be points of T, f be a path from b to c, and g be a path from a to b. Then $\mathrm{rng}\, f \subseteq \mathrm{rng}(g + f)$.

(46) Let T be a non empty topological space, a, b, c be points of T, f be a path from a to b, and g be a path from b to c. If a, b are connected and b, c are connected, then $\mathrm{rng}(f + g) = \mathrm{rng}\, f \cup \mathrm{rng}\, g$.

(47) Let T be an arcwise connected non empty topological space, a, b, c be points of T, f be a path from a to b, and g be a path from b to c. Then $\operatorname{rng}(f+g) = \operatorname{rng} f \cup \operatorname{rng} g$.

(48) Let T be a non empty topological space, a, b, c, d be points of T, f be a path from a to b, g be a path from b to c, and h be a path from c to d. Suppose a, b are connected and b, c are connected and c, d are connected. Then $\operatorname{rng}(f+g+h) = \operatorname{rng} f \cup \operatorname{rng} g \cup \operatorname{rng} h$.

(49) Let T be an arcwise connected non empty topological space, a, b, c, d be points of T, f be a path from a to b, g be a path from b to c, and h be a path from c to d. Then $\operatorname{rng}(f+g+h) = \operatorname{rng} f \cup \operatorname{rng} g \cup \operatorname{rng} h$.

(50) For every non empty topological space T and for every point a of T holds $\mathbb{I} \longmapsto a$ is a path from a to a.

(51) Let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$ and P be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2. Then there exists a path F from p_1 to p_2 and there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}}) \restriction P$ such that $\operatorname{rng} f = P$ and $F = f$.

(52) Let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Then there exists a path F from p_1 to p_2 and there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}}) \restriction \mathcal{L}(p_1, p_2)$ such that $\operatorname{rng} f = \mathcal{L}(p_1, p_2)$ and $F = f$.

(53) Let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_2 \in P$ and $q_1 \neq p_1$ and $q_1 \neq p_2$ and $q_2 \neq p_1$ and $q_2 \neq p_2$. Then there exists a path f from q_1 to q_2 such that $\operatorname{rng} f \subseteq P$ and $\operatorname{rng} f$ misses $\{p_1, p_2\}$.

3. Rectangles

Next we state three propositions:

(54) If $a \leq b$ and $c \leq d$, then $\mathrm{Rectangle}(a, b, c, d) \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$.

(55) $\mathrm{InsideOfRectangle}(a, b, c, d) \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$.

(56) $\mathrm{ClosedInsideOfRectangle}(a, b, c, d) = (\mathrm{OutsideOfRectangle}(a, b, c, d))^{\mathrm{c}}$.

Let a, b, c, d be real numbers. Note that $\mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ is closed.

One can prove the following propositions:

(57) $\mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ misses $\mathrm{OutsideOfRectangle}(a, b, c, d)$.

(58) $\mathrm{ClosedInsideOfRectangle}(a, b, c, d) \cap \mathrm{InsideOfRectangle}(a, b, c, d) = \mathrm{InsideOfRectangle}(a, b, c, d)$.

(59) If $a < b$ and $c < d$, then $\operatorname{Int} \mathrm{ClosedInsideOfRectangle}(a, b, c, d) = \mathrm{InsideOfRectangle}(a, b, c, d)$.

(60) If $a \leq b$ and $c \leq d$, then $\mathrm{ClosedInsideOfRectangle}(a, b, c, d) \setminus \mathrm{InsideOfRectangle}(a, b, c, d) = \mathrm{Rectangle}(a, b, c, d)$.

(61) If $a < b$ and $c < d$, then $\operatorname{Fr} \mathrm{ClosedInsideOfRectangle}(a, b, c, d) = \mathrm{Rectangle}(a, b, c, d)$.

(62) If $a \leq b$ and $c \leq d$, then $\text{W-bound}(\mathrm{ClosedInsideOfRectangle}(a, b, c, d)) = a$.

(63) If $a \leq b$ and $c \leq d$, then $\text{S-bound}(\mathrm{ClosedInsideOfRectangle}(a, b, c, d)) = c$.

(64) If $a \leq b$ and $c \leq d$, then $\text{E-bound}(\mathrm{ClosedInsideOfRectangle}(a, b, c, d)) = b$.

(65) If $a \leq b$ and $c \leq d$, then $\text{N-bound}(\mathrm{ClosedInsideOfRectangle}(a, b, c, d)) = d$.

(66) If $a < b$ and $c < d$ and $p_1 \in \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $p_2 \notin \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and P is an arc from p_1 to p_2, then $\mathrm{Segment}(P, p_1, p_2, p_1, \mathrm{FPoint}(P, p_1, p_2, \mathrm{Rectangle}(a, b, c, d))) \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$.

4. Some Useful Functions

Let S, T be non empty topological spaces and let x be a point of $[\!: S, T :\!]$. Then $x_{\mathbf{1}}$ is an element of S, and $x_{\mathbf{2}}$ is an element of T.

Let o be a point of $\mathcal{E}^2_{\mathrm{T}}$. The functor $(\square_2)_1 - o_1$ yielding a real map of $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ is defined as follows:

(Def. 1) For every point x of $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ holds $((\square_2)_1 - o_1)(x) = (x_{\mathbf{2}})_{\mathbf{1}} - o_{\mathbf{1}}$.

The functor $(\square_2)_2 - o_2$ yields a real map of $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ and is defined as follows:

(Def. 2) For every point x of $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ holds $((\square_2)_2 - o_2)(x) = (x_{\mathbf{2}})_{\mathbf{2}} - o_{\mathbf{2}}$.

The real map $(\square_1)_1 - (\square_2)_1$ of $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ is defined as follows:

(Def. 3) For every point x of $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ holds $((\square_1)_1 - (\square_2)_1)(x) = (x_{\mathbf{1}})_{\mathbf{1}} - (x_{\mathbf{2}})_{\mathbf{1}}$.

The real map $(\square_1)_2 - (\square_2)_2$ of $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ is defined as follows:

(Def. 4) For every point x of $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ holds $((\square_1)_2 - (\square_2)_2)(x) = (x_{\mathbf{1}})_{\mathbf{2}} - (x_{\mathbf{2}})_{\mathbf{2}}$.

The real map $(\square_2)_1$ of $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ is defined as follows:

(Def. 5) For every point x of $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ holds $(\square_2)_1(x) = (x_{\mathbf{2}})_{\mathbf{1}}$.

The real map $(\square_2)_2$ of $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ is defined by:

(Def. 6) For every point x of $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ holds $(\square_2)_2(x) = (x_{\mathbf{2}})_{\mathbf{2}}$.

One can prove the following propositions:

(67) For every point o of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\square_2)_1 - o_1$ is a continuous map from $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ into $\mathbb{R}^{\mathbf{1}}$.

(68) For every point o of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\square_2)_2 - o_2$ is a continuous map from $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ into $\mathbb{R}^{\mathbf{1}}$.

(69) $(\square_1)_1 - (\square_2)_1$ is a continuous map from $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$ into $\mathbb{R}^{\mathbf{1}}$.

(70) $(\square_1)_2 - (\square_2)_2$ is a continuous map from $[:\mathcal{E}^2_{\rm T}, \mathcal{E}^2_{\rm T}:]$ into $\mathbb{R}^{\mathbf{1}}$.

(71) $(\square_2)_1$ is a continuous map from $[:\mathcal{E}^2_{\rm T}, \mathcal{E}^2_{\rm T}:]$ into $\mathbb{R}^{\mathbf{1}}$.

(72) $(\square_2)_2$ is a continuous map from $[:\mathcal{E}^2_{\rm T}, \mathcal{E}^2_{\rm T}:]$ into $\mathbb{R}^{\mathbf{1}}$.

Let o be a point of $\mathcal{E}^2_{\rm T}$. One can check that $(\square_2)_1 - o_1$ is continuous and $(\square_2)_2 - o_2$ is continuous.

One can check the following observations:

* $(\square_1)_1 - (\square_2)_1$ is continuous,
* $(\square_1)_2 - (\square_2)_2$ is continuous,
* $(\square_2)_1$ is continuous, and
* $(\square_2)_2$ is continuous.

Let n be a non empty natural number, let o, p be points of $\mathcal{E}^n_{\rm T}$, and let r be a positive real number. Let us assume that p is a point of $\mathrm{Tdisk}(o, r)$. The functor $\mathrm{DiskProj}(o, r, p)$ yielding a map from $(\mathcal{E}^n_{\rm T}) \restriction (\overline{\mathrm{Ball}}(o, r) \setminus \{p\})$ into $\mathrm{Tcircle}(o, r)$ is defined by:

(Def. 7) For every point x of $(\mathcal{E}^n_{\rm T}) \restriction (\overline{\mathrm{Ball}}(o, r) \setminus \{p\})$ there exists a point y of $\mathcal{E}^n_{\rm T}$ such that $x = y$ and $(\mathrm{DiskProj}(o, r, p))(x) = \mathrm{HC}(p, y, o, r)$.

The following propositions are true:

(73) Let o, p be points of $\mathcal{E}^2_{\rm T}$ and r be a positive real number. If p is a point of $\mathrm{Tdisk}(o, r)$, then $\mathrm{DiskProj}(o, r, p)$ is continuous.

(74) Let n be a non empty natural number, o, p be points of $\mathcal{E}^n_{\rm T}$, and r be a positive real number. If $p \in \mathrm{Ball}(o, r)$, then $\mathrm{DiskProj}(o, r, p) \restriction \mathrm{Sphere}(o, r) = \mathrm{id}_{\mathrm{Sphere}(o, r)}$.

Let n be a non empty natural number, let o, p be points of $\mathcal{E}^n_{\rm T}$, and let r be a positive real number. Let us assume that $p \in \mathrm{Ball}(o, r)$. The functor $\mathrm{RotateCircle}(o, r, p)$ yields a map from $\mathrm{Tcircle}(o, r)$ into $\mathrm{Tcircle}(o, r)$ and is defined by:

(Def. 8) For every point x of $\mathrm{Tcircle}(o, r)$ there exists a point y of $\mathcal{E}^n_{\rm T}$ such that $x = y$ and $(\mathrm{RotateCircle}(o, r, p))(x) = \mathrm{HC}(y, p, o, r)$.

One can prove the following propositions:

(75) For all points o, p of $\mathcal{E}^2_{\rm T}$ and for every positive real number r such that $p \in \mathrm{Ball}(o, r)$ holds $\mathrm{RotateCircle}(o, r, p)$ is continuous.

(76) Let n be a non empty natural number, o, p be points of $\mathcal{E}^n_{\rm T}$, and r be a positive real number. If $p \in \mathrm{Ball}(o, r)$, then $\mathrm{RotateCircle}(o, r, p)$ has no fixpoint.

5. Jordan Curve Theorem

The following propositions are true:

(77) If $U = P$ and U is a component of $(\mathcal{E}^2_{\mathrm{T}})\restriction C^{\mathrm{c}}$ and V is a component of $(\mathcal{E}^2_{\mathrm{T}})\restriction C^{\mathrm{c}}$ and $U \neq V$, then $\overline{P}$ misses V.

(78) If U is a component of $(\mathcal{E}^2_{\mathrm{T}})\restriction C^{\mathrm{c}}$, then $(\mathcal{E}^2_{\mathrm{T}})\restriction C^{\mathrm{c}}\restriction U$ is arcwise connected.

(79) If $U = P$ and U is a component of $(\mathcal{E}^2_{\mathrm{T}})\restriction C^{\mathrm{c}}$, then $C = \mathrm{Fr}\,P$.

One can prove the following propositions:

(80) For every homeomorphism h of $\mathcal{E}^2_{\mathrm{T}}$ holds $h^\circ C$ satisfies conditions of simple closed curve.

(81) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then $P \subseteq \mathrm{ClosedInsideOfRectangle}(-1, 1, -3, 3)$.

(82) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then P misses $\mathcal{L}([-1, 3], [1, 3])$.

(83) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then P misses $\mathcal{L}([-1, -3], [1, -3])$.

(84) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then $P \cap \mathrm{Rectangle}(-1, 1, -3, 3) = \{[-1, 0], [1, 0]\}$.

(85) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then $\text{W-bound}(P) = -1$.

(86) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then $\text{E-bound}(P) = 1$.

(87) For every compact subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $[-1, 0]$ and $[1, 0]$ realize maximal distance in P holds $\mathrm{W}_{\mathrm{most}}(P) = \{[-1, 0]\}$.

(88) For every compact subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $[-1, 0]$ and $[1, 0]$ realize maximal distance in P holds $\mathrm{E}_{\mathrm{most}}(P) = \{[1, 0]\}$.

(89) Let P be a compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $[-1, 0]$ and $[1, 0]$ realize maximal distance in P. Then $\mathrm{W}_{\mathrm{min}}(P) = [-1, 0]$ and $\mathrm{W}_{\mathrm{max}}(P) = [-1, 0]$.

(90) Let P be a compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $[-1, 0]$ and $[1, 0]$ realize maximal distance in P. Then $\mathrm{E}_{\mathrm{min}}(P) = [1, 0]$ and $\mathrm{E}_{\mathrm{max}}(P) = [1, 0]$.

(91) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then $\mathcal{L}([0, 3], \mathrm{UMP}\,P)$ is vertical.

(92) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P, then $\mathcal{L}(\mathrm{LMP}\,P, [0, -3])$ is vertical.

(93) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P and $p \in P$, then $p_{\mathbf{2}} < 3$.

(94) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in P and $p \in P$, then $-3 < p_{\mathbf{2}}$.

(95) If $[-1, 0]$ and $[1, 0]$ realize maximal distance in D and $p \in \mathcal{L}([0, 3], \mathrm{UMP}\,D)$, then $(\mathrm{UMP}\,D)_{\mathbf{2}} \leq p_{\mathbf{2}}$.

(96) If $[-1,0]$ and $[1,0]$ realize maximal distance in D and $p \in \mathcal{L}(\mathrm{LMP}\, D, [0, -3])$, then $p_{\mathbf{2}} \leq (\mathrm{LMP}\, D)_{\mathbf{2}}$.

(97) If $[-1,0]$ and $[1,0]$ realize maximal distance in D, then $\mathcal{L}([0, 3], \mathrm{UMP}\, D) \subseteq \mathrm{NorthHalfline}\, \mathrm{UMP}\, D$.

(98) If $[-1,0]$ and $[1,0]$ realize maximal distance in D, then $\mathcal{L}(\mathrm{LMP}\, D, [0, -3]) \subseteq \mathrm{SouthHalfline}\, \mathrm{LMP}\, D$.

(99) If $[-1,0]$ and $[1,0]$ realize maximal distance in C and P is an inside component of C, then $\mathcal{L}([0,3], \mathrm{UMP}\, C)$ misses P.

(100) If $[-1,0]$ and $[1,0]$ realize maximal distance in C and P is an inside component of C, then $\mathcal{L}(\mathrm{LMP}\, C, [0,-3])$ misses P.

(101) If $[-1,0]$ and $[1,0]$ realize maximal distance in D, then $\mathcal{L}([0,3], \mathrm{UMP}\, D) \cap D = \{\mathrm{UMP}\, D\}$.

(102) If $[-1,0]$ and $[1,0]$ realize maximal distance in D, then $\mathcal{L}([0, -3], \mathrm{LMP}\, D) \cap D = \{\mathrm{LMP}\, D\}$.

(103) Suppose P is compact and $[-1,0]$ and $[1,0]$ realize maximal distance in P and A is an inside component of P. Then $A \subseteq \mathrm{ClosedInsideOfRectangle}(-1, 1, -3, 3)$.

(104) If $[-1,0]$ and $[1,0]$ realize maximal distance in C, then $\mathcal{L}([0,3],[0,-3])$ meets C.

(105) Suppose $[-1,0]$ and $[1,0]$ realize maximal distance in C. Let J_1, J_2 be compact middle-intersecting subsets of T_2. Suppose that J_1 is an arc from $[-1,0]$ to $[1,0]$ and J_2 is an arc from $[-1,0]$ to $[1,0]$ and $C = J_1 \cup J_2$ and $J_1 \cap J_2 = \{[-1,0],[1,0]\}$ and $\mathrm{UMP}\, C \in J_1$ and $\mathrm{LMP}\, C \in J_2$ and $\mathrm{W\text{-}bound}(C) = \mathrm{W\text{-}bound}(J_1)$ and $\mathrm{E\text{-}bound}(C) = \mathrm{E\text{-}bound}(J_1)$. Let U_1 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $U_1 = \mathrm{Component}(\mathrm{Down}(\frac{1}{2} \cdot (\mathrm{UMP}(\mathcal{L}(\mathrm{LMP}\, J_1, [0, -3]) \cap J_2) + \mathrm{LMP}\, J_1), C^{\mathrm{c}}))$. Then U_1 is an inside component of C and for every subset V of T_2 such that V is an inside component of C holds $V = U_1$, where $T_2 = \mathcal{E}^2_{\mathrm{T}}$.

(106) Suppose $[-1,0]$ and $[1,0]$ realize maximal distance in C. Let J_1, J_2 be compact middle-intersecting subsets of T_2. Suppose that J_1 is an arc from $[-1,0]$ to $[1,0]$ and J_2 is an arc from $[-1,0]$ to $[1,0]$ and $C = J_1 \cup J_2$ and $J_1 \cap J_2 = \{[-1,0],[1,0]\}$ and $\mathrm{UMP}\, C \in J_1$ and $\mathrm{LMP}\, C \in J_2$ and $\mathrm{W\text{-}bound}(C) = \mathrm{W\text{-}bound}(J_1)$ and $\mathrm{E\text{-}bound}(C) = \mathrm{E\text{-}bound}(J_1)$. Then $\mathrm{BDD}\, C = \mathrm{Component}(\mathrm{Down}(\frac{1}{2} \cdot (\mathrm{UMP}(\mathcal{L}(\mathrm{LMP}\, J_1, [0, -3]) \cap J_2) + \mathrm{LMP}\, J_1), C^{\mathrm{c}}))$, where $T_2 = \mathcal{E}^2_{\mathrm{T}}$.

(107) Let C be a simple closed curve. Then there exist subsets A_1, A_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that

(i) $C^{\mathrm{c}} = A_1 \cup A_2$,

(ii) A_1 misses A_2,

(iii) $\overline{A_1} \setminus A_1 = \overline{A_2} \setminus A_2$, and

(iv) for all subsets C_1, C_2 of $(\mathcal{E}^2_{\mathrm{T}})\restriction C^{\mathrm{c}}$ such that $C_1 = A_1$ and $C_2 = A_2$ holds C_1 is a component of $(\mathcal{E}^2_{\mathrm{T}})\restriction C^{\mathrm{c}}$ and C_2 is a component of $(\mathcal{E}^2_{\mathrm{T}})\restriction C^{\mathrm{c}}$.

(108) Every simple closed curve is Jordan.

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The Inner Product and Conjugate of Matrix of Complex Numbers

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Summary. Concepts of the inner product and conjugate of matrix of complex numbers are defined here. Operations such as addition, subtraction, scalar multiplication and inner product are introduced using correspondent definitions of the conjugate of a matrix of a complex field. Many equations for such operations consist like a case of the conjugate of matrix of a field and some operations on the set of sum of complex numbers are introduced.

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The papers [20], [24], [18], [25], [7], [8], [9], [3], [19], [2], [4], [11], [5], [10], [6], [17], [1], [13], [14], [23], [12], [15], [16], [22], and [21] provide the notation and terminology for this paper.

We follow the rules: i, j denote natural numbers, a denotes an element of $\mathbb{C}$, and R_1, R_2 denote elements of $\mathbb{C}^i$.

Let M be a matrix over $\mathbb{C}$. The functor $\overline{M}$ yields a matrix over $\mathbb{C}$ and is defined by:

(Def. 1) $\operatorname{len} \overline{M} = \operatorname{len} M$ and $\operatorname{width} \overline{M} = \operatorname{width} M$ and for all natural numbers i, j such that $\langle i, j \rangle \in$ the indices of M holds $\overline{M} \circ (i,j) = \overline{M \circ (i,j)}$.

One can prove the following propositions:

(1) For every matrix M over $\mathbb{C}$ holds $\langle i, j \rangle \in$ the indices of M iff $1 \leq i$ and $i \leq \operatorname{len} M$ and $1 \leq j$ and $j \leq \operatorname{width} M$.

(2) For every matrix M over $\mathbb{C}$ holds $\overline{\overline{M}} = M$.

(3) For every complex number a and for every matrix M over $\mathbb{C}$ holds $\operatorname{len}(a \cdot M) = \operatorname{len} M$ and $\operatorname{width}(a \cdot M) = \operatorname{width} M$.

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(4) Let i, j be natural numbers, a be a complex number, and M be a matrix over $\mathbb{C}$. Suppose $\mathrm{len}(a \cdot M) = \mathrm{len}\, M$ and $\mathrm{width}(a \cdot M) = \mathrm{width}\, M$ and $\langle i, j\rangle \in$ the indices of M. Then $(a \cdot M) \circ (i, j) = a \cdot (M \circ (i, j))$.

(5) For every complex number a and for every matrix M over $\mathbb{C}$ holds $\overline{a \cdot M} = \overline{a} \cdot \overline{M}$.

(6) For all matrices M_1, M_2 over $\mathbb{C}$ holds $\mathrm{len}(M_1 + M_2) = \mathrm{len}\, M_1$ and $\mathrm{width}(M_1 + M_2) = \mathrm{width}\, M_1$.

(7) Let i, j be natural numbers and M_1, M_2 be matrices over $\mathbb{C}$. Suppose $\mathrm{len}\, M_1 = \mathrm{len}\, M_2$ and $\mathrm{width}\, M_1 = \mathrm{width}\, M_2$ and $\langle i, j\rangle \in$ the indices of M_1. Then $(M_1 + M_2) \circ (i, j) = (M_1 \circ (i, j)) + (M_2 \circ (i, j))$.

(8) For all matrices M_1, M_2 over $\mathbb{C}$ such that $\mathrm{len}\, M_1 = \mathrm{len}\, M_2$ and $\mathrm{width}\, M_1 = \mathrm{width}\, M_2$ holds $\overline{M_1 + M_2} = \overline{M_1} + \overline{M_2}$.

(9) For every matrix M over $\mathbb{C}$ holds $\mathrm{len}(-M) = \mathrm{len}\, M$ and $\mathrm{width}(-M) = \mathrm{width}\, M$.

(10) Let i, j be natural numbers and M be a matrix over $\mathbb{C}$. If $\mathrm{len}(-M) = \mathrm{len}\, M$ and $\mathrm{width}(-M) = \mathrm{width}\, M$ and $\langle i, j\rangle \in$ the indices of M, then $(-M) \circ (i, j) = -(M \circ (i, j))$.

(11) For every matrix M over $\mathbb{C}$ holds $(-1) \cdot M = -M$.

(12) For every matrix M over $\mathbb{C}$ holds $\overline{-M} = -\overline{M}$.

(13) For all matrices M_1, M_2 over $\mathbb{C}$ holds $\mathrm{len}(M_1 - M_2) = \mathrm{len}\, M_1$ and $\mathrm{width}(M_1 - M_2) = \mathrm{width}\, M_1$.

(14) Let i, j be natural numbers and M_1, M_2 be matrices over $\mathbb{C}$. Suppose $\mathrm{len}\, M_1 = \mathrm{len}\, M_2$ and $\mathrm{width}\, M_1 = \mathrm{width}\, M_2$ and $\langle i, j\rangle \in$ the indices of M_1. Then $(M_1 - M_2) \circ (i, j) = (M_1 \circ (i, j)) - (M_2 \circ (i, j))$.

(15) For all matrices M_1, M_2 over $\mathbb{C}$ such that $\mathrm{len}\, M_1 = \mathrm{len}\, M_2$ and $\mathrm{width}\, M_1 = \mathrm{width}\, M_2$ holds $\overline{M_1 - M_2} = \overline{M_1} - \overline{M_2}$.

Let M be a matrix over $\mathbb{C}$. The functor M^* yields a matrix over $\mathbb{C}$ and is defined by:

(Def. 2) $M^* = \overline{M^{\mathrm{T}}}$.

Let x be a finite sequence of elements of $\mathbb{C}$. Let us assume that $\mathrm{len}\, x > 0$. The functor FinSeq2Matrix x yielding a matrix over $\mathbb{C}$ is defined as follows:

(Def. 3) $\mathrm{len}\,\mathrm{FinSeq2Matrix}\, x = \mathrm{len}\, x$ and $\mathrm{width}\,\mathrm{FinSeq2Matrix}\, x = 1$ and for every j such that $j \in \mathrm{Seg}\,\mathrm{len}\, x$ holds $(\mathrm{FinSeq2Matrix}\, x)(j) = \langle x(j) \rangle$.

Let M be a matrix over $\mathbb{C}$. The functor Matrix2FinSeq M yields a finite sequence of elements of $\mathbb{C}$ and is defined as follows:

(Def. 4) $\mathrm{Matrix2FinSeq}\, M = M_{\square,1}$.

Let F_1, F_2 be finite sequences of elements of $\mathbb{C}$. The functor $F_1 \bullet F_2$ yielding a finite sequence of elements of $\mathbb{C}$ is defined as follows:

(Def. 5) $F_1 \bullet F_2 = (\cdot_{\mathbb{C}})^{\circ}(F_1, F_2)$.

Let us observe that the functor $F_1 \bullet F_2$ is commutative.

Let F be a finite sequence of elements of $\mathbb{C}$. The functor $\sum F$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 6) $\sum F = +_{\mathbb{C}} \circledast F$.

Let M be a matrix over $\mathbb{C}$ and let F be a finite sequence of elements of $\mathbb{C}$. The functor $M \cdot F$ yielding a finite sequence of elements of $\mathbb{C}$ is defined as follows:

(Def. 7) $\mathrm{len}(M \cdot F) = \mathrm{len}\, M$ and for every i such that $i \in \mathrm{Seg}\,\mathrm{len}\, M$ holds $(M \cdot F)(i) = \sum(\mathrm{Line}(M, i) \bullet F)$.

We now state the proposition

(16) $a \cdot (R_1 \bullet R_2) = a \cdot R_1 \bullet R_2$.

Let M be a matrix over $\mathbb{C}$ and let a be a complex number. The functor $M \cdot a$ yielding a matrix over $\mathbb{C}$ is defined by:

(Def. 8) $M \cdot a = a \cdot M$.

We now state three propositions:

(17) For every element a of $\mathbb{C}$ and for every matrix M over $\mathbb{C}$ holds $\overline{M \cdot a} = \overline{a} \cdot \overline{M}$.

(18) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\mathrm{len}\, x = \mathrm{len}\, y$ holds $\mathrm{len}(x \bullet y) = \mathrm{len}\, x$ and $\mathrm{len}(x \bullet y) = \mathrm{len}\, y$.

(19) Let F_1, F_2 be finite sequences of elements of $\mathbb{C}$ and i be a natural number. If $i \in \mathrm{dom}(F_1 \bullet F_2)$, then $(F_1 \bullet F_2)(i) = F_1(i) \cdot F_2(i)$.

Let us consider i, R_1, R_2. Then $R_1 \bullet R_2$ is an element of $\mathbb{C}^i$.

We now state a number of propositions:

(20) $(R_1 \bullet R_2)(j) = R_1(j) \cdot R_2(j)$.

(21) For all elements a, b of $\mathbb{C}$ holds $\overline{+_{\mathbb{C}}(a, \overline{b})} = +_{\mathbb{C}}(\overline{a}, b)$.

(22) Let F be a finite sequence of elements of $\mathbb{C}$. Then there exists a function G from $\mathbb{N}$ into $\mathbb{C}$ such that for every natural number n if $1 \leq n$ and $n \leq \mathrm{len}\, F$, then $G(n) = F(n)$.

(23) For every finite sequence F of elements of $\mathbb{C}$ such that $\mathrm{len}\, \overline{F} \geq 1$ holds $+_{\mathbb{C}} \circledast \overline{F} = \overline{+_{\mathbb{C}} \circledast F}$.

(24) For every finite sequence F of elements of $\mathbb{C}$ such that $\mathrm{len}\, F \geq 1$ holds $\sum \overline{F} = \overline{\sum F}$.

(25) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\mathrm{len}\, x = \mathrm{len}\, y$ holds $\overline{x \bullet \overline{y}} = y \bullet \overline{x}$.

(26) For all finite sequences x, y of elements of $\mathbb{C}$ and for every element a of $\mathbb{C}$ such that $\mathrm{len}\, x = \mathrm{len}\, y$ holds $x \bullet a \cdot y = a \cdot (x \bullet y)$.

(27) For all finite sequences x, y of elements of $\mathbb{C}$ and for every element a of $\mathbb{C}$ such that $\mathrm{len}\, x = \mathrm{len}\, y$ holds $a \cdot x \bullet y = a \cdot (x \bullet y)$.

(28) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $\overline{x \bullet y} = \overline{x} \bullet \overline{y}$.

(29) For every finite sequence F of elements of $\mathbb{C}$ and for every element a of $\mathbb{C}$ holds $\sum(a \cdot F) = a \cdot \sum F$.

Let x be a finite sequence of elements of $\mathbb{R}$. The functor $\mathrm{FR2FC}\, x$ yielding a finite sequence of elements of $\mathbb{C}$ is defined as follows:

(Def. 9) $\mathrm{FR2FC}\, x = x$.

Next we state a number of propositions:

(30) Let R be a finite sequence of elements of $\mathbb{R}$ and F be a finite sequence of elements of $\mathbb{C}$. If $R = F$ and $\operatorname{len} R \geq 1$, then $+_{\mathbb{R}} \circledast R = +_{\mathbb{C}} \circledast F$.

(31) Let x be a finite sequence of elements of $\mathbb{R}$ and y be a finite sequence of elements of $\mathbb{C}$. If $x = y$ and $\operatorname{len} x \geq 1$, then $\sum x = \sum y$.

(32) For all finite sequences F_1, F_2 of elements of $\mathbb{C}$ such that $\operatorname{len} F_1 = \operatorname{len} F_2$ holds $\sum(F_1 - F_2) = \sum F_1 - \sum F_2$.

(33) Let F_1, F_2 be finite sequences of elements of $\mathbb{C}$ and i be a natural number. If $i \in \operatorname{dom}(F_1 + F_2)$, then $(F_1 + F_2)(i) = F_1(i) + F_2(i)$.

(34) Let F_1, F_2 be finite sequences of elements of $\mathbb{C}$ and i be a natural number. If $i \in \operatorname{dom}(F_1 - F_2)$, then $(F_1 - F_2)(i) = F_1(i) - F_2(i)$.

(35) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $(x - y) \bullet z = x \bullet z - y \bullet z$.

(36) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $x \bullet (y - z) = x \bullet y - x \bullet z$.

(37) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $x \bullet (y + z) = x \bullet y + x \bullet z$.

(38) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $(x + y) \bullet z = x \bullet z + y \bullet z$.

(39) For all finite sequences F_1, F_2 of elements of $\mathbb{C}$ such that $\operatorname{len} F_1 = \operatorname{len} F_2$ holds $\sum(F_1 + F_2) = \sum F_1 + \sum F_2$.

(40) Let x_1, y_1 be finite sequences of elements of $\mathbb{C}$ and x_2, y_2 be finite sequences of elements of $\mathbb{R}$. If $x_1 = x_2$ and $y_1 = y_2$ and $\operatorname{len} x_1 = \operatorname{len} y_2$, then $(\cdot_{\mathbb{C}})^{\circ}(x_1, y_1) = (\cdot_{\mathbb{R}})^{\circ}(x_2, y_2)$.

(41) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $\mathrm{FR2FC}(x \bullet y) = \mathrm{FR2FC}\, x \bullet \mathrm{FR2FC}\, y$.

(42) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} x > 0$ holds $|(x, y)| = \sum(x \bullet \overline{y})$.

(43) For all matrices A, B over $\mathbb{C}$ such that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ holds the indices of $A =$ the indices of B.

(44) Let i, j be natural numbers and M_1, M_2 be matrices over $\mathbb{C}$. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $j \in \operatorname{Seg} \operatorname{len} M_1$, then $\operatorname{Line}(M_1 +$

$M_2, j) = \mathrm{Line}(M_1, j) + \mathrm{Line}(M_2, j)$.

(45) For every matrix M over $\mathbb{C}$ such that $i \in \mathrm{Seg}\,\mathrm{len}\,M$ holds $\overline{\mathrm{Line}(M, i)} = \mathrm{Line}(\overline{M}, i)$.

(46) Let F be a finite sequence of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If $\mathrm{len}\,F = \mathrm{width}\,M$, then $F \bullet \overline{\mathrm{Line}(\overline{M}, i)} = \overline{\mathrm{Line}(\overline{M}, i) \bullet \overline{F}}$.

(47) Let F be a finite sequence of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If $\mathrm{len}\,F = \mathrm{width}\,M$ and $\mathrm{len}\,F \geq 1$, then $\overline{M \cdot F} = \overline{M} \cdot \overline{F}$.

(48) For all finite sequences F_1, F_2, F_3 of elements of $\mathbb{C}$ such that $\mathrm{len}\,F_1 = \mathrm{len}\,F_2$ and $\mathrm{len}\,F_2 = \mathrm{len}\,F_3$ holds $F_1 \bullet (F_2 \bullet F_3) = (F_1 \bullet F_2) \bullet F_3$.

(49) For every finite sequence F of elements of $\mathbb{C}$ holds $\sum(-F) = -\sum F$.

(50) For every element z of $\mathbb{C}$ holds $\sum\langle z\rangle = z$.

(51) For all finite sequences F_1, F_2 of elements of $\mathbb{C}$ holds $\sum(F_1 \frown F_2) = \sum F_1 + \sum F_2$.

Let M be a matrix over $\mathbb{C}$. The functor $\mathrm{LineSum}\,M$ yielding a finite sequence of elements of $\mathbb{C}$ is defined as follows:

(Def. 10) $\mathrm{len}\,\mathrm{LineSum}\,M = \mathrm{len}\,M$ and for every natural number i such that $i \in \mathrm{Seg}\,\mathrm{len}\,M$ holds $(\mathrm{LineSum}\,M)(i) = \sum \mathrm{Line}(M, i)$.

Let M be a matrix over $\mathbb{C}$. The functor $\mathrm{ColSum}\,M$ yielding a finite sequence of elements of $\mathbb{C}$ is defined by:

(Def. 11) $\mathrm{len}\,\mathrm{ColSum}\,M = \mathrm{width}\,M$ and for every natural number j such that $j \in \mathrm{Seg}\,\mathrm{width}\,M$ holds $(\mathrm{ColSum}\,M)(j) = \sum(M_{\square,j})$.

Next we state three propositions:

(52) For every finite sequence F of elements of $\mathbb{C}$ such that $\mathrm{len}\,F = 1$ holds $\sum F = F(1)$.

(53) Let f, g be finite sequences of elements of $\mathbb{C}$ and n be a natural number. If $\mathrm{len}\,f = n + 1$ and $g = f \restriction n$, then $\sum f = \sum g + f_{\mathrm{len}\,f}$.

(54) For every matrix M over $\mathbb{C}$ such that $\mathrm{len}\,M > 0$ holds $\sum \mathrm{LineSum}\,M = \sum \mathrm{ColSum}\,M$.

Let M be a matrix over $\mathbb{C}$. The functor $\mathrm{SumAll}\,M$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 12) $\mathrm{SumAll}\,M = \sum \mathrm{LineSum}\,M$.

Next we state two propositions:

(55) For every matrix M over $\mathbb{C}$ holds $\mathrm{ColSum}\,M = \mathrm{LineSum}(M^{\mathrm{T}})$.

(56) For every matrix M over $\mathbb{C}$ such that $\mathrm{len}\,M > 0$ holds $\mathrm{SumAll}\,M = \mathrm{SumAll}(M^{\mathrm{T}})$.

Let x, y be finite sequences of elements of $\mathbb{C}$ and let M be a matrix over $\mathbb{C}$. Let us assume that $\mathrm{len}\,x = \mathrm{len}\,M$ and $\mathrm{len}\,y = \mathrm{width}\,M$. The functor $\mathrm{QuadraticForm}(x, M, y)$ yielding a matrix over $\mathbb{C}$ is defined by the conditions (Def. 13).

(Def. 13)(i) $\operatorname{len} \operatorname{QuadraticForm}(x, M, y) = \operatorname{len} x$,

(ii) $\operatorname{width} \operatorname{QuadraticForm}(x, M, y) = \operatorname{len} y$, and

(iii) for all natural numbers i, j such that $\langle i, j \rangle \in$ the indices of M holds $\operatorname{QuadraticForm}(x, M, y) \circ (i, j) = x(i) \cdot (M \circ (i, j)) \cdot \overline{y(j)}$.

The following propositions are true:

(57) Let x, y be finite sequences of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If $\operatorname{len} x = \operatorname{len} M$ and $\operatorname{len} y = \operatorname{width} M$ and $\operatorname{len} x > 0$ and $\operatorname{len} y > 0$, then $(\operatorname{QuadraticForm}(x, M, y))^{\mathrm{T}} = \overline{\operatorname{QuadraticForm}(y, M^*, x)}$.

(58) Let x, y be finite sequences of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If $\operatorname{len} x = \operatorname{len} M$ and $\operatorname{len} y = \operatorname{width} M$, then $\overline{\operatorname{QuadraticForm}(x, M, y)} = \operatorname{QuadraticForm}(\overline{x}, \overline{M}, \overline{y})$.

(59) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $0 < \operatorname{len} y$ holds $|(x, y)| = \overline{|(y, x)|}$.

(60) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $0 < \operatorname{len} y$ holds $\overline{|(x, y)|} = |(\overline{x}, \overline{y})|$.

(61) For every matrix M over $\mathbb{C}$ such that $\operatorname{width} M > 0$ holds $\overline{M^{\mathrm{T}}} = \overline{M}^{\mathrm{T}}$.

(62) Let x, y be finite sequences of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If $\operatorname{len} x = \operatorname{width} M$ and $\operatorname{len} y = \operatorname{len} M$ and $\operatorname{len} x > 0$ and $\operatorname{len} y > 0$, then $|(x, M^* \cdot y)| = \operatorname{SumAll} \operatorname{QuadraticForm}(x, M^{\mathrm{T}}, y)$.

(63) Let x, y be finite sequences of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If $\operatorname{len} y = \operatorname{len} M$ and $\operatorname{len} x = \operatorname{width} M$ and $\operatorname{len} x > 0$ and $\operatorname{len} y > 0$ and $\operatorname{len} M > 0$, then $|(M \cdot x, y)| = \operatorname{SumAll} \operatorname{QuadraticForm}(x, M^{\mathrm{T}}, y)$.

(64) Let x, y be finite sequences of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If $\operatorname{len} x = \operatorname{width} M$ and $\operatorname{len} y = \operatorname{len} M$ and $\operatorname{width} M > 0$ and $\operatorname{len} M > 0$, then $|(M \cdot x, y)| = |(x, M^* \cdot y)|$.

(65) Let x, y be finite sequences of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If $\operatorname{len} x = \operatorname{len} M$ and $\operatorname{len} y = \operatorname{width} M$ and $\operatorname{width} M > 0$ and $\operatorname{len} M > 0$ and $\operatorname{len} x > 0$, then $|(x, M \cdot y)| = |(M^* \cdot x, y)|$.

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Partial Sum and Partial Product of Some Series

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Summary. This article contains partial sum and partial product of some series which are often used.

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The notation and terminology used in this paper have been introduced in the following articles: [2], [1], [3], [4], [5], [6], and [7].

We use the following convention: n is a natural number, a, b, c, d are real numbers, and s is a sequence of real numbers.

We now state a number of propositions:

(1) $(a+b+c)^2 = a^2+b^2+c^2+2\cdot a\cdot b+2\cdot a\cdot c+2\cdot b\cdot c.$

(2) $(a+b)^3 = a^3+3\cdot a^2\cdot b+3\cdot b^2\cdot a+b^3.$

(3) $((a-b)+c)^2 = (((a^2+b^2+c^2)-2\cdot a\cdot b)+2\cdot a\cdot c)-2\cdot b\cdot c.$

(4) $(a-b-c)^2 = ((a^2+b^2+c^2)-2\cdot a\cdot b-2\cdot a\cdot c)+2\cdot b\cdot c.$

(5) $(a-b)^3 = ((a^3-3\cdot a^2\cdot b)+3\cdot b^2\cdot a)-b^3.$

(6) $(a+b)^4 = a^4+4\cdot a^3\cdot b+6\cdot a^2\cdot b^2+4\cdot b^3\cdot a+b^4.$

(7) $(a+b+c+d)^2 = a^2+b^2+c^2+d^2+(2\cdot a\cdot b+2\cdot a\cdot c+2\cdot a\cdot d)+(2\cdot b\cdot c+2\cdot b\cdot d)+2\cdot c\cdot d.$

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(8) $(a+b+c)^3 = a^3+b^3+c^3+(3\cdot a^2\cdot b+3\cdot a^2\cdot c)+(3\cdot b^2\cdot a+3\cdot b^2\cdot c)+(3\cdot c^2\cdot a+3\cdot c^2\cdot b)+6\cdot a\cdot b\cdot c$.

(9) If $a \neq 0$, then $((\frac{1}{a})^{n+1}+a^{n+1})^2 = (\frac{1}{a})^{2\cdot n+2}+a^{2\cdot n+2}+2$.

(10) If $a \neq 1$ and for every n holds $s(n) = a^n$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{1-a^{n+1}}{1-a}$.

(11) If $a \neq 1$ and $a \neq 0$ and for every n holds $s(n) = (\frac{1}{a})^n$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{(\frac{1}{a})^n-a}{1-a}$.

(12) If for every n holds $s(n) = 10^n+2\cdot n+1$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = (\frac{10^{n+1}}{9}-\frac{1}{9})+(n+1)^2$.

(13) If for every n holds $s(n) = (2\cdot n-1)+(\frac{1}{2})^n$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = (n^2+1)-(\frac{1}{2})^n$.

(14) If for every n holds $s(n) = n\cdot(\frac{1}{2})^n$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = 2-(2+n)\cdot(\frac{1}{2})^n$.

(15) If for every n holds $s(n) = ((\frac{1}{2})^n+2^n)^2$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = -\frac{(\frac{1}{4})^n}{3}+\frac{4^{n+1}}{3}+2\cdot n+3$.

(16) If for every n holds $s(n) = ((\frac{1}{3})^n+3^n)^2$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = -\frac{(\frac{1}{9})^n}{8}+\frac{9^{n+1}}{8}+2\cdot n+3$.

(17) If for every n holds $s(n) = n\cdot 2^n$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = (n\cdot 2^{n+1}-2^{n+1})+2$.

(18) If for every n holds $s(n) = (2\cdot n+1)\cdot 3^n$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = n\cdot 3^{n+1}+1$.

(19) If $a \neq 1$ and for every n holds $s(n) = n\cdot a^n$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{a\cdot(1-a^n)}{(1-a)^2}-\frac{n\cdot a^{n+1}}{1-a}$.

(20) If for every n holds $s(n) = \frac{1}{(\mathrm{root}_2(n+1))+(\mathrm{root}_2(n))}$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \mathrm{root}_2(n+1)$.

(21) If for every n holds $s(n) = 2^n+(\frac{1}{2})^n$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = (2^{n+1}-(\frac{1}{2})^n)+1$.

(22) If for every n holds $s(n) = n!\cdot n+\frac{n}{(n+1)!}$, then for every n such that $n \geq 1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = (n+1)!-\frac{1}{(n+1)!}$.

(23) Suppose $a \neq 1$ and for every n such that $n \geq 1$ holds $s(n) = (\frac{a}{a-1})^n$ and $s(0) = 0$. Let given n. If $n \geq 1$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = a\cdot((\frac{a}{a-1})^n-1)$.

(24) If for every n such that $n \geq 1$ holds $s(n) = 2^n\cdot\frac{3\cdot n-1}{4}$ and $s(0) = 0$, then for every n such that $n \geq 1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = 2^n\cdot\frac{3\cdot n-4}{2}+2$.

(25) If for every n holds $s(n) = \frac{n+1}{n+2}$, then (the partial product of s)$(n) = \frac{1}{n+2}$.

(26) If for every n holds $s(n) = \frac{1}{n+1}$, then (the partial product of s)$(n) = \frac{1}{(n+1)!}$.

(27) Suppose that for every n such that $n \geq 1$ holds $s(n) = n$ and $s(0) = 1$. Let given n. If $n \geq 1$, then (the partial product of $s)(n) = n!$.

(28) Suppose that for every n such that $n \geq 1$ holds $s(n) = \frac{a}{n}$ and $s(0) = 1$. Let given n. If $n \geq 1$, then (the partial product of $s)(n) = \frac{a^n}{n!}$.

(29) Suppose that for every n such that $n \geq 1$ holds $s(n) = a$ and $s(0) = 1$. Let given n. If $n \geq 1$, then (the partial product of $s)(n) = a^n$.

(30) Suppose that for every n such that $n \geq 2$ holds $s(n) = 1 - \frac{1}{n^2}$ and $s(0) = 1$ and $s(1) = 1$. Let given n. If $n \geq 2$, then (the partial product of $s)(n) = \frac{n+1}{2 \cdot n}$.

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Some Differentiable Formulas of Special Functions

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Summary. This article contains some differentiable formulas of special functions.

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The terminology and notation used in this paper are introduced in the following papers: [13], [15], [16], [2], [4], [10], [12], [3], [1], [6], [9], [7], [8], [11], [17], [5], and [14].

For simplicity, we use the following convention: x, a, b are real numbers, n is a natural number, Z is an open subset of $\mathbb{R}$, and f, f_1, f_2, g are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Next we state a number of propositions:

(1) Suppose $Z \subseteq \mathrm{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_2(x) = a - x$ and $f_2(x) \neq 0$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{2 \cdot a}{(a-x)^{\mathbf{2}}}$.

(2) Suppose $Z \subseteq \mathrm{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_2(x) = x + a$ and $f_2(x) \neq 0$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{2 \cdot a}{(x+a)^{\mathbf{2}}}$.

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(3) Suppose $Z \subseteq \mathrm{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_2(x) = x - b$ and $f_2(x) \neq 0$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{a-b}{(x-b)^{\mathbf{2}}}$.

(4) Suppose $Z \subseteq \mathrm{dom}\, f$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then $\frac{1}{f}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{x^{\mathbf{2}}}$.

(5) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin}) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) (the function sin) $\cdot \frac{1}{f}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function sin}) \cdot \frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{x^{\mathbf{2}}} \cdot (\text{the function cos})(\frac{1}{x})$.

(6) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos}) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) (the function cos) $\cdot \frac{1}{f}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function cos}) \cdot \frac{1}{f})'_{\restriction Z}(x) = \frac{1}{x^{\mathbf{2}}} \cdot (\text{the function sin})(\frac{1}{x})$.

(7) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z\, ((\text{the function sin}) \cdot \frac{1}{f}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\mathrm{id}_Z\, ((\text{the function sin}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z\, ((\text{the function sin}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = (\text{the function sin})(\frac{1}{x}) - \frac{1}{x} \cdot (\text{the function cos})(\frac{1}{x})$.

(8) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z\, ((\text{the function cos}) \cdot \frac{1}{f}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\mathrm{id}_Z\, ((\text{the function cos}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z\, ((\text{the function cos}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = (\text{the function cos})(\frac{1}{x}) + \frac{1}{x} \cdot (\text{the function sin})(\frac{1}{x})$.

(9) Suppose $Z \subseteq \mathrm{dom}(((\text{the function sin}) \cdot \frac{1}{f})\, ((\text{the function cos}) \cdot \frac{1}{f}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $((\text{the function sin}) \cdot \frac{1}{f})\, ((\text{the function cos}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function sin}) \cdot \frac{1}{f})\, ((\text{the function cos}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = \frac{1}{x^{\mathbf{2}}} \cdot ((\text{the function sin})(\frac{1}{x})^{\mathbf{2}} - (\text{the function cos})(\frac{1}{x})^{\mathbf{2}})$.

(10) Suppose $Z \subseteq \mathrm{dom}(((\text{the function sin}) \cdot f)\, ((^{n}_{\mathbb{Z}}) \cdot (\text{the function sin})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function sin}) \cdot f)\, ((^{n}_{\mathbb{Z}}) \cdot (\text{the function sin}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function sin}) \cdot f)\, ((^{n}_{\mathbb{Z}}) \cdot (\text{the function sin})))'_{\restriction Z}(x) = n \cdot (\text{the function sin})(x)^{n-1}_{\mathbb{Z}} \cdot (\text{the function sin})((n+1) \cdot x)$.

(11) Suppose $Z \subseteq \mathrm{dom}(((\text{the function cos})\cdot f)\,((^{n}_{\mathbb{Z}})\cdot(\text{the function sin})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function cos})\cdot f)\,((^{n}_{\mathbb{Z}})\cdot(\text{the function sin}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function cos})\cdot f)\,((^{n}_{\mathbb{Z}})\cdot(\text{the function sin})))'_{\restriction Z}(x) = n\cdot(\text{the function sin})(x)^{n-1}_{\mathbb{Z}}\cdot(\text{the function cos})((n+1)\cdot x)$.

(12) Suppose $Z \subseteq \mathrm{dom}(((\text{the function cos})\cdot f)\,((^{n}_{\mathbb{Z}})\cdot(\text{the function cos})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function cos})\cdot f)\,((^{n}_{\mathbb{Z}})\cdot(\text{the function cos}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function cos})\cdot f)\,((^{n}_{\mathbb{Z}})\cdot(\text{the function cos})))'_{\restriction Z}(x) = -n\cdot(\text{the function cos})(x)^{n-1}_{\mathbb{Z}}\cdot(\text{the function sin})((n+1)\cdot x)$.

(13) Suppose $Z \subseteq \mathrm{dom}(((\text{the function sin})\cdot f)\,((^{n}_{\mathbb{Z}})\cdot(\text{the function cos})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function sin})\cdot f)\,((^{n}_{\mathbb{Z}})\cdot(\text{the function cos}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function sin})\cdot f)\,((^{n}_{\mathbb{Z}})\cdot(\text{the function cos})))'_{\restriction Z}(x) = n\cdot(\text{the function cos})(x)^{n-1}_{\mathbb{Z}}\cdot(\text{the function cos})((n+1)\cdot x)$.

(14) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{f}\,(\text{the function sin}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\frac{1}{f}\,(\text{the function sin})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{f}\,(\text{the function sin}))'_{\restriction Z}(x) = \frac{1}{x}\cdot(\text{the function cos})(x) - \frac{1}{x^{\mathbf{2}}}\cdot(\text{the function sin})(x)$.

(15) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{f}\,(\text{the function cos}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\frac{1}{f}\,(\text{the function cos})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{f}\,(\text{the function cos}))'_{\restriction Z}(x) = -\frac{1}{x}\cdot(\text{the function sin})(x) - \frac{1}{x^{\mathbf{2}}}\cdot(\text{the function cos})(x)$.

(16) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin}) + (^{\frac{1}{2}}_{\mathbb{R}})\cdot f)$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) > 0$. Then

(i) $(\text{the function sin}) + (^{\frac{1}{2}}_{\mathbb{R}})\cdot f$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function sin}) + (^{\frac{1}{2}}_{\mathbb{R}})\cdot f)'_{\restriction Z}(x) = (\text{the function cos})(x) + \frac{1}{2}\cdot x_{\mathbb{R}}^{-\frac{1}{2}}$.

(17) Suppose $Z \subseteq \mathrm{dom}(g\,((\text{the function sin})\cdot\frac{1}{f}))$ and $g = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $g\,((\text{the function sin})\cdot\frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(g\,((\text{the function sin}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = 2 \cdot x \cdot (\text{the function sin})(\frac{1}{x}) - (\text{the function cos})(\frac{1}{x})$.

(18) Suppose $Z \subseteq \operatorname{dom}(g\,((\text{the function cos}) \cdot \frac{1}{f}))$ and $g = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $g\,((\text{the function cos}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(g\,((\text{the function cos}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = 2 \cdot x \cdot (\text{the function cos})(\frac{1}{x}) + (\text{the function sin})(\frac{1}{x})$.

(19) Suppose $Z \subseteq \operatorname{dom}(\log\_(e) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) > 0$. Then $\log\_(e) \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot f)'_{\restriction Z}(x) = \frac{1}{x}$.

(20) Suppose $Z \subseteq \operatorname{dom}(\operatorname{id}_Z f)$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$. Then $\operatorname{id}_Z f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\operatorname{id}_Z f)'_{\restriction Z}(x) = 1 + (\log\_(e))(x)$.

(21) Suppose $Z \subseteq \operatorname{dom}(g\,f)$ and $g = \frac{2}{\mathbb{Z}}$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$. Then $g\,f$ is differentiable on Z and for every x such that $x \in Z$ holds $(g\,f)'_{\restriction Z}(x) = x + 2 \cdot x \cdot (\log\_(e))(x)$.

(22) Suppose $Z \subseteq \operatorname{dom}(\frac{f_1+f_2}{f_1-f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = a$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $(f_1 - f_2)(x) > 0$. Then $\frac{f_1+f_2}{f_1-f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1+f_2}{f_1-f_2})'_{\restriction Z}(x) = \frac{4 \cdot a \cdot x}{(a-x^2)^2}$.

(23) Suppose that

(i) $Z \subseteq \operatorname{dom}(\log\_(e) \cdot \frac{f_1+f_2}{f_1-f_2})$,

(ii) for every x such that $x \in Z$ holds $f_1(x) = a$,

(iii) $f_2 = \frac{2}{\mathbb{Z}}$,

(iv) for every x such that $x \in Z$ holds $(f_1 - f_2)(x) > 0$, and

(v) for every x such that $x \in Z$ holds $(f_1 + f_2)(x) > 0$.

Then $\log\_(e) \cdot \frac{f_1+f_2}{f_1-f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot \frac{f_1+f_2}{f_1-f_2})'_{\restriction Z}(x) = \frac{4 \cdot a \cdot x}{a^2-x^4}$.

(24) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{f}\,g)$ and for every x such that $x \in Z$ holds $f(x) = x$ and $g = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$. Then $\frac{1}{f}\,g$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f}\,g)'_{\restriction Z}(x) = \frac{1}{x^{\mathbf{2}}} \cdot (1 - (\log\_(e))(x))$.

(25) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{f})$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$ and for every x such that $x \in Z$ holds $f(x) \neq 0$. Then $\frac{1}{f}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{x \cdot (\log\_(e))(x)^{\mathbf{2}}}$.

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Formulas and Identities of Hyperbolic Functions

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Summary. In this article, we proved formulas of hyperbolic sine, hyperbolic cosine and hyperbolic tangent, and their identities.

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The papers [1], [3], [6], [5], [7], [4], and [2] provide the terminology and notation for this paper.

We follow the rules: x, y, z, w are real numbers and n is a natural number.

One can prove the following propositions:

(1) $\tanh x = \frac{\sinh x}{\cosh x}$ and $\tanh 0 = 0$.

(2) $\sinh x = \frac{1}{\operatorname{cosech} x}$ and $\cosh x = \frac{1}{\operatorname{sech} x}$ and $\tanh x = \frac{1}{\coth x}$.

(3) $\operatorname{sech} x \leq 1$ and $0 < \operatorname{sech} x$ and $\operatorname{sech} 0 = 1$.

(4) If $x \geq 0$, then $\tanh x \geq 0$.

(5) $\cosh x = \frac{1}{\sqrt{1-(\tanh x)^{\mathbf{2}}}}$ and $\sinh x = \frac{\tanh x}{\sqrt{1-(\tanh x)^{\mathbf{2}}}}$.

(6) $(\cosh x + \sinh x)^n = \cosh(n \cdot x) + \sinh(n \cdot x)$ and $(\cosh x - \sinh x)^n = \cosh(n \cdot x) - \sinh(n \cdot x)$.

(7)(i) $\exp x = \cosh x + \sinh x$,

(ii) $\exp(-x) = \cosh x - \sinh x$,

(iii) $\exp x = \frac{\cosh(\frac{x}{2})+\sinh(\frac{x}{2})}{\cosh(\frac{x}{2})-\sinh(\frac{x}{2})}$,

(iv) $\exp(-x) = \frac{\cosh(\frac{x}{2})-\sinh(\frac{x}{2})}{\cosh(\frac{x}{2})+\sinh(\frac{x}{2})}$,

(v) $\exp x = \frac{1+\tanh(\frac{x}{2})}{1-\tanh(\frac{x}{2})}$, and

(vi) $\exp(-x) = \frac{1-\tanh(\frac{x}{2})}{1+\tanh(\frac{x}{2})}$.

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(8) If $x \neq 0$, then $\exp x = \frac{\coth(\frac{x}{2})+1}{\coth(\frac{x}{2})-1}$ and $\exp(-x) = \frac{\coth(\frac{x}{2})-1}{\coth(\frac{x}{2})+1}$.

(9) $\frac{\cosh x+\sinh x}{\cosh x-\sinh x} = \frac{1+\tanh x}{1-\tanh x}$.

(10) If $y \neq 0$, then $\coth y + \tanh z = \frac{\cosh(y+z)}{\sinh y \cdot \cosh z}$ and $\coth y - \tanh z = \frac{\cosh(y-z)}{\sinh y \cdot \cosh z}$.

(11) $\sinh y \cdot \sinh z = \frac{1}{2} \cdot (\cosh(y+z) - \cosh(y-z))$ and $\sinh y \cdot \cosh z = \frac{1}{2} \cdot (\sinh(y+z) + \sinh(y-z))$ and $\cosh y \cdot \sinh z = \frac{1}{2} \cdot (\sinh(y+z) - \sinh(y-z))$ and $\cosh y \cdot \cosh z = \frac{1}{2} \cdot (\cosh(y+z) + \cosh(y-z))$.

(12) $(\sinh y)^{\mathbf{2}} - (\cosh z)^{\mathbf{2}} = \sinh(y+z) \cdot \sinh(y-z) - 1$.

(13) $(\sinh y - \sinh z)^{\mathbf{2}} - (\cosh y - \cosh z)^{\mathbf{2}} = 4 \cdot (\sinh(\frac{y-z}{2}))^{\mathbf{2}}$ and $(\cosh y + \cosh z)^{\mathbf{2}} - (\sinh y + \sinh z)^{\mathbf{2}} = 4 \cdot (\cosh(\frac{y-z}{2}))^{\mathbf{2}}$.

(14) $\frac{\sinh y+\sinh z}{\sinh y-\sinh z} = \tanh(\frac{y+z}{2}) \cdot \coth(\frac{y-z}{2})$.

(15) $\frac{\cosh y+\cosh z}{\cosh y-\cosh z} = \coth(\frac{y+z}{2}) \cdot \coth(\frac{y-z}{2})$.

(16) If $y - z \neq 0$, then $\frac{\sinh y+\sinh z}{\cosh y+\cosh z} = \frac{\cosh y-\cosh z}{\sinh y-\sinh z}$.

(17) If $y + z \neq 0$, then $\frac{\sinh y-\sinh z}{\cosh y+\cosh z} = \frac{\cosh y-\cosh z}{\sinh y+\sinh z}$.

(18) $\frac{\sinh y+\sinh z}{\cosh y+\cosh z} = \tanh(\frac{y}{2} + \frac{z}{2})$ and $\frac{\sinh y-\sinh z}{\cosh y+\cosh z} = \tanh(\frac{y}{2} - \frac{z}{2})$.

(19) $\frac{\tanh y+\tanh z}{\tanh y-\tanh z} = \frac{\sinh(y+z)}{\sinh(y-z)}$.

(20) $\frac{\sinh(y-z)+\sinh y+\sinh(y+z)}{\cosh(y-z)+\cosh y+\cosh(y+z)} = \tanh y$.

(21)(i) $\sinh(y+z+w) = (\tanh y + \tanh z + \tanh w + \tanh y \cdot \tanh z \cdot \tanh w) \cdot \cosh y \cdot \cosh z \cdot \cosh w$,

(ii) $\cosh(y+z+w) = (1 + \tanh y \cdot \tanh z + \tanh z \cdot \tanh w + \tanh w \cdot \tanh y) \cdot \cosh y \cdot \cosh z \cdot \cosh w$, and

(iii) $\tanh(y+z+w) = \frac{\tanh y+\tanh z+\tanh w+\tanh y \cdot \tanh z \cdot \tanh w}{1+\tanh z \cdot \tanh w+\tanh w \cdot \tanh y+\tanh y \cdot \tanh z}$.

(22) $\cosh(2 \cdot y) + \cosh(2 \cdot z) + \cosh(2 \cdot w) + \cosh(2 \cdot (y+z+w)) = 4 \cdot \cosh(z+w) \cdot \cosh(w+y) \cdot \cosh(y+z)$.

(23) $\sinh y \cdot \sinh z \cdot \sinh(z-y) + \sinh z \cdot \sinh w \cdot \sinh(w-z) + \sinh w \cdot \sinh y \cdot \sinh(y-w) + \sinh(z-y) \cdot \sinh(w-z) \cdot \sinh(y-w) = 0$.

(24) If $x \geq 0$, then $\sinh(\frac{x}{2}) = \sqrt{\frac{\cosh x-1}{2}}$.

(25) If $x < 0$, then $\sinh(\frac{x}{2}) = -\sqrt{\frac{\cosh x-1}{2}}$.

(26) $\sinh(2 \cdot x) = 2 \cdot \sinh x \cdot \cosh x$ and $\cosh(2 \cdot x) = 2 \cdot (\cosh x)^{\mathbf{2}} - 1$ and $\tanh(2 \cdot x) = \frac{2 \cdot \tanh x}{1+(\tanh x)^{\mathbf{2}}}$.

(27) $\sinh(2 \cdot x) = \frac{2 \cdot \tanh x}{1-(\tanh x)^{\mathbf{2}}}$ and $\sinh(3 \cdot x) = \sinh x \cdot (4 \cdot (\cosh x)^{\mathbf{2}} - 1)$ and $\sinh(3 \cdot x) = 3 \cdot \sinh x - 2 \cdot \sinh x \cdot (1 - \cosh(2 \cdot x))$ and $\cosh(2 \cdot x) = 1 + 2 \cdot (\sinh x)^{\mathbf{2}}$ and $\cosh(2 \cdot x) = (\cosh x)^{\mathbf{2}} + (\sinh x)^{\mathbf{2}}$ and $\cosh(2 \cdot x) = \frac{1+(\tanh x)^{\mathbf{2}}}{1-(\tanh x)^{\mathbf{2}}}$ and $\cosh(3 \cdot x) = \cosh x \cdot (4 \cdot (\sinh x)^{\mathbf{2}} + 1)$ and $\tanh(3 \cdot x) = \frac{3 \cdot \tanh x+(\tanh x)^3}{1+3 \cdot (\tanh x)^{\mathbf{2}}}$.

(28) $\frac{\sinh(5\cdot x)+2\cdot\sinh(3\cdot x)+\sinh x}{\sinh(7\cdot x)+2\cdot\sinh(5\cdot x)+\sinh(3\cdot x)} = \frac{\sinh(3\cdot x)}{\sinh(5\cdot x)}$.

(29) If $x \geq 0$, then $\tanh(\frac{x}{2}) = \sqrt{\frac{\cosh x-1}{\cosh x+1}}$.

(30) If $x < 0$, then $\tanh(\frac{x}{2}) = -\sqrt{\frac{\cosh x-1}{\cosh x+1}}$.

(31)(i) $(\sinh x)^3 = \frac{\sinh(3\cdot x)-3\cdot\sinh x}{4}$,

(ii) $(\sinh x)^4 = \frac{(\cosh(4\cdot x)-4\cdot\cosh(2\cdot x))+3}{8}$,

(iii) $(\sinh x)^5 = \frac{(\sinh(5\cdot x)-5\cdot\sinh(3\cdot x))+10\cdot\sinh x}{16}$,

(iv) $(\sinh x)^6 = \frac{((\cosh(6\cdot x)-6\cdot\cosh(4\cdot x))+15\cdot\cosh(2\cdot x))-10}{32}$,

(v) $(\sinh x)^7 = \frac{((\sinh(7\cdot x)-7\cdot\sinh(5\cdot x))+21\cdot\sinh(3\cdot x))-35\cdot\sinh x}{64}$, and

(vi) $(\sinh x)^8 = \frac{(((\cosh(8\cdot x)-8\cdot\cosh(6\cdot x))+28\cdot\cosh(4\cdot x))-56\cdot\cosh(2\cdot x))+35}{128}$.

(32)(i) $(\cosh x)^3 = \frac{\cosh(3\cdot x)+3\cdot\cosh x}{4}$,

(ii) $(\cosh x)^4 = \frac{\cosh(4\cdot x)+4\cdot\cosh(2\cdot x)+3}{8}$,

(iii) $(\cosh x)^5 = \frac{\cosh(5\cdot x)+5\cdot\cosh(3\cdot x)+10\cdot\cosh x}{16}$,

(iv) $(\cosh x)^6 = \frac{\cosh(6\cdot x)+6\cdot\cosh(4\cdot x)+15\cdot\cosh(2\cdot x)+10}{32}$,

(v) $(\cosh x)^7 = \frac{\cosh(7\cdot x)+7\cdot\cosh(5\cdot x)+21\cdot\cosh(3\cdot x)+35\cdot\cosh x}{64}$, and

(vi) $(\cosh x)^8 = \frac{\cosh(8\cdot x)+8\cdot\cosh(6\cdot x)+28\cdot\cosh(4\cdot x)+56\cdot\cosh(2\cdot x)+35}{128}$.

(33) $\cosh(2 \cdot y) + \cos(2 \cdot z) = 2 + 2 \cdot ((\sinh y)^{\mathbf{2}} - (\sin z)^{\mathbf{2}})$ and $\cosh(2 \cdot y) - \cos(2 \cdot z) = 2 \cdot ((\sinh y)^{\mathbf{2}} + (\sin z)^{\mathbf{2}})$.

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Niemytzki Plane - an Example of Tychonoff Space Which Is Not T_4

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Summary. We continue Mizar formalization of General Topology according to the book [20] by Engelking. Niemytzki plane is defined as halfplane $y \geq 0$ with topology introduced by a neighborhood system. Niemytzki plane is not T_4. Next, the definition of Tychonoff space is given. The characterization of Tychonoff space by prebasis and the fact that Tychonoff spaces are between T_3 and T_4 is proved. The final result is that Niemytzki plane is also a Tychonoff space.

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The notation and terminology used here are introduced in the following papers: [38], [34], [15], [41], [17], [40], [35], [42], [11], [14], [12], [8], [13], [33], [10], [37], [4], [2], [1], [3], [5], [32], [39], [22], [25], [23], [29], [27], [26], [28], [43], [18], [31], [30], [36], [19], [24], [9], [16], [21], [7], and [6].

1. Preliminaries

In this paper x, y are elements of $\mathbb{R}$.

One can prove the following propositions:

(1) For all functions f, g such that $f \approx g$ and for every set A holds $(f{+\cdot}g)^{-1}(A) = f^{-1}(A) \cup g^{-1}(A)$.

(2) For all functions f, g such that $\operatorname{dom} f$ misses $\operatorname{dom} g$ and for every set A holds $(f{+\cdot}g)^{-1}(A) = f^{-1}(A) \cup g^{-1}(A)$.

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Let X be a set and let Y be a non empty real-membered set. Note that every relation between X and Y is real-yielding.

Next we state several propositions:

(3) For all sets x, a and for every function f such that $a \in \operatorname{dom} f$ holds $(\operatorname{commute}(x \dot{\longmapsto} f))(a) = x \dot{\longmapsto} f(a)$.

(4) Let b be a set and f be a function. Then $b \in \operatorname{dom} \operatorname{commute}(f)$ if and only if there exists a set a and there exists a function g such that $a \in \operatorname{dom} f$ and $g = f(a)$ and $b \in \operatorname{dom} g$.

(5) Let a, b be sets and f be a function. Then $a \in \operatorname{dom}(\operatorname{commute}(f))(b)$ if and only if there exists a function g such that $a \in \operatorname{dom} f$ and $g = f(a)$ and $b \in \operatorname{dom} g$.

(6) For all sets a, b and for all functions f, g such that $a \in \operatorname{dom} f$ and $g = f(a)$ and $b \in \operatorname{dom} g$ holds $(\operatorname{commute}(f))(b)(a) = g(b)$.

(7) For every set a and for all functions f, g, h such that $h = f \cup g$ holds $(\operatorname{commute}(h))(a) = (\operatorname{commute}(f))(a) \cup (\operatorname{commute}(g))(a)$.

Let us note that every finite subset of $\mathbb{R}$ is bounded.

The following propositions are true:

(8) For all real numbers a, b, c, d such that $a < b$ and $c \leq d$ holds $]a, c[\cap [b, d] = [b, c[$.

(9) For all real numbers a, b, c, d such that $a \geq b$ and $c > d$ holds $]a, c[\cap [b, d] =]a, d]$.

(10) For all real numbers a, b, c, d such that $a \leq b$ and $b < c$ and $c \leq d$ holds $[a, c[\cup]b, d] = [a, d]$.

(11) For all real numbers a, b, c, d such that $a \leq b$ and $b < c$ and $c \leq d$ holds $[a, c[\cap]b, d] =]b, c[$.

(12) For all sets X, Y holds $\prod\langle X, Y\rangle \approx [: X, Y :]$ and $\overline{\overline{\prod\langle X, Y\rangle}} = \overline{\overline{X}} \cdot \overline{\overline{Y}}$.

In this article we present several logical schemes. The scheme *SCH1* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding sets, and a unary predicate $\mathcal{P}$, and states that:

> There exists a function f from $\mathcal{C}$ into $\mathcal{B}$ such that for every element a of $\mathcal{A}$ holds
>
> (i) if $\mathcal{P}[a]$, then $f(a) = \mathcal{F}(a)$, and
>
> (ii) if not $\mathcal{P}[a]$, then $f(a) = \mathcal{G}(a)$

provided the parameters meet the following conditions:

- $\mathcal{C} \subseteq \mathcal{A}$, and
- For every element a of $\mathcal{A}$ such that $a \in \mathcal{C}$ holds if $\mathcal{P}[a]$, then $\mathcal{F}(a) \in \mathcal{B}$ and if not $\mathcal{P}[a]$, then $\mathcal{G}(a) \in \mathcal{B}$.

The scheme *SCH2* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, three unary functors $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ yielding sets, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

There exists a function f from $\mathcal{C}$ into $\mathcal{B}$ such that for every element a of $\mathcal{A}$ holds

(i) if $\mathcal{P}[a]$, then $f(a) = \mathcal{F}(a)$,

(ii) if not $\mathcal{P}[a]$ and $\mathcal{Q}[a]$, then $f(a) = \mathcal{G}(a)$, and

(iii) if not $\mathcal{P}[a]$ and not $\mathcal{Q}[a]$, then $f(a) = \mathcal{H}(a)$

provided the parameters meet the following conditions:

- $\mathcal{C} \subseteq \mathcal{A}$, and
- For every element a of $\mathcal{A}$ such that $a \in \mathcal{C}$ holds if $\mathcal{P}[a]$, then $\mathcal{F}(a) \in \mathcal{B}$ and if not $\mathcal{P}[a]$ and $\mathcal{Q}[a]$, then $\mathcal{G}(a) \in \mathcal{B}$ and if not $\mathcal{P}[a]$ and not $\mathcal{Q}[a]$, then $\mathcal{H}(a) \in \mathcal{B}$.

The following four propositions are true:

(13) For all real numbers a, b holds $|[a,b]|^{\mathbf{2}} = a^{\mathbf{2}} + b^{\mathbf{2}}$.

(14) Let X be a topological space, Y be a non empty topological space, A, B be closed subsets of X, f be a continuous function from $X \restriction A$ into Y, and g be a continuous function from $X \restriction B$ into Y. If $f \approx g$, then $f {+\cdot} g$ is a continuous function from $X \restriction (A \cup B)$ into Y.

(15) Let X be a topological space, Y be a non empty topological space, and A, B be closed subsets of X. Suppose A misses B. Let f be a continuous function from $X \restriction A$ into Y and g be a continuous function from $X \restriction B$ into Y. Then $f {+\cdot} g$ is a continuous function from $X \restriction (A \cup B)$ into Y.

(16) Let X be a topological space, Y be a non empty topological space, A be an open closed subset of X, f be a continuous function from $X \restriction A$ into Y, and g be a continuous function from $X \restriction A^{\mathrm{c}}$ into Y. Then $f {+\cdot} g$ is a continuous function from X into Y.

2. Niemytzki Plane

One can prove the following proposition

(17) For every natural number n and for every point a of $\mathcal{E}^n_{\mathrm{T}}$ and for every positive real number r holds $a \in \mathrm{Ball}(a, r)$.

The subset $(y = 0)$-line of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 1) $(y = 0)\text{-line} = \{[x, 0]\}$.

The subset $(y \geq 0)$-plane of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 2) $(y \geq 0)\text{-plane} = \{[x, y] : y \geq 0\}$.

We now state several propositions:

(18) For all sets a, b holds $\langle a, b \rangle \in (y = 0)$-line iff $a \in \mathbb{R}$ and $b = 0$.

(19) For all real numbers a, b holds $[a, b] \in (y = 0)$-line iff $b = 0$.

(20) $\overline{\overline{(y = 0)\text{-line}}} = \mathfrak{c}$.

(21) For all sets a, b holds $\langle a, b\rangle \in (y \geq 0)$-plane iff $a \in \mathbb{R}$ and there exists y such that $b = y$ and $y \geq 0$.

(22) For all real numbers a, b holds $[a, b] \in (y \geq 0)$-plane iff $b \geq 0$.

Let us note that $(y = 0)$-line is non empty and $(y \geq 0)$-plane is non empty.

We now state several propositions:

(23) $(y = 0)$-line $\subseteq (y \geq 0)$-plane.

(24) For all real numbers a, b, r such that $r > 0$ holds $\mathrm{Ball}([a, b], r) \subseteq (y \geq 0)$-plane iff $r \leq b$.

(25) For all real numbers a, b, r such that $r > 0$ and $b \geq 0$ holds $\mathrm{Ball}([a, b], r)$ misses $(y = 0)$-line iff $r \leq b$.

(26) Let n be a natural number, a, b be elements of $\mathcal{E}^n_{\mathrm{T}}$, and r_1, r_2 be positive real numbers. If $|a - b| \leq r_1 - r_2$, then $\mathrm{Ball}(b, r_2) \subseteq \mathrm{Ball}(a, r_1)$.

(27) For every real number a and for all positive real numbers r_1, r_2 such that $r_1 \leq r_2$ holds $\mathrm{Ball}([a, r_1], r_1) \subseteq \mathrm{Ball}([a, r_2], r_2)$.

(28) Let T_1, T_2 be non empty topological spaces, B_1 be a neighborhood system of T_1, and B_2 be a neighborhood system of T_2. Suppose $B_1 = B_2$. Then the topological structure of $T_1 =$ the topological structure of T_2.

In the sequel r is an element of $\mathbb{R}$.

Niemytzki plane is a strict non empty topological space and is defined by the conditions (Def. 3).

(Def. 3)(i) The carrier of Niemytzki plane $= (y \geq 0)$-plane, and

(ii) there exists a neighborhood system B of Niemytzki plane such that for every x holds $B([x, 0]) = \{\mathrm{Ball}([x, r], r) \cup \{[x, 0]\} : r > 0\}$ and for all x, y such that $y > 0$ holds $B([x, y]) = \{\mathrm{Ball}([x, y], r) \cap (y \geq 0)\text{-plane} : r > 0\}$.

The following propositions are true:

(29) $(y \geq 0)$-plane $\setminus$ $(y = 0)$-line is an open subset of Niemytzki plane.

(30) $(y = 0)$-line is a closed subset of Niemytzki plane.

(31) Let x be a real number and r be a positive real number. Then $\mathrm{Ball}([x, r], r) \cup \{[x, 0]\}$ is an open subset of Niemytzki plane.

(32) Let x be a real number and y, r be positive real numbers. Then $\mathrm{Ball}([x, y], r) \cap (y \geq 0)$-plane is an open subset of Niemytzki plane.

(33) Let x, y be real numbers and r be a positive real number. If $r \leq y$, then $\mathrm{Ball}([x, y], r)$ is an open subset of Niemytzki plane.

(34) Let p be a point of Niemytzki plane and r be a positive real number. Then there exists a point a of $\mathcal{E}^2_{\mathrm{T}}$ and there exists an open subset U of Niemytzki plane such that $p \in U$ and $a \in U$ and for every point b of $\mathcal{E}^2_{\mathrm{T}}$ such that $b \in U$ holds $|b - a| < r$.

(35) Let x, y be real numbers and r be a positive real number. Then there exist rational numbers w, v such that $[w, v] \in \mathrm{Ball}([x, y], r)$ and $[w, v] \neq [x,$

$y]$.

(36) Let A be a subset of Niemytzki plane. If $A = ((y \geq 0)\text{-plane} \setminus (y = 0)\text{-line}) \cap \prod\langle\mathbb{Q}, \mathbb{Q}\rangle$, then for every set x holds $\overline{A \setminus \{x\}} = \Omega_{\text{Niemytzki plane}}$.

(37) Let A be a subset of Niemytzki plane. If $A = (y \geq 0)\text{-plane} \setminus (y = 0)\text{-line}$, then for every set x holds $\overline{A \setminus \{x\}} = \Omega_{\text{Niemytzki plane}}$.

(38) For every subset A of Niemytzki plane such that $A = (y \geq 0)\text{-plane} \setminus (y = 0)\text{-line}$ holds $\overline{A} = \Omega_{\text{Niemytzki plane}}$.

(39) For every subset A of Niemytzki plane such that $A = (y = 0)\text{-line}$ holds $\overline{A} = A$ and $\operatorname{Int} A = \emptyset$.

(40) $((y \geq 0)\text{-plane} \setminus (y = 0)\text{-line}) \cap \prod\langle\mathbb{Q}, \mathbb{Q}\rangle$ is a dense subset of Niemytzki plane.

(41) $((y \geq 0)\text{-plane} \setminus (y = 0)\text{-line}) \cap \prod\langle\mathbb{Q}, \mathbb{Q}\rangle$ is a dense-in-itself subset of Niemytzki plane.

(42) $(y \geq 0)\text{-plane} \setminus (y = 0)\text{-line}$ is a dense subset of Niemytzki plane.

(43) $(y \geq 0)\text{-plane} \setminus (y = 0)\text{-line}$ is a dense-in-itself subset of Niemytzki plane.

(44) $(y = 0)\text{-line}$ is a nowhere dense subset of Niemytzki plane.

(45) For every subset A of Niemytzki plane such that $A = (y = 0)\text{-line}$ holds $\operatorname{Der} A$ is empty.

(46) Every subset of $(y = 0)\text{-line}$ is a closed subset of Niemytzki plane.

(47) $\mathbb{Q}$ is a dense subset of Sorgenfrey line.

(48) Sorgenfrey line is separable.

(49) Niemytzki plane is separable.

(50) Niemytzki plane is a T_1 space.

(51) Niemytzki plane is not T_4.

3. Tychonoff Spaces

Let T be a topological space. We say that T is Tychonoff if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) T is a T_1 space, and

(ii) for every closed subset A of T and for every point a of T such that $a \in A^{c}$ there exists a continuous function f from T into $\mathbb{I}$ such that $f(a) = 0$ and $f^{\circ}A \subseteq \{1\}$.

Let us observe that every topological space which is Tychonoff is also T_1 and T_3 and every non empty topological space which is T_1 and T_4 is also Tychonoff.

We now state the proposition

(52) Let X be a T_1 topological space. Suppose X is Tychonoff. Let B be a prebasis of X, x be a point of X, and V be a subset of X. Suppose $x \in V$

and $V \in B$. Then there exists a continuous function f from X into $\mathbb{I}$ such that $f(x) = 0$ and $f^{\circ}V^{\mathrm{c}} \subseteq \{1\}$.

Let X be a set and let Y be a non empty real-membered set. Observe that every relation between X and Y is real-yielding.

The following propositions are true:

(53) Let X be a topological space, R be a non empty subspace of $\mathbb{R}^{\mathbf{1}}$, f, g be continuous functions from X into R, and A be a subset of X. Suppose that for every point x of X holds $x \in A$ iff $f(x) \leq g(x)$. Then A is closed.

(54) Let X be a topological space, R be a non empty subspace of $\mathbb{R}^{\mathbf{1}}$, and f, g be continuous functions from X into R. Then there exists a continuous function h from X into R such that for every point x of X holds $h(x) = \max(f(x), g(x))$.

(55) Let X be a non empty topological space, R be a non empty subspace of $\mathbb{R}^{\mathbf{1}}$, A be a finite non empty set, and F be a many sorted function indexed by A. Suppose that for every set a such that $a \in A$ holds $F(a)$ is a continuous function from X into R. Then there exists a continuous function f from X into R such that for every point x of X and for every finite non empty subset S of $\mathbb{R}$ if $S = \mathrm{rng}(\mathrm{commute}(F))(x)$, then $f(x) = \max S$.

(56) Let X be a T_1 non empty topological space and B be a prebasis of X. Suppose that for every point x of X and for every subset V of X such that $x \in V$ and $V \in B$ there exists a continuous function f from X into $\mathbb{I}$ such that $f(x) = 0$ and $f^{\circ}V^{\mathrm{c}} \subseteq \{1\}$. Then X is Tychonoff.

(57) Sorgenfrey line is a T_1 space.

(58) For every real number x holds $]-\infty, x[$ is a closed subset of Sorgenfrey line.

(59) For every real number x holds $]-\infty, x]$ is a closed subset of Sorgenfrey line.

(60) For every real number x holds $[x, +\infty[$ is a closed subset of Sorgenfrey line.

(61) For all real numbers x, y holds $[x, y[$ is a closed subset of Sorgenfrey line.

(62) Let x be a real number and w be a rational number. Suppose $x < w$. Then there exists a continuous function f from Sorgenfrey line into $\mathbb{I}$ such that for every point a of Sorgenfrey line holds

(i) if $a \in [x, w[$, then $f(a) = 0$, and

(ii) if $a \notin [x, w[$, then $f(a) = 1$.

(63) Sorgenfrey line is Tychonoff.

4. Niemytzki Plane is Tychonoff Space

Let x be a real number and let r be a positive real number. The functor $+(x,r)$ yielding a function from Niemytzki plane into $\mathbb{I}$ is defined by the conditions (Def. 5).

(Def. 5)(i) $(+(x,r))([x,0]) = 0$, and

(ii) for every real number a and for every non negative real number b holds if $a \neq x$ or $b \neq 0$ and if $[a,b] \notin \mathrm{Ball}([x,r],r)$, then $(+(x,r))([a,b]) = 1$ and if $[a,b] \in \mathrm{Ball}([x,r],r)$, then $(+(x,r))([a,b]) = \frac{|[x,0]-[a,b]|^{\mathbf{2}}}{2\cdot r\cdot b}$.

One can prove the following propositions:

(64) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} \geq 0$. Let x be a real number and r be a positive real number. If $(+(x,r))(p) = 0$, then $p = [x,0]$.

(65) For all real numbers x, y and for every positive real number r such that $x \neq y$ holds $(+(x,r))([y,0]) = 1$.

(66) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$, x be a real number, and a, r be positive real numbers. If $a \leq 1$ and $|p-[x,r\cdot a]| = r\cdot a$ and $p_{\mathbf{2}} \neq 0$, then $(+(x,r))(p) = a$.

(67) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$, x, a be real numbers, and r be a positive real number. If $0 \leq a$ and $a \leq 1$ and $|p-[x,r\cdot a]| < r\cdot a$, then $(+(x,r))(p) < a$.

(68) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} \geq 0$. Let x, a be real numbers and r be a positive real number. If $0 \leq a$ and $a < 1$ and $|p-[x,r\cdot a]| > r\cdot a$, then $(+(x,r))(p) > a$.

(69) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} \geq 0$. Let x, a, b be real numbers and r be a positive real number. Suppose $0 \leq a$ and $b \leq 1$ and $(+(x,r))(p) \in\,]a,b[$. Then there exists a positive real number r_1 such that $r_1 \leq p_{\mathbf{2}}$ and $\mathrm{Ball}(p,r_1) \subseteq (+(x,r))^{-1}(]a,b[)$.

(70) For every real number x and for all positive real numbers a, r holds $\mathrm{Ball}([x,r\cdot a],r\cdot a) \subseteq (+(x,r))^{-1}(]0,a[)$.

(71) For every real number x and for all positive real numbers a, r holds $\mathrm{Ball}([x,r\cdot a],r\cdot a) \cup \{[x,0]\} \subseteq (+(x,r))^{-1}([0,a[)$.

(72) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} \geq 0$. Let x, a be real numbers and r be a positive real number. If $0 < (+(x,r))(p)$ and $(+(x,r))(p) < a$ and $a \leq 1$, then $p \in \mathrm{Ball}([x,r\cdot a],r\cdot a)$.

(73) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} > 0$. Let x, a be real numbers and r be a positive real number. Suppose $0 \leq a$ and $a < (+(x,r))(p)$. Then there exists a positive real number r_1 such that $r_1 \leq p_{\mathbf{2}}$ and $\mathrm{Ball}(p,r_1) \subseteq (+(x,r))^{-1}(]a,1])$.

(74) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} = 0$. Let x be a real number and r be a positive real number. Suppose $(+(x,r))(p) = 1$. Then there exists a positive real number r_1 such that $\mathrm{Ball}([p_{\mathbf{1}},r_1],r_1) \cup \{p\} \subseteq (+(x,r))^{-1}(\{1\})$.

(75) Let T be a non empty topological space, S be a subspace of T, and B be a basis of T. Then $\{A \cap \Omega_S; A$ ranges over subsets of T: $A \in B \ \wedge \ A$ meets $\Omega_S\}$ is a basis of S.

(76) $\{]a, b[; a$ ranges over real numbers, b ranges over real numbers: $a < b\}$ is a basis of $\mathbb{R}^{\mathbf{1}}$.

(77) Let T be a topological space, U, V be subsets of T, and B be a set. If $U \in B$ and $V \in B$ and $B \cup \{U \cup V\}$ is a basis of T, then B is a basis of T.

(78) $\{[0, a[; a$ ranges over real numbers: $0 < a \ \wedge \ a \leq 1\} \cup \{]a, 1]; a$ ranges over real numbers: $0 \leq a \ \wedge \ a < 1\} \cup \{]a, b[; a$ ranges over real numbers, b ranges over real numbers: $0 \leq a \ \wedge \ a < b \ \wedge \ b \leq 1\}$ is a basis of $\mathbb{I}$.

(79) Let T be a non empty topological space and f be a function from T into $\mathbb{I}$. Then f is continuous if and only if for all real numbers a, b such that $0 \leq a$ and $a < 1$ and $0 < b$ and $b \leq 1$ holds $f^{-1}([0, b[)$ is open and $f^{-1}(]a, 1])$ is open.

Let x be a real number and let r be a positive real number. Note that $+(x, r)$ is continuous.

We now state the proposition

(80) Let U be a subset of Niemytzki plane and given x, r. Suppose $U = \mathrm{Ball}([x, r], r) \cup \{[x, 0]\}$. Then there exists a continuous function f from Niemytzki plane into $\mathbb{I}$ such that

(i) $f([x, 0]) = 0$, and

(ii) for all real numbers a, b holds if $[a, b] \in U^{\mathrm{c}}$, then $f([a, b]) = 1$ and if $[a, b] \in U \setminus \{[x, 0]\}$, then $f([a, b]) = \frac{|[x,0]-[a,b]|^{\mathbf{2}}}{2 \cdot r \cdot b}$.

Let x, y be real numbers and let r be a positive real number. The functor $+(x, y, r)$ yields a function from Niemytzki plane into $\mathbb{I}$ and is defined by the condition (Def. 6).

(Def. 6) Let a be a real number and b be a non negative real number. Then

(i) if $[a, b] \notin \mathrm{Ball}([x, y], r)$, then $(+(x, y, r))([a, b]) = 1$, and

(ii) if $[a, b] \in \mathrm{Ball}([x, y], r)$, then $(+(x, y, r))([a, b]) = \frac{|[x,y]-[a,b]|}{r}$.

The following propositions are true:

(81) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} \geq 0$. Let x be a real number, y be a non negative real number, and r be a positive real number. Then $(+(x, y, r))(p) = 0$ if and only if $p = [x, y]$.

(82) Let x be a real number, y be a non negative real number, and r, a be positive real numbers. If $a \leq 1$, then $(+(x, y, r))^{-1}([0, a[) = \mathrm{Ball}([x, y], r \cdot a) \cap (y \geq 0)$-plane.

(83) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} > 0$. Let x be a real number, a be a non negative real number, and y, r be positive real numbers. If $(+(x, y, r))(p) > a$, then $|[x, y] - p| > r \cdot a$ and $\mathrm{Ball}(p, |[x, y] - p| - r \cdot a) \cap (y \geq 0)$-plane $\subseteq (+(x, y, r))^{-1}(]a, 1])$.

(84) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $p_{\mathbf{2}} = 0$. Let x be a real number, a be a non negative real number, and y, r be positive real numbers. Suppose $(+(x, y, r))(p) > a$. Then $|[x, y] - p| > r \cdot a$ and there exists a positive real number r_1 such that $r_1 = \frac{|[x,y]-p|-r \cdot a}{2}$ and $\mathrm{Ball}([p_{\mathbf{1}}, r_1], r_1) \cup \{p\} \subseteq (+(x, y, r))^{-1}(]a, 1])$.

Let x be a real number and let y, r be positive real numbers. One can verify that $+(x, y, r)$ is continuous.

We now state three propositions:

(85) Let U be a subset of Niemytzki plane and given x, y, r. Suppose $y > 0$ and $U = \mathrm{Ball}([x, y], r) \cap (y \geq 0)$-plane. Then there exists a continuous function f from Niemytzki plane into $\mathbb{I}$ such that $f([x, y]) = 0$ and for all real numbers a, b holds if $[a, b] \in U^{\mathrm{c}}$, then $f([a, b]) = 1$ and if $[a, b] \in U$, then $f([a, b]) = \frac{|[x,y]-[a,b]|}{r}$.

(86) Niemytzki plane is a T_1 space.

(87) Niemytzki plane is Tychonoff.

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On the Partial Product and Partial Sum of Series and Related Basic Inequalities

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Summary. This article introduced some important inequalities on partial sum and partial product, as well as some basic inequalities.

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The notation and terminology used in this paper are introduced in the following papers: [2], [1], [9], [6], [3], [5], [7], [8], and [4].

For simplicity, we adopt the following rules: a, b, c, d are positive real numbers, m, u, w, x, y, z are real numbers, n, k are natural numbers, and s, s_1 are sequences of real numbers.

Next we state a number of propositions:

(1) $(a+b)\cdot(\frac{1}{a}+\frac{1}{b}) \geq 4$.

(2) $a^4+b^4 \geq a^3\cdot b+a\cdot b^3$.

(3) If $a<b$, then $1<\frac{b+c}{a+c}$.

(4) If $a<b$, then $\frac{a}{b}<\sqrt{\frac{a}{b}}$.

(5) If $a<b$, then $\sqrt{\frac{a}{b}}<\frac{b+\sqrt{\frac{a^{\mathbf{2}}+b^{\mathbf{2}}}{2}}}{a+\sqrt{\frac{a^{\mathbf{2}}+b^{\mathbf{2}}}{2}}}$.

(6) If $a<b$, then $\frac{a}{b}<\frac{b+\sqrt{\frac{a^{\mathbf{2}}+b^{\mathbf{2}}}{2}}}{a+\sqrt{\frac{a^{\mathbf{2}}+b^{\mathbf{2}}}{2}}}$.

(7) $\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{a\cdot b}$.

(8) $\frac{a+b}{2} \leq \sqrt{\frac{a^{\mathbf{2}}+b^{\mathbf{2}}}{2}}$.

(9) $x+y \leq \sqrt{2\cdot(x^{\mathbf{2}}+y^{\mathbf{2}})}$.

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(10) $\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \frac{a+b}{2}$.

(11) $\sqrt{a \cdot b} \leq \sqrt{\frac{a^{\mathbf{2}}+b^{\mathbf{2}}}{2}}$.

(12) $\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{\frac{a^{\mathbf{2}}+b^{\mathbf{2}}}{2}}$.

(13) If $|x| < 1$ and $|y| < 1$, then $|\frac{x+y}{1+x \cdot y}| \leq 1$.

(14) $\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$.

(15) $\frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d} > 1$.

(16) $\frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d} < 2$.

(17) If $a+b > c$ and $b+c > a$ and $a+c > b$, then $\frac{1}{(a+b)-c} + \frac{1}{(b+c)-a} + \frac{1}{(c+a)-b} \geq \frac{9}{a+b+c}$.

(18) $\sqrt{(a+b) \cdot (c+d)} \geq \sqrt{a \cdot c} + \sqrt{b \cdot d}$.

(19) $(a \cdot b + c \cdot d) \cdot (a \cdot c + b \cdot d) \geq 4 \cdot a \cdot b \cdot c \cdot d$.

(20) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$.

(21) If $a \cdot b + b \cdot c + c \cdot a = 1$, then $a+b+c \geq \sqrt{3}$.

(22) $\frac{(b+c)-a}{a} + \frac{(c+a)-b}{b} + \frac{(a+b)-c}{c} \geq 3$.

(23) $(a + \frac{1}{a}) \cdot (b + \frac{1}{b}) \geq (\sqrt{a \cdot b} + \frac{1}{\sqrt{a \cdot b}})^{\mathbf{2}}$.

(24) $\frac{b \cdot c}{a} + \frac{a \cdot c}{b} + \frac{a \cdot b}{c} \geq a+b+c$.

(25) If $x > y$ and $y > z$, then $x^{\mathbf{2}} \cdot y + y^{\mathbf{2}} \cdot z + z^{\mathbf{2}} \cdot x > x \cdot y^{\mathbf{2}} + y \cdot z^{\mathbf{2}} + z \cdot x^{\mathbf{2}}$.

(26) If $a > b$ and $b > c$, then $\frac{b}{a-b} > \frac{c}{a-c}$.

(27) If $b > a$ and $c > d$, then $\frac{c}{c+a} > \frac{d}{d+b}$.

(28) $m \cdot x + z \cdot y \leq \sqrt{m^{\mathbf{2}} + z^{\mathbf{2}}} \cdot \sqrt{x^{\mathbf{2}} + y^{\mathbf{2}}}$.

(29) $(m \cdot x + u \cdot y + w \cdot z)^{\mathbf{2}} \leq (m^{\mathbf{2}} + u^{\mathbf{2}} + w^{\mathbf{2}}) \cdot (x^{\mathbf{2}} + y^{\mathbf{2}} + z^{\mathbf{2}})$.

(30) $\frac{9 \cdot a \cdot b \cdot c}{a^{\mathbf{2}}+b^{\mathbf{2}}+c^{\mathbf{2}}} \leq a+b+c$.

(31) $a+b+c \leq \sqrt{\frac{a^{\mathbf{2}}+a \cdot b+b^{\mathbf{2}}}{3}} + \sqrt{\frac{b^{\mathbf{2}}+b \cdot c+c^{\mathbf{2}}}{3}} + \sqrt{\frac{c^{\mathbf{2}}+c \cdot a+a^{\mathbf{2}}}{3}}$.

(32) $\sqrt{\frac{a^{\mathbf{2}}+a \cdot b+b^{\mathbf{2}}}{3}} + \sqrt{\frac{b^{\mathbf{2}}+b \cdot c+c^{\mathbf{2}}}{3}} + \sqrt{\frac{c^{\mathbf{2}}+c \cdot a+a^{\mathbf{2}}}{3}} \leq \sqrt{\frac{a^{\mathbf{2}}+b^{\mathbf{2}}}{2}} + \sqrt{\frac{b^{\mathbf{2}}+c^{\mathbf{2}}}{2}} + \sqrt{\frac{c^{\mathbf{2}}+a^{\mathbf{2}}}{2}}$.

(33) $\sqrt{\frac{a^{\mathbf{2}}+b^{\mathbf{2}}}{2}} + \sqrt{\frac{b^{\mathbf{2}}+c^{\mathbf{2}}}{2}} + \sqrt{\frac{c^{\mathbf{2}}+a^{\mathbf{2}}}{2}} \leq \sqrt{3 \cdot (a^{\mathbf{2}} + b^{\mathbf{2}} + c^{\mathbf{2}})}$.

(34) $\sqrt{3 \cdot (a^{\mathbf{2}} + b^{\mathbf{2}} + c^{\mathbf{2}})} \leq \frac{b \cdot c}{a} + \frac{c \cdot a}{b} + \frac{a \cdot b}{c}$.

(35) If $a+b = 1$, then $(\frac{1}{a^{\mathbf{2}}} - 1) \cdot (\frac{1}{b^{\mathbf{2}}} - 1) \geq 9$.

(36) If $a+b = 1$, then $a \cdot b + \frac{1}{a \cdot b} \geq \frac{17}{4}$.

(37) If $a+b+c = 1$, then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$.

(38) If $a+b+c = 1$, then $(\frac{1}{a} - 1) \cdot (\frac{1}{b} - 1) \cdot (\frac{1}{c} - 1) \geq 8$.

(39) If $a+b+c = 1$, then $(1 + \frac{1}{a}) \cdot (1 + \frac{1}{b}) \cdot (1 + \frac{1}{c}) \geq 64$.

(40) If $x+y+z = 1$, then $x^{\mathbf{2}} + y^{\mathbf{2}} + z^{\mathbf{2}} \geq \frac{1}{3}$.

(41) If $x+y+z = 1$, then $x \cdot y + y \cdot z + z \cdot x \leq \frac{1}{3}$.

(42) If $a \cdot b \cdot c = 1$, then $\sqrt{a} + \sqrt{b} + \sqrt{c} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

(43) If $a > b$ and $b > c$, then $a^{2 \cdot a} \cdot b^{2 \cdot b} \cdot c^{2 \cdot c} > a^{b+c} \cdot b^{a+c} \cdot c^{a+b}$.

(44) If $n \geq 1$, then $a^{n+1} + b^{n+1} \geq a^n \cdot b + a \cdot b^n$.

(45) If $a^{\mathbf{2}} + b^{\mathbf{2}} = c^{\mathbf{2}}$ and $n \geq 3$, then $a^{n+2} + b^{n+2} < c^{n+2}$.

(46) If $n \geq 1$, then $(1 + \frac{1}{n+1})^n < (1 + \frac{1}{n})^{n+1}$.

(47) If $n \geq 1$ and $k \geq 1$, then $(a^k + b^k) \cdot (a^n + b^n) \leq 2 \cdot (a^{k+n} + b^{k+n})$.

(48) If for every n holds $s(n) = \frac{1}{\sqrt{n+1}}$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) < 2 \cdot \sqrt{n+1}$.

(49) If for every n holds $s(n) = \frac{1}{(n+1)^{\mathbf{2}}}$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) \leq 2 - \frac{1}{n+1}$.

(50) If for every n holds $s(n) = \frac{1}{(n+1)^{\mathbf{2}}}$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) < 2$.

(51) If for every n holds $s(n) < 1$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) < n + 1$.

(52) If for every n holds $s(n) > 0$ and $s(n) < 1$, then for every n holds (the partial product of $s)(n) \geq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) - n$.

(53) If for every n holds $s(n) > 0$ and $s_1(n) = \frac{1}{s(n)}$, then for every n holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}(n) > 0$.

(54) If for every n holds $s(n) > 0$ and $s_1(n) = \frac{1}{s(n)}$, then for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}(n) \geq (n+1)^{\mathbf{2}}$.

(55) If for every n such that $n \geq 1$ holds $s(n) = \sqrt{n}$ and $s(0) = 0$, then for every n such that $n \geq 1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) < \frac{1}{6} \cdot (4 \cdot n + 3) \cdot \sqrt{n}$.

(56) If for every n such that $n \geq 1$ holds $s(n) = \sqrt{n}$ and $s(0) = 0$, then for every n such that $n \geq 1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) > \frac{2}{3} \cdot n \cdot \sqrt{n}$.

(57) Suppose that for every n such that $n \geq 1$ holds $s(n) = 1 + \frac{1}{2 \cdot n+1}$ and $s(0) = 1$. Let given n. If $n \geq 1$, then (the partial product of $s)(n) > \frac{1}{2} \cdot \sqrt{2 \cdot n + 3}$.

(58) If for every n such that $n \geq 1$ holds $s(n) = \sqrt{n \cdot (n+1)}$ and $s(0) = 0$, then for every n such that $n \geq 1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) > \frac{n \cdot (n+1)}{2}$.

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Several Differentiable Formulas of Special Functions. Part II

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Summary. In this article, we give several other differentiable formulas of special functions.

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The papers [11], [13], [14], [1], [8], [10], [2], [4], [7], [5], [6], [9], [15], [3], and [12] provide the notation and terminology for this paper.

For simplicity, we use the following convention: x, a denote real numbers, n denotes a natural number, Z denotes an open subset of $\mathbb{R}$, and f, f_1, f_2 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

One can prove the following propositions:

(1) If $a > 0$, then $\exp(x \cdot \log_e a) = a^x_{\mathbb{R}}$.

(2) If $a > 0$, then $\exp(-x \cdot \log_e a) = a^{-x}_{\mathbb{R}}$.

(3) Suppose $Z \subseteq \mathrm{dom}(f_1 - f_2)$ and for every x such that $x \in Z$ holds $f_1(x) = a^{\mathbf{2}}$ and $f_2 = \frac{2}{\mathbb{Z}}$. Then $f_1 - f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 - f_2)'_{\restriction Z}(x) = -2 \cdot x$.

(4) Suppose $Z \subseteq \mathrm{dom}(\frac{f_1+f_2}{f_1-f_2})$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = a^{\mathbf{2}}$ and $(f_1 - f_2)(x) \neq 0$. Then $\frac{f_1+f_2}{f_1-f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1+f_2}{f_1-f_2})'_{\restriction Z}(x) = \frac{4 \cdot a^{\mathbf{2}} \cdot x}{(a^{\mathbf{2}} - x^2)^{\mathbf{2}}}$.

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(5) Suppose $Z \subseteq \operatorname{dom} f$ and $f = \log\_(e) \cdot \frac{f_1+f_2}{f_1-f_2}$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = a^{\mathbf{2}}$ and $(f_1-f_2)(x) > 0$ and $a \neq 0$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{\restriction Z}(x) = \frac{4 \cdot a^{\mathbf{2}} \cdot x}{a^4 - x^4}$.

(6) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{4 \cdot a^{\mathbf{2}}}\, f)$ and $f = \log\_(e) \cdot \frac{f_1+f_2}{f_1-f_2}$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_1(x) = a^{\mathbf{2}}$ and $(f_1 - f_2)(x) > 0$ and $a \neq 0$. Then $\frac{1}{4 \cdot a^{\mathbf{2}}}\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{4 \cdot a^{\mathbf{2}}}\, f)'_{\restriction Z}(x) = \frac{x}{a^4 - x^4}$.

(7) Suppose $Z \subseteq \operatorname{dom}(\frac{f_1}{f_2+f_1})$ and $f_1 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_2(x) = 1$ and $x \neq 0$. Then $\frac{f_1}{f_2+f_1}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2+f_1})'_{\restriction Z}(x) = \frac{2 \cdot x}{(1+x^{\mathbf{2}})^{\mathbf{2}}}$.

(8) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{2}\, f)$ and $f = \log\_(e) \cdot \frac{f_1}{f_2+f_1}$ and $f_1 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f_2(x) = 1$ and $x \neq 0$. Then $\frac{1}{2}\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{2}\, f)'_{\restriction Z}(x) = \frac{1}{x \cdot (1+x^{\mathbf{2}})}$.

(9) Suppose $Z \subseteq \operatorname{dom}(\log\_(e) \cdot \frac{n}{\mathbb{Z}})$ and for every x such that $x \in Z$ holds $x > 0$. Then $\log\_(e) \cdot \frac{n}{\mathbb{Z}}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot \frac{n}{\mathbb{Z}})'_{\restriction Z}(x) = \frac{n}{x}$.

(10) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{f_2} + \log\_(e) \cdot \frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_2(x) = x$ and $f_2(x) > 0$ and $f_1(x) = x - 1$ and $f_1(x) > 0$. Then $\frac{1}{f_2} + \log\_(e) \cdot \frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f_2} + \log\_(e) \cdot \frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{1}{x^{\mathbf{2}} \cdot (x-1)}$.

(11) Suppose $Z \subseteq \operatorname{dom}(\exp \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = x \cdot \log_e a$ and $a > 0$. Then $\exp \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\exp \cdot f)'_{\restriction Z}(x) = (a^x_{\mathbb{R}}) \cdot \log_e a$.

(12) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{\log_e a}\, ((\exp \cdot f_1)\, f_2))$ and for every x such that $x \in Z$ holds $f_1(x) = x \cdot \log_e a$ and $f_2(x) = x - \frac{1}{\log_e a}$ and $a > 0$ and $a \neq 1$. Then $\frac{1}{\log_e a}\, ((\exp \cdot f_1)\, f_2)$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{\log_e a}\, ((\exp \cdot f_1)\, f_2))'_{\restriction Z}(x) = x \cdot a^x_{\mathbb{R}}$.

(13) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{1+\log_e a}\, ((\exp \cdot f)\, \exp))$ and for every x such that $x \in Z$ holds $f(x) = x \cdot \log_e a$ and $a > 0$ and $a \neq \frac{1}{e}$. Then $\frac{1}{1+\log_e a}\, ((\exp \cdot f)\, \exp)$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{1+\log_e a}\, ((\exp \cdot f)\, \exp))'_{\restriction Z}(x) = (a^x_{\mathbb{R}}) \cdot \exp(x)$.

(14) Suppose $Z \subseteq \operatorname{dom}(\exp \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = -x$. Then $\exp \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\exp \cdot f)'_{\restriction Z}(x) = -\exp(-x)$.

(15) Suppose $Z \subseteq \operatorname{dom}(f_1\, (\exp \cdot f_2))$ and for every x such that $x \in Z$ holds $f_1(x) = -x - 1$ and $f_2(x) = -x$. Then $f_1\, (\exp \cdot f_2)$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1\, (\exp \cdot f_2))'_{\restriction Z}(x) = \frac{x}{\exp x}$.

(16) Suppose $Z \subseteq \operatorname{dom}(-\exp \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = -x \cdot \log_e a$ and $a > 0$. Then $-\exp \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $(-\exp \cdot f)'_{\restriction Z}(x) = (a_{\mathbb{R}}^{-x}) \cdot \log_e a$.

(17) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{\log_e a}((-\exp \cdot f_1)\, f_2))$ and for every x such that $x \in Z$ holds $f_1(x) = -x \cdot \log_e a$ and $f_2(x) = x + \frac{1}{\log_e a}$ and $a > 0$ and $a \neq 1$. Then $\frac{1}{\log_e a}((-\exp \cdot f_1)\, f_2)$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{\log_e a}((-\exp \cdot f_1)\, f_2))'_{\restriction Z}(x) = \frac{x}{a_{\mathbb{R}}^{x}}$.

(18) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{\log_e a - 1}\, \frac{\exp \cdot f}{\exp})$ and for every x such that $x \in Z$ holds $f(x) = x \cdot \log_e a$ and $a > 0$ and $a \neq e$. Then $\frac{1}{\log_e a - 1}\, \frac{\exp \cdot f}{\exp}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{\log_e a - 1}\, \frac{\exp \cdot f}{\exp})'_{\restriction Z}(x) = \frac{a_{\mathbb{R}}^{x}}{\exp(x)}$.

(19) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{1 - \log_e a}\, \frac{\exp}{\exp \cdot f})$ and for every x such that $x \in Z$ holds $f(x) = x \cdot \log_e a$ and $a > 0$ and $a \neq e$. Then $\frac{1}{1 - \log_e a}\, \frac{\exp}{\exp \cdot f}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{1 - \log_e a}\, \frac{\exp}{\exp \cdot f})'_{\restriction Z}(x) = \frac{\exp(x)}{a_{\mathbb{R}}^{x}}$.

(20) Suppose $Z \subseteq \operatorname{dom}(\log\_(e) \cdot (\exp + f))$ and for every x such that $x \in Z$ holds $f(x) = 1$. Then $\log\_(e) \cdot (\exp + f)$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot (\exp + f))'_{\restriction Z}(x) = \frac{\exp(x)}{\exp(x) + 1}$.

(21) Suppose $Z \subseteq \operatorname{dom}(\log\_(e) \cdot (\exp - f))$ and for every x such that $x \in Z$ holds $f(x) = 1$ and $(\exp - f)(x) > 0$. Then $\log\_(e) \cdot (\exp - f)$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot (\exp - f))'_{\restriction Z}(x) = \frac{\exp(x)}{\exp(x) - 1}$.

(22) Suppose $Z \subseteq \operatorname{dom}(-\log\_(e) \cdot (f - \exp))$ and for every x such that $x \in Z$ holds $f(x) = 1$ and $(f - \exp)(x) > 0$. Then $-\log\_(e) \cdot (f - \exp)$ is differentiable on Z and for every x such that $x \in Z$ holds $(-\log\_(e) \cdot (f - \exp))'_{\restriction Z}(x) = \frac{\exp(x)}{1 - \exp(x)}$.

(23) Suppose $Z \subseteq \operatorname{dom}((\frac{2}{\mathbb{Z}}) \cdot \exp + f)$ and for every x such that $x \in Z$ holds $f(x) = 1$. Then $(\frac{2}{\mathbb{Z}}) \cdot \exp + f$ is differentiable on Z and for every x such that $x \in Z$ holds $((\frac{2}{\mathbb{Z}}) \cdot \exp + f)'_{\restriction Z}(x) = 2 \cdot \exp(2 \cdot x)$.

(24) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{2}(\log\_(e) \cdot f))$ and $f = (\frac{2}{\mathbb{Z}}) \cdot \exp + f_1$ and for every x such that $x \in Z$ holds $f_1(x) = 1$. Then $\frac{1}{2}(\log\_(e) \cdot f)$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{2}(\log\_(e) \cdot f))'_{\restriction Z}(x) = \frac{\exp x}{\exp x + \exp(-x)}$.

(25) Suppose $Z \subseteq \operatorname{dom}((\frac{2}{\mathbb{Z}}) \cdot \exp - f)$ and for every x such that $x \in Z$ holds $f(x) = 1$. Then $(\frac{2}{\mathbb{Z}}) \cdot \exp - f$ is differentiable on Z and for every x such that $x \in Z$ holds $((\frac{2}{\mathbb{Z}}) \cdot \exp - f)'_{\restriction Z}(x) = 2 \cdot \exp(2 \cdot x)$.

(26) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{2}(\log\_(e) \cdot f))$ and $f = (\frac{2}{\mathbb{Z}}) \cdot \exp - f_1$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f(x) > 0$. Then $\frac{1}{2}(\log\_(e) \cdot f)$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{2}(\log\_(e) \cdot f))'_{\restriction Z}(x) = \frac{\exp x}{\exp x - \exp(-x)}$.

(27) Suppose $Z \subseteq \mathrm{dom}((\frac{2}{\mathbb{Z}}) \cdot (\exp - f))$ and for every x such that $x \in Z$ holds $f(x) = 1$. Then $(\frac{2}{\mathbb{Z}}) \cdot (\exp - f)$ is differentiable on Z and for every x such that $x \in Z$ holds $((\frac{2}{\mathbb{Z}}) \cdot (\exp - f))'_{\restriction Z}(x) = 2 \cdot \exp(x) \cdot (\exp(x) - 1)$.

(28) Suppose $Z \subseteq \mathrm{dom}\, f$ and $f = \log\_(e) \cdot \frac{(\frac{2}{\mathbb{Z}}) \cdot (\exp - f_1)}{\exp}$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $(\exp - f_1)(x) > 0$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{\restriction Z}(x) = \frac{\exp(x)+1}{\exp(x)-1}$.

(29) Suppose $Z \subseteq \mathrm{dom}((\frac{2}{\mathbb{Z}}) \cdot (\exp + f))$ and for every x such that $x \in Z$ holds $f(x) = 1$. Then $(\frac{2}{\mathbb{Z}}) \cdot (\exp + f)$ is differentiable on Z and for every x such that $x \in Z$ holds $((\frac{2}{\mathbb{Z}}) \cdot (\exp + f))'_{\restriction Z}(x) = 2 \cdot \exp(x) \cdot (\exp(x) + 1)$.

(30) Suppose $Z \subseteq \mathrm{dom}\, f$ and $f = \log\_(e) \cdot \frac{(\frac{2}{\mathbb{Z}}) \cdot (\exp + f_1)}{\exp}$ and for every x such that $x \in Z$ holds $f_1(x) = 1$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{\restriction Z}(x) = \frac{\exp(x)-1}{\exp(x)+1}$.

(31) Suppose $Z \subseteq \mathrm{dom}((\frac{2}{\mathbb{Z}}) \cdot (f - \exp))$ and for every x such that $x \in Z$ holds $f(x) = 1$. Then $(\frac{2}{\mathbb{Z}}) \cdot (f - \exp)$ is differentiable on Z and for every x such that $x \in Z$ holds $((\frac{2}{\mathbb{Z}}) \cdot (f - \exp))'_{\restriction Z}(x) = -2 \cdot \exp(x) \cdot (1 - \exp(x))$.

(32) Suppose $Z \subseteq \mathrm{dom}\, f$ and $f = \log\_(e) \cdot \frac{\exp}{(\frac{2}{\mathbb{Z}}) \cdot (f_1 - \exp)}$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $(f_1 - \exp)(x) > 0$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{\restriction Z}(x) = \frac{1+\exp(x)}{1-\exp(x)}$.

(33) Suppose $Z \subseteq \mathrm{dom}\, f$ and $f = \log\_(e) \cdot \frac{\exp}{(\frac{2}{\mathbb{Z}}) \cdot (f_1 + \exp)}$ and for every x such that $x \in Z$ holds $f_1(x) = 1$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{\restriction Z}(x) = \frac{1-\exp(x)}{1+\exp(x)}$.

(34) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot f)$ and $f = \exp + \exp \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = -x$. Then $\log\_(e) \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot f)'_{\restriction Z}(x) = \frac{\exp x - \exp(-x)}{\exp x + \exp(-x)}$.

(35) Suppose $Z \subseteq \mathrm{dom}(\log\_(e) \cdot f)$ and $f = \exp - \exp \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = -x$ and $f(x) > 0$. Then $\log\_(e) \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot f)'_{\restriction Z}(x) = \frac{\exp x + \exp(-x)}{\exp x - \exp(-x)}$.

(36) Suppose $Z \subseteq \mathrm{dom}(\frac{2}{3}\, ((\frac{\frac{3}{2}}{\mathbb{R}}) \cdot (f + \exp)))$ and for every x such that $x \in Z$ holds $f(x) = 1$. Then $\frac{2}{3}\, ((\frac{\frac{3}{2}}{\mathbb{R}}) \cdot (f + \exp))$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{2}{3}\, ((\frac{\frac{3}{2}}{\mathbb{R}}) \cdot (f + \exp)))'_{\restriction Z}(x) = \exp(x) \cdot (1 + \exp(x))^{\frac{1}{2}}_{\mathbb{R}}$.

(37) Suppose $Z \subseteq \mathrm{dom}(\frac{2}{3 \cdot \log_e a}\, ((\frac{\frac{3}{2}}{\mathbb{R}}) \cdot (f + \exp \cdot f_1)))$ and for every x such that $x \in Z$ holds $f(x) = 1$ and $f_1(x) = x \cdot \log_e a$ and $a > 0$ and $a \neq 1$. Then $\frac{2}{3 \cdot \log_e a}\, ((\frac{\frac{3}{2}}{\mathbb{R}}) \cdot (f + \exp \cdot f_1))$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{2}{3 \cdot \log_e a}\, ((\frac{\frac{3}{2}}{\mathbb{R}}) \cdot (f + \exp \cdot f_1)))'_{\restriction Z}(x) = (a^{x}_{\mathbb{R}}) \cdot (1 + a^{x}_{\mathbb{R}})^{\frac{1}{2}}_{\mathbb{R}}$.

(38) Suppose $Z \subseteq \mathrm{dom}((-\frac{1}{2})\,((\text{the function cos})\cdot f))$ and for every x such that $x \in Z$ holds $f(x) = 2 \cdot x$. Then

(i) $(-\frac{1}{2})\,((\text{the function cos})\cdot f)$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((-\frac{1}{2})\,((\text{the function cos})\cdot f))'_{\restriction Z}(x) = \sin(2 \cdot x)$.

(39) Suppose that

(i) $Z \subseteq \mathrm{dom}(2\,((\frac{1}{2}_{\mathbb{R}})\cdot(f - \text{the function cos})))$, and

(ii) for every x such that $x \in Z$ holds $f(x) = 1$ and $(\text{the function sin})(x) > 0$ and $(\text{the function cos})(x) < 1$ and $(\text{the function cos})(x) > -1$.

Then

(iii) $2\,((\frac{1}{2}_{\mathbb{R}})\cdot(f - \text{the function cos}))$ is differentiable on Z, and

(iv) for every x such that $x \in Z$ holds $(2\,((\frac{1}{2}_{\mathbb{R}})\cdot(f - \text{the function cos})))'_{\restriction Z}(x) = (1 + (\text{the function cos})(x))^{\frac{1}{2}}_{\mathbb{R}}$.

(40) Suppose that

(i) $Z \subseteq \mathrm{dom}((-2)\,((\frac{1}{2}_{\mathbb{R}})\cdot(f + \text{the function cos})))$, and

(ii) for every x such that $x \in Z$ holds $f(x) = 1$ and $(\text{the function sin})(x) > 0$ and $(\text{the function cos})(x) < 1$ and $(\text{the function cos})(x) > -1$.

Then

(iii) $(-2)\,((\frac{1}{2}_{\mathbb{R}})\cdot(f + \text{the function cos}))$ is differentiable on Z, and

(iv) for every x such that $x \in Z$ holds $((-2)\,((\frac{1}{2}_{\mathbb{R}})\cdot(f + \text{the function cos})))'_{\restriction Z}(x) = (1 - (\text{the function cos})(x))^{\frac{1}{2}}_{\mathbb{R}}$.

(41) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{2}\,(\log\_(e)\cdot f))$ and $f = f_1 + 2\,(\text{the function sin})$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f(x) > 0$. Then

(i) $\frac{1}{2}\,(\log\_(e)\cdot f)$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{2}\,(\log\_(e)\cdot f))'_{\restriction Z}(x) = \frac{(\text{the function cos})(x)}{1+2\cdot(\text{the function sin})(x)}$.

(42) Suppose $Z \subseteq \mathrm{dom}((-\frac{1}{2})\,(\log\_(e)\cdot f))$ and $f = f_1 + 2\,(\text{the function cos})$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f(x) > 0$. Then

(i) $(-\frac{1}{2})\,(\log\_(e)\cdot f)$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((-\frac{1}{2})\,(\log\_(e)\cdot f))'_{\restriction Z}(x) = \frac{(\text{the function sin})(x)}{1+2\cdot(\text{the function cos})(x)}$.

(43) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{4\cdot a}\,((\text{the function sin})\cdot f))$ and for every x such that $x \in Z$ holds $f(x) = 2 \cdot a \cdot x$ and $a \neq 0$. Then

(i) $\frac{1}{4\cdot a}\,((\text{the function sin})\cdot f)$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{4\cdot a}\,((\text{the function sin})\cdot f))'_{\restriction Z}(x) = \frac{1}{2}\cdot\cos(2\cdot a\cdot x)$.

(44) Suppose $Z \subseteq \mathrm{dom}(f_1 - \frac{1}{4\cdot a}\,((\text{the function sin})\cdot f))$ and for every x such that $x \in Z$ holds $f_1(x) = \frac{x}{2}$ and $f(x) = 2 \cdot a \cdot x$ and $a \neq 0$. Then

(i) $f_1 - \frac{1}{4 \cdot a}$ ((the function sin) $\cdot f$) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(f_1 - \frac{1}{4 \cdot a}$ ((the function sin) $\cdot f))'_{\restriction Z}(x) = (\sin(a \cdot x))^{\mathbf{2}}$.

(45) Suppose $Z \subseteq \mathrm{dom}(f_1 + \frac{1}{4 \cdot a}$ ((the function sin) $\cdot f))$ and for every x such that $x \in Z$ holds $f_1(x) = \frac{x}{2}$ and $f(x) = 2 \cdot a \cdot x$ and $a \neq 0$. Then

(i) $f_1 + \frac{1}{4 \cdot a}$ ((the function sin) $\cdot f$) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(f_1 + \frac{1}{4 \cdot a}$ ((the function sin) $\cdot f))'_{\restriction Z}(x) = (\cos(a \cdot x))^{\mathbf{2}}$.

(46) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{n}$ $((\frac{n}{\mathbb{Z}})$ $\cdot$ (the function cos))) and $n > 0$. Then

(i) $\frac{1}{n}$ $((\frac{n}{\mathbb{Z}})$ $\cdot$ (the function cos)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{n}$ $((\frac{n}{\mathbb{Z}})$ $\cdot$ (the function cos)))$'_{\restriction Z}(x) = -(($the function cos$)(x)^{n-1}_{\mathbb{Z}}) \cdot ($the function sin$)(x)$.

(47) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{3}$ $((\frac{3}{\mathbb{Z}})$ $\cdot$ (the function cos))$-$the function cos) and $n > 0$. Then

(i) $\frac{1}{3}$ $((\frac{3}{\mathbb{Z}})$ $\cdot$ (the function cos))$-$the function cos is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{3}$ $((\frac{3}{\mathbb{Z}})$ $\cdot$ (the function cos))$-$the function cos$)'_{\restriction Z}(x) = ($the function sin$)(x)^3$.

(48) Suppose $Z \subseteq \mathrm{dom}(($the function sin$)-\frac{1}{3}$ $((\frac{3}{\mathbb{Z}})$ $\cdot$ (the function sin))) and $n > 0$. Then

(i) (the function sin)$-\frac{1}{3}$ $((\frac{3}{\mathbb{Z}})$ $\cdot$ (the function sin)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin)$-\frac{1}{3}$ $((\frac{3}{\mathbb{Z}})$ $\cdot$ (the function sin)))$'_{\restriction Z}(x) = ($the function cos$)(x)^3$.

(49) Suppose $Z \subseteq \mathrm{dom}(($the function sin$) \cdot \log_(e))$. Then

(i) (the function sin) $\cdot \log\_(e)$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(($the function sin$) \cdot \log_(e))'_{\restriction Z}(x) = \frac{(\text{the function cos})(\log_e x)}{x}$.

(50) Suppose $Z \subseteq \mathrm{dom}(-($the function cos$) \cdot \log_(e))$. Then

(i) $-$(the function cos) $\cdot \log\_(e)$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(-($the function cos$) \cdot \log_(e))'_{\restriction Z}(x) = \frac{(\text{the function sin})(\log_e x)}{x}$.

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On the Calculus of Binary Arithmetics. Part II

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Summary. In this paper, we introduce binary arithmetic and its related operations. We include some theorems concerning logical operators.

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The terminology and notation used in this paper are introduced in the following articles: [4], [3], [2], and [1].

In this paper x, y, z denote boolean sets.

Next we state a number of propositions:

(1) $\mathit{true} \Rightarrow x = x$.

(2) $\mathit{false} \Rightarrow x = \mathit{true}$.

(3) $x \Rightarrow x = \mathit{true}$ and $\neg(x \Rightarrow x) = \mathit{false}$.

(4) $\neg(x \Rightarrow y) = x \wedge \neg y$.

(5) $x \Rightarrow \neg x = \neg x$ and $\neg(x \Rightarrow \neg x) = x$.

(6) $\neg x \Rightarrow x = x$.

(7) $\mathit{true} \Leftrightarrow x = x$.

(8) $\mathit{false} \Leftrightarrow x = \neg x$.

(9) $x \Leftrightarrow x = \mathit{true}$ and $\neg(x \Leftrightarrow x) = \mathit{false}$.

(10) $\neg x \Leftrightarrow x = \mathit{false}$.

(11) $x \wedge (y \Leftrightarrow z) = x \wedge (\neg y \vee z) \wedge (\neg z \vee y)$.

(12) $x \wedge (y \ '\mathrm{nand}'\ z) = x \wedge \neg y \vee x \wedge \neg z$.

(13) $x \wedge (y \ '\mathrm{nor}'\ z) = x \wedge \neg y \wedge \neg z$.

(14) $x \wedge (x \wedge y) = x \wedge y$.

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(15) $x \wedge (x \vee y) = x \vee x \wedge y$.

(16) $x \wedge (x \oplus y) = x \wedge \neg y$.

(17) $x \wedge (x \Rightarrow y) = x \wedge y$.

(18) $x \wedge (x \Leftrightarrow y) = x \wedge y$.

(19) $x \wedge (x\ '\mathrm{nand}'\ y) = x \wedge \neg y$.

(20) $x \wedge (x\ '\mathrm{nor}'\ y) = \mathit{false}$.

(21) $x \vee (y \oplus z) = x \vee \neg y \wedge z \vee y \wedge \neg z$.

(22) $x \vee (y \Leftrightarrow z) = (x \vee \neg y \vee z) \wedge (x \vee \neg z \vee y)$.

(23) $x \vee (y\ '\mathrm{nand}'\ z) = x \vee \neg y \vee \neg z$.

(24) $x \vee (y\ '\mathrm{nor}'\ z) = (x \vee \neg y) \wedge (x \vee \neg z)$ and $x \vee (y\ '\mathrm{nor}'\ z) = (y \Rightarrow x) \wedge (z \Rightarrow x)$.

(25) $x \vee (x \vee y) = x \vee y$.

(26) $x \vee (x \Rightarrow y) = \mathit{true}$.

(27) $x \vee (x \Leftrightarrow y) = y \Rightarrow x$.

(28) $x \vee (x\ '\mathrm{nand}'\ y) = \mathit{true}$.

(29) $x \vee (x\ '\mathrm{nor}'\ y) = y \Rightarrow x$.

(30) $x \Rightarrow y \oplus z = \neg x \vee \neg y \wedge z \vee y \wedge \neg z$.

(31) $x \Rightarrow y \Leftrightarrow z = (\neg x \vee \neg y \vee z) \wedge (\neg x \vee y \vee \neg z)$.

(32) $x \Rightarrow y\ '\mathrm{nand}'\ z = \neg x \vee \neg y \vee \neg z$.

(33) $x \Rightarrow y\ '\mathrm{nor}'\ z = (\neg x \vee \neg y) \wedge (\neg x \vee \neg z)$ and $x \Rightarrow y\ '\mathrm{nor}'\ z = (x \Rightarrow \neg y) \wedge (x \Rightarrow \neg z)$.

(34) $x \Rightarrow x \wedge y = x \Rightarrow y$.

(35) $x \Rightarrow x \vee y = \mathit{true}$.

(36) $x \Rightarrow x \oplus y = \neg x \vee \neg y$.

(37) $x \Rightarrow x \Rightarrow y = x \Rightarrow y$.

(38) $x \Rightarrow x \Leftrightarrow y = x \Rightarrow y$ and $x \Rightarrow x \Leftrightarrow y = x \Rightarrow x \Rightarrow y$.

(39) $x \Rightarrow x\ '\mathrm{nand}'\ y = \neg(x \wedge y)$.

(40) $x \Rightarrow x\ '\mathrm{nor}'\ y = \neg x$.

(41) $x\ '\mathrm{nand}'\ (y \Rightarrow z) = (\neg x \vee y) \wedge (\neg x \vee \neg z)$ and $x\ '\mathrm{nand}'\ (y \Rightarrow z) = (x \Rightarrow y) \wedge (x \Rightarrow \neg z)$.

(42) $x\ '\mathrm{nand}'\ (y \Leftrightarrow z) = \neg(x \wedge (\neg y \vee z) \wedge (\neg z \vee y))$.

(43) $x\ '\mathrm{nand}'\ (y\ '\mathrm{nand}'\ z) = (\neg x \vee y) \wedge (\neg x \vee z)$ and $x\ '\mathrm{nand}'\ (y\ '\mathrm{nand}'\ z) = (x \Rightarrow y) \wedge (x \Rightarrow z)$.

(44) $x\ '\mathrm{nand}'\ (y\ '\mathrm{nor}'\ z) = \neg x \vee y \vee z$.

(45) $x\ '\mathrm{nand}'\ x \wedge y = \neg(x \wedge y)$.

(46) $x\ '\mathrm{nand}'\ (x \oplus y) = x \Rightarrow y$.

(47) $x\ '\mathrm{nand}'\ (x \Rightarrow y) = \neg(x \wedge y)$.

(48) $x\ '\mathrm{nand}'\ (x \Leftrightarrow y) = \neg(x \wedge y)$.

(49) $x\ '\mathrm{nand}'\ (x\ '\mathrm{nand}'\ y) = x \Rightarrow y$.

(50) $x\ '\mathrm{nand}'\ (x\ '\mathrm{nor}'\ y) = \mathit{true}$.

(51) $x\ '\mathrm{nor}'\ (y \oplus z) = \neg(x \vee \neg y \wedge z \vee y \wedge \neg z)$.

(52) $x\ '\mathrm{nor}'\ (y \Leftrightarrow z) = \neg((x \vee \neg y \vee z) \wedge (x \vee \neg z \vee y))$.

(53) $x\ '\mathrm{nor}'\ (y\ '\mathrm{nand}'\ z) = \neg x \wedge y \wedge z$.

(54) $x\ '\mathrm{nor}'\ (y\ '\mathrm{nor}'\ z) = \neg x \wedge y \vee \neg x \wedge z$.

(55) $x\ '\mathrm{nor}'\ x \wedge y = \neg x$.

(56) $x\ '\mathrm{nor}'\ (x \vee y) = \neg x \wedge \neg y$.

(57) $x\ '\mathrm{nor}'\ (x \oplus y) = \neg x \wedge \neg y$.

(58) $x\ '\mathrm{nor}'\ (x \Rightarrow y) = \mathit{false}$.

(59) $x\ '\mathrm{nor}'\ (x \Leftrightarrow y) = \neg x \wedge y$.

(60) $x\ '\mathrm{nor}'\ (x\ '\mathrm{nand}'\ y) = \mathit{false}$.

(61) $x\ '\mathrm{nor}'\ (x\ '\mathrm{nor}'\ y) = \neg x \wedge y$.

(62) $x \oplus y \wedge z = (x \vee y \wedge z) \wedge (\neg x \vee \neg(y \wedge z))$.

(63) $x \oplus x \wedge y = x \wedge \neg y$.

(64) $x \oplus (x \vee y) = \neg x \wedge y$.

(65) $\neg x \wedge (x \oplus y) = \neg x \wedge y$.

(66) $x \wedge \neg(x \oplus y) = x \wedge y$.

(67) $x \oplus (x \oplus y) = y$.

(68) $x \wedge \neg(x \Rightarrow y) = x \wedge \neg y$.

(69) $x \oplus (x \Rightarrow y) = \neg x \vee \neg y$.

(70) $\neg x \wedge (x \Leftrightarrow y) = \neg x \wedge \neg y$.

(71) $x \wedge \neg(x \Leftrightarrow y) = x \wedge \neg y$.

(72) $x \oplus (x \Leftrightarrow y) = \neg y$.

(73) $x \oplus (x\ '\mathrm{nand}'\ y) = x \Rightarrow y$.

(74) $x \oplus (x\ '\mathrm{nor}'\ y) = y \Rightarrow x$.

(75) $\neg x \wedge (x \Rightarrow y) = \neg x \vee \neg x \wedge y$.

(76) $\neg x \wedge (y \Leftrightarrow z) = \neg x \wedge (\neg y \vee z) \wedge (\neg z \vee y)$.

(77) $\neg x \wedge (x \Leftrightarrow y) = \neg x \wedge \neg y \wedge (\neg x \vee y)$.

(78) $\neg x \wedge (x\ '\mathrm{nand}'\ y) = \neg x \vee \neg x \wedge \neg y$.

(79) $\neg x \wedge (x\ '\mathrm{nor}'\ y) = \neg x \wedge \neg y$.

(80) $\neg x \vee (x \Rightarrow y) = \neg x \vee y$.

(81) $\neg x \vee (x \Leftrightarrow y) = \neg x \vee y$.

(82) $\neg x \vee (x\ '\mathrm{nand}'\ y) = \neg x \vee \neg y$.

(83) $\neg x \oplus (x \Rightarrow y) = x \wedge y$.

(84) $\neg x \oplus (y \Rightarrow x) = x \wedge (x \vee \neg y) \vee \neg x \wedge y$.

(85) $\neg(x \Rightarrow y) = x \wedge \neg y$.

(86) $\neg(x \Leftrightarrow y) = x \wedge \neg y \vee y \wedge \neg x$.

(87) $\neg x \oplus (x \Leftrightarrow y) = y$.

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Some Properties of Some Special Matrices

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Summary. This article describes definitions of reversible matrix, symmetrical matrix, antisymmetric matrix, orthogonal matrix and their main properties.

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The terminology and notation used in this paper have been introduced in the following articles: [8], [3], [11], [12], [1], [10], [9], [6], [2], [4], [5], [13], and [7].

For simplicity, we adopt the following convention: n denotes a natural number, K denotes a field, a denotes an element of K, and M, M_1, M_2, M_3, M_4 denote matrices over K of dimension n.

Let n be a natural number, let K be a field, and let M_1, M_2 be matrices over K of dimension n. We say that M_1 is permutable with M_2 if and only if:

(Def. 1) $M_1 \cdot M_2 = M_2 \cdot M_1$.

Let us note that the predicate M_1 is permutable with M_2 is symmetric.

Let n be a natural number, let K be a field, and let M_1, M_2 be matrices over K of dimension n. We say that M_1 is reverse of M_2 if and only if:

(Def. 2) $M_1 \cdot M_2 = M_2 \cdot M_1$ and $M_1 \cdot M_2 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$.

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Let us note that the predicate M_1 is reverse of M_2 is symmetric.

Let n be a natural number, let K be a field, and let M_1 be a matrix over K of dimension n. We say that M_1 is reversible if and only if:

(Def. 3) There exists a matrix M_2 over K of dimension n such that M_1 is reverse of M_2.

Let us consider n, K and let M_1 be a matrix over K of dimension n. Then $-M_1$ is a matrix over K of dimension n.

Let us consider n, K and let M_1, M_2 be matrices over K of dimension n. Then $M_1 + M_2$ is a matrix over K of dimension n.

Let us consider n, K and let M_1, M_2 be matrices over K of dimension n. Then $M_1 - M_2$ is a matrix over K of dimension n.

Let us consider n, K and let M_1, M_2 be matrices over K of dimension n. Then $M_1 \cdot M_2$ is a matrix over K of dimension n.

The following propositions are true:

(1) For every field K and for every matrix A over K such that $\operatorname{len} A > 0$ and $\operatorname{width} A > 0$ holds $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{(\operatorname{len} A)\times(\operatorname{len} A)} \cdot A = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{(\operatorname{len} A)\times(\operatorname{width} A)}$.

(2) For every field K and for every matrix A over K such that $\operatorname{len} A > 0$ and $\operatorname{width} A > 0$ holds $A \cdot \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{(\operatorname{width} A)\times(\operatorname{width} A)} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{(\operatorname{len} A)\times(\operatorname{width} A)}$.

(3) If $n > 0$, then M_1 is permutable with $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times n}$.

(4) If M_1 is permutable with M_2 and M_2 is permutable with M_3 and M_1 is permutable with M_3, then M_1 is permutable with $M_2 \cdot M_3$.

(5) If M_1 is permutable with M_2 and permutable with M_3 and $n > 0$, then M_1 is permutable with $M_2 + M_3$.

(6) M_1 is permutable with $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$.

(7) If M_2 is reverse of M_3 and M_1 is reverse of M_3, then $M_1 = M_2$.

Let n be a natural number, let K be a field, and let M_1 be a matrix over K of dimension n. Let us assume that M_1 is reversible. The functor $M_1{}^\smile$ yields a matrix over K of dimension n and is defined by:

(Def. 4) $M_1{}^\smile$ is reverse of M_1.

We now state a number of propositions:

(8) $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})^\smile = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ and $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ is reversible.

(9) $((\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})^\smile)^\smile = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$.

(10) If $n > 0$, then $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})^{\mathrm{T}} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$.

(11) Let K be a field, n be a natural number, and M be a matrix over K of dimension n. If $M = (\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})^{\mathrm{T}}$ and $n > 0$, then $M^\smile = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$.

(12) If $M_1{}^{\mathrm{T}} = M_2$ and M_3 is reverse of M_1 and $M = M_3{}^{\mathrm{T}}$ and $n > 0$, then M_2 is reverse of M.

(13) If M is reversible and $n > 0$ and $M_1 = M^{\mathrm{T}}$ and $M_2 = (M^\smile)^{\mathrm{T}}$, then $M_1{}^\smile = M_2$.

(14) Let K be a field, n be a natural number, and M_1, M_2, M_3, M_4 be matrices over K of dimension n. If M_3 is reverse of M_1 and M_4 is reverse of M_2, then $M_3 \cdot M_4$ is reverse of $M_2 \cdot M_1$.

(15) Let K be a field, n be a natural number, and M_1, M_2 be matrices over K of dimension n. If M_2 is reverse of M_1, then M_1 is permutable with M_2.

(16) If M is reversible, then $M^\smile$ is reversible and $(M^\smile)^\smile = M$.

(17) If $n > 0$ and $M_1 \cdot M_2 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n \times n}$ and M_1 is reversible, then M_1 is permutable with M_2.

(18) If $n > 0$ and $M_1 = M_1 \cdot M_2$ and M_1 is reversible, then M_1 is permutable with M_2.

(19) If $n > 0$ and $M_1 = M_2 \cdot M_1$ and M_1 is reversible, then M_1 is permutable with M_2.

Let n be a natural number, let K be a field, and let M_1 be a matrix over K of dimension n. We say that M_1 is symmetrical if and only if:

(Def. 5) $M_1{}^{\mathrm{T}} = M_1$.

The following propositions are true:

(20) If $n > 0$, then $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ is symmetrical.

(21) If $n > 0$, then $(\begin{pmatrix} a & \ldots & a \\ \vdots & \ddots & \vdots \\ a & \ldots & a \end{pmatrix}^{n \times n})^{\mathrm{T}} = \begin{pmatrix} a & \ldots & a \\ \vdots & \ddots & \vdots \\ a & \ldots & a \end{pmatrix}^{n \times n}$.

(22) If $n > 0$, then $\begin{pmatrix} a & \ldots & a \\ \vdots & \ddots & \vdots \\ a & \ldots & a \end{pmatrix}^{n \times n}$ is symmetrical.

(23) If $n > 0$ and M_1 is symmetrical and M_2 is symmetrical, then M_1 is permutable with M_2 iff $M_1 \cdot M_2$ is symmetrical.

(24) If $n > 0$, then $(M_1 + M_2)^{\mathrm{T}} = M_1{}^{\mathrm{T}} + M_2{}^{\mathrm{T}}$.

(25) If $n > 0$ and M_1 is symmetrical and M_2 is symmetrical, then $M_1 + M_2$ is symmetrical.

(26) Suppose that

(i) M_1 is an upper triangular matrix over K of dimension n and a lower triangular matrix over K of dimension n, and

(ii) $n > 0$.

Then M_1 is symmetrical.

(27) Let K be a field, n be a natural number, and M_1, M_2 be matrices over K of dimension n. If $n > 0$, then $(-M_1)^{\mathrm{T}} = -M_1{}^{\mathrm{T}}$.

(28) Let K be a field, n be a natural number, and M_1, M_2 be matrices over K of dimension n. If M_1 is symmetrical and $n > 0$, then $-M_1$ is symmetrical.

(29) Let K be a field, n be a natural number, and M_1, M_2 be matrices over K of dimension n. Suppose $n > 0$ and M_1 is symmetrical and M_2 is symmetrical. Then $M_1 - M_2$ is symmetrical.

Let n be a natural number, let K be a field, and let M_1 be a matrix over K of dimension n. We say that M_1 is antisymmetric if and only if:

(Def. 6) $M_1{}^{\mathrm{T}} = -M_1$.

We now state a number of propositions:

(30) Let K be a Fanoian field, n be a natural number, and M_1 be a matrix over K of dimension n. If M_1 is symmetrical and antisymmetric and $n > 0$, then $M_1 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n \times n}$.

(31) Let K be a Fanoian field, n, i be natural numbers, and M_1 be a matrix over K of dimension n. If M_1 is antisymmetric and $n > 0$ and $i \in \operatorname{Seg} n$, then $M_1 \circ (i, i) = 0_K$.

(32) Let K be a field, n be a natural number, and M_1, M_2 be matrices over K of dimension n. Suppose $n > 0$ and M_1 is antisymmetric and M_2 is antisymmetric. Then $M_1 + M_2$ is antisymmetric.

(33) Let K be a field, n be a natural number, and M_1, M_2 be matrices over K of dimension n. If M_1 is antisymmetric and $n > 0$, then $-M_1$ is antisymmetric.

(34) Let K be a field, n be a natural number, and M_1, M_2 be matrices over K of dimension n. Suppose $n > 0$ and M_1 is antisymmetric and M_2 is antisymmetric. Then $M_1 - M_2$ is antisymmetric.

(35) If $M_2 = M_1 - M_1{}^{\mathrm{T}}$ and $n > 0$, then M_2 is antisymmetric.

(36) If $n > 0$, then M_1 is permutable with M_2 iff $(M_1 + M_2) \cdot (M_1 + M_2) = M_1 \cdot M_1 + M_1 \cdot M_2 + M_1 \cdot M_2 + M_2 \cdot M_2$.

(37) If $n > 0$ and M_1 is reversible and M_2 is reversible, then $M_1 \cdot M_2$ is reversible and $(M_1 \cdot M_2)^{\smile} = M_2{}^{\smile} \cdot M_1{}^{\smile}$.

(38) If $n > 0$ and M_1 is reversible and M_2 is reversible and M_1 is permutable with M_2, then $M_1 \cdot M_2$ is reversible and $(M_1 \cdot M_2)^{\smile} = M_1{}^{\smile} \cdot M_2{}^{\smile}$.

(39) If $n > 0$ and M_1 is reversible and M_2 is reversible and $M_1 \cdot M_2 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$, then M_1 is reverse of M_2.

(40) If $n > 0$ and M_1 is reversible and M_2 is reversible and $M_2 \cdot M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$, then M_1 is reverse of M_2.

(41) If $n > 0$ and M_1 is reversible and permutable with M_2, then $M_1{}^{\smile}$ is permutable with M_2.

Let n be a natural number, let K be a field, and let M_1 be a matrix over K of dimension n. We say that M_1 is orthogonal if and only if:

(Def. 7) M_1 is reversible and $M_1{}^{\mathrm{T}} = M_1{}^{\smile}$.

The following propositions are true:

(42) If $n > 0$, then $M_1 \cdot M_1{}^{\mathrm{T}} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ and M_1 is reversible iff M_1 is orthogonal.

(43) If $n > 0$, then M_1 is reversible and $M_1{}^{\mathrm{T}} \cdot M_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ iff M_1 is orthogonal.

(44) If $n > 0$ and M_1 is orthogonal, then $M_1{}^{\mathrm{T}} \cdot M_1 = M_1 \cdot M_1{}^{\mathrm{T}}$.

(45) If $n > 0$ and M_1 is orthogonal and permutable with M_2 and $M_3 = M_1{}^{\mathrm{T}}$, then M_3 is permutable with M_2.

(46) If $n > 0$ and M_1 is reversible and M_2 is reversible, then $M_1 \cdot M_2$ is reversible and $(M_1 \cdot M_2)^{\smile} = M_2{}^{\smile} \cdot M_1{}^{\smile}$.

(47) If $n > 0$ and M_1 is orthogonal and M_2 is orthogonal, then $M_1 \cdot M_2$ is orthogonal.

(48) If $n > 0$ and M_1 is orthogonal and permutable with M_2 and $M_3 = M_1{}^{\mathrm{T}}$, then M_3 is permutable with M_2.

(49) If $n > 0$ and M_1 is permutable with M_2, then $M_1 + M_1$ is permutable with M_2.

(50) If $n > 0$ and M_1 is permutable with M_2, then $M_1 + M_2$ is permutable with M_2.

(51) If $n > 0$ and M_1 is permutable with M_2, then $M_1 + M_1$ is permutable with $M_2 + M_2$.

(52) If $n > 0$ and M_1 is permutable with M_2, then $M_1 + M_2$ is permutable with $M_2 + M_2$.

(53) If $n > 0$ and M_1 is permutable with M_2, then $M_1 + M_2$ is permutable with $M_1 + M_2$.

(54) If $n > 0$ and M_1 is permutable with M_2, then $M_1 \cdot M_2$ is permutable with M_2.

(55) If $n > 0$ and M_1 is permutable with M_2, then $M_1 \cdot M_1$ is permutable with M_2.

(56) If $n > 0$ and M_1 is permutable with M_2, then $M_1 \cdot M_1$ is permutable with $M_2 \cdot M_2$.

(57) If $n > 0$ and M_1 is permutable with M_2 and $M_3 = M_1{}^{\mathrm{T}}$ and $M_4 = M_2{}^{\mathrm{T}}$,

then M_3 is permutable with M_4.

(58) Suppose $n > 0$ and M_1 is reversible and M_2 is reversible and M_3 is reversible. Then $M_1 \cdot M_2 \cdot M_3$ is reversible and $(M_1 \cdot M_2 \cdot M_3)^{\smile} = M_3{}^{\smile} \cdot M_2{}^{\smile} \cdot M_1{}^{\smile}$.

(59) If $n > 0$ and M_1 is orthogonal and M_2 is orthogonal and M_3 is orthogonal, then $M_1 \cdot M_2 \cdot M_3$ is orthogonal.

(60) If $n > 0$, then $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{K}^{n \times n}$ is orthogonal.

(61) If $n > 0$ and M_1 is orthogonal and M_2 is orthogonal, then $M_1{}^{\smile} \cdot M_2$ is orthogonal.

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Generalized Full Adder Circuits (GFAs). Part I

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Summary. In the article we formalized the concept of the Generalized Full Addition and Subtraction circuits (GFAs), defined the structures of calculation units for the redundant signed digit (RSD) operations, and proved the stability of the circuits. Generally, 1-bit binary full adder assumes positive weights to all of its three binary inputs and two outputs. We obtained four type of 1-bit GFA to constract the RSD arithmetic logical units that we generalized full adder to have both positive and negative weights to inputs and outputs.

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The articles [15], [14], [18], [13], [1], [21], [5], [6], [7], [2], [4], [16], [20], [8], [12], [17], [11], [10], [9], [3], and [19] provide the terminology and notation for this paper.

1. Preliminaries

In this article we present several logical schemes. The scheme *1AryBooleEx* deals with a unary functor $\mathcal{F}$ yielding an element of *Boolean*, and states that:

There exists a function f from $Boolean^1$ into *Boolean* such that for every element x of *Boolean* holds $f(\langle x \rangle) = \mathcal{F}(x)$

for all values of the parameter.

The scheme *1AryBooleUniq* deals with a unary functor $\mathcal{F}$ yielding an element of *Boolean*, and states that:

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Let f_1, f_2 be functions from $Boolean^1$ into $Boolean$. Suppose for every element x of $Boolean$ holds $f_1(\langle x\rangle) = \mathcal{F}(x)$ and for every element x of $Boolean$ holds $f_2(\langle x\rangle) = \mathcal{F}(x)$. Then $f_1 = f_2$

for all values of the parameter.

The scheme *1AryBooleDef* deals with a unary functor $\mathcal{F}$ yielding an element of *Boolean*, and states that:

(i) There exists a function f from $Boolean^1$ into $Boolean$ such that for every element x of $Boolean$ holds $f(\langle x\rangle) = \mathcal{F}(x)$, and

(ii) for all functions f_1, f_2 from $Boolean^1$ into $Boolean$ such that for every element x of $Boolean$ holds $f_1(\langle x\rangle) = \mathcal{F}(x)$ and for every element x of $Boolean$ holds $f_2(\langle x\rangle) = \mathcal{F}(x)$ holds $f_1 = f_2$

for all values of the parameter.

The function inv1 from $Boolean^1$ into $Boolean$ is defined by:

(Def. 1) For every element x of $Boolean$ holds $(\mathrm{inv1})(\langle x\rangle) = \neg x$.

Next we state the proposition

(1) For every element x of $Boolean$ holds $(\mathrm{inv1})(\langle x\rangle) = \neg x$ and $(\mathrm{inv1})(\langle x\rangle) = \mathrm{nand}_2(\langle x, x\rangle)$ and $(\mathrm{inv1})(\langle 0\rangle) = 1$ and $(\mathrm{inv1})(\langle 1\rangle) = 0$.

The function buf1 from $Boolean^1$ into $Boolean$ is defined by:

(Def. 2) For every element x of $Boolean$ holds $(\mathrm{buf1})(\langle x\rangle) = x$.

One can prove the following proposition

(2) For every element x of $Boolean$ holds $(\mathrm{buf1})(\langle x\rangle) = x$ and $(\mathrm{buf1})(\langle x\rangle) = \mathrm{and}_2(\langle x, x\rangle)$ and $(\mathrm{buf1})(\langle 0\rangle) = 0$ and $(\mathrm{buf1})(\langle 1\rangle) = 1$.

The function and2c from $Boolean^2$ into $Boolean$ is defined by:

(Def. 3) For all elements x, y of $Boolean$ holds $(\mathrm{and2c})(\langle x, y\rangle) = x \wedge \neg y$.

Next we state the proposition

(3) Let x, y be elements of $Boolean$. Then $(\mathrm{and2c})(\langle x, y\rangle) = x \wedge \neg y$ and $(\mathrm{and2c})(\langle x, y\rangle) = (\mathrm{and}_{2a})(\langle y, x\rangle)$ and $(\mathrm{and2c})(\langle x, y\rangle) = (\mathrm{nor}_{2a})(\langle x, y\rangle)$ and $(\mathrm{and2c})(\langle 0, 0\rangle) = 0$ and $(\mathrm{and2c})(\langle 0, 1\rangle) = 0$ and $(\mathrm{and2c})(\langle 1, 0\rangle) = 1$ and $(\mathrm{and2c})(\langle 1, 1\rangle) = 0$.

The function xor2c from $Boolean^2$ into $Boolean$ is defined by:

(Def. 4) For all elements x, y of $Boolean$ holds $(\mathrm{xor2c})(\langle x, y\rangle) = x \oplus \neg y$.

We now state several propositions:

(4) Let x, y be elements of $Boolean$. Then $(\mathrm{xor2c})(\langle x, y\rangle) = x \oplus \neg y$ and $(\mathrm{xor2c})(\langle x, y\rangle) = (\mathrm{xor}_{2a})(\langle x, y\rangle)$ and $(\mathrm{xor2c})(\langle x, y\rangle) = \mathrm{or}_2(\langle (\mathrm{and}_{2b})(\langle x, y\rangle), \mathrm{and}_2(\langle x, y\rangle)\rangle)$ and $(\mathrm{xor2c})(\langle 0, 0\rangle) = 1$ and $(\mathrm{xor2c})(\langle 0, 1\rangle) = 0$ and $(\mathrm{xor2c})(\langle 1, 0\rangle) = 0$ and $(\mathrm{xor2c})(\langle 1, 1\rangle) = 1$.

(5) For all elements x, y of $Boolean$ holds $\neg(x \oplus y) = \neg x \oplus y$ and $\neg(x \oplus y) = x \oplus \neg y$ and $\neg x \oplus \neg y = x \oplus y$.

(6) For all elements x, y of *Boolean* holds $(\mathrm{inv1})(\langle \mathrm{xor}_2(\langle x, y\rangle)\rangle) = (\mathrm{xor}_{2a})(\langle x, y\rangle)$ and $(\mathrm{inv1})(\langle \mathrm{xor}_2(\langle x, y\rangle)\rangle) = (\mathrm{xor2c})(\langle x, y\rangle)$ and $\mathrm{xor}_2(\langle (\mathrm{inv1})(\langle x\rangle), (\mathrm{inv1})(\langle y\rangle)\rangle) = \mathrm{xor}_2(\langle x, y\rangle)$.

(7) For all elements x, y, z of *Boolean* holds $\neg(x \oplus \neg y \oplus z) = x \oplus \neg y \oplus \neg z$.

(8) For all elements x, y, z of *Boolean* holds $(\mathrm{inv1})(\langle \mathrm{xor}_2(\langle (\mathrm{xor2c})(\langle x, y\rangle), z\rangle)\rangle) = (\mathrm{xor2c})(\langle (\mathrm{xor2c})(\langle x, y\rangle), z\rangle)$.

(9) For all elements x, y, z of *Boolean* holds $\neg x \oplus y \oplus \neg z = x \oplus \neg y \oplus \neg z$.

(10) For all elements x, y, z of *Boolean* holds $(\mathrm{xor2c})(\langle (\mathrm{xor}_{2a})(\langle x, y\rangle), z\rangle) = (\mathrm{xor2c})(\langle (\mathrm{xor2c})(\langle x, y\rangle), z\rangle)$.

(11) For all elements x, y, z of *Boolean* holds $\neg(\neg x \oplus \neg y \oplus \neg z) = x \oplus y \oplus z$.

(12) For all elements x, y, z of *Boolean* holds $(\mathrm{inv1})(\langle (\mathrm{xor2c})(\langle (\mathrm{xor}_{2b})(\langle x, y\rangle), z\rangle)\rangle) = \mathrm{xor}_2(\langle \mathrm{xor}_2(\langle x, y\rangle), z\rangle)$.

2. Generalized Full Adder (GFA) Circuit (TYPE-0)

Let x, y, z be sets. The functor $\mathrm{GFA0CarryIStr}(x, y, z)$ yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined by:

(Def. 5) $\mathrm{GFA0CarryIStr}(x, y, z) = \mathrm{1GateCircStr}(\langle x, y\rangle, \mathrm{and}_2) +\cdot \mathrm{1GateCircStr}(\langle y, z\rangle, \mathrm{and}_2) +\cdot \mathrm{1GateCircStr}(\langle z, x\rangle, \mathrm{and}_2)$.

Let x, y, z be sets. The functor $\mathrm{GFA0CarryICirc}(x, y, z)$ yields a strict Boolean circuit of $\mathrm{GFA0CarryIStr}(x, y, z)$ with denotation held in gates and is defined as follows:

(Def. 6) $\mathrm{GFA0CarryICirc}(x, y, z) = \mathrm{1GateCircuit}(x, y, \mathrm{and}_2) +\cdot \mathrm{1GateCircuit}(y, z, \mathrm{and}_2) +\cdot \mathrm{1GateCircuit}(z, x, \mathrm{and}_2)$.

Let x, y, z be sets. The functor $\mathrm{GFA0CarryStr}(x, y, z)$ yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined as follows:

(Def. 7) $\mathrm{GFA0CarryStr}(x, y, z) = \mathrm{GFA0CarryIStr}(x, y, z) +\cdot \mathrm{1GateCircStr}(\langle \langle \langle x, y\rangle, \mathrm{and}_2 \rangle, \langle \langle y, z\rangle, \mathrm{and}_2 \rangle, \langle \langle z, x\rangle, \mathrm{and}_2 \rangle\rangle, \mathrm{or}_3)$.

Let x, y, z be sets. The functor $\mathrm{GFA0CarryCirc}(x, y, z)$ yields a strict Boolean circuit of $\mathrm{GFA0CarryStr}(x, y, z)$ with denotation held in gates and is defined as follows:

(Def. 8) $\mathrm{GFA0CarryCirc}(x, y, z) = \mathrm{GFA0CarryICirc}(x, y, z) +\cdot \mathrm{1GateCircuit}(\langle \langle x, y\rangle, \mathrm{and}_2 \rangle, \langle \langle y, z\rangle, \mathrm{and}_2 \rangle, \langle \langle z, x\rangle, \mathrm{and}_2 \rangle, \mathrm{or}_3)$.

Let x, y, z be sets. The functor $\mathrm{GFA0CarryOutput}(x, y, z)$ yielding an element of $\mathrm{InnerVertices}(\mathrm{GFA0CarryStr}(x, y, z))$ is defined as follows:

(Def. 9) $\mathrm{GFA0CarryOutput}(x, y, z) = \langle \langle \langle \langle x, y\rangle, \mathrm{and}_2 \rangle, \langle \langle y, z\rangle, \mathrm{and}_2 \rangle, \langle \langle z, x\rangle, \mathrm{and}_2 \rangle\rangle, \mathrm{or}_3 \rangle$.

One can prove the following propositions:

(13) For all sets x, y, z holds InnerVertices(GFA0CarryIStr(x, y, z)) $= \{\langle\langle x, y\rangle, \mathrm{and}_2\rangle, \langle\langle y, z\rangle, \mathrm{and}_2\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\}$.

(14) For all sets x, y, z holds InnerVertices(GFA0CarryStr(x, y, z)) $= \{\langle\langle x, y\rangle, \mathrm{and}_2\rangle, \langle\langle y, z\rangle, \mathrm{and}_2\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\} \cup \{$GFA0CarryOutput($x, y, z$)$\}$.

(15) For all sets x, y, z holds InnerVertices(GFA0CarryStr(x, y, z)) is a binary relation.

(16) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_2\rangle$ holds InputVertices(GFA0CarryIStr(x, y, z)) $= \{x, y, z\}$.

(17) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_2\rangle$ holds InputVertices(GFA0CarryStr(x, y, z)) $= \{x, y, z\}$.

(18) For all non pair sets x, y, z holds InputVertices(GFA0CarryStr(x, y, z)) has no pairs.

(19) Let x, y, z be sets. Then $x \in$ the carrier of GFA0CarryStr(x, y, z) and $y \in$ the carrier of GFA0CarryStr(x, y, z) and $z \in$ the carrier of GFA0CarryStr(x, y, z) and $\langle\langle x, y\rangle, \mathrm{and}_2\rangle \in$ the carrier of GFA0CarryStr(x, y, z) and $\langle\langle y, z\rangle, \mathrm{and}_2\rangle \in$ the carrier of GFA0CarryStr(x, y, z) and $\langle\langle z, x\rangle, \mathrm{and}_2\rangle \in$ the carrier of GFA0CarryStr(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \mathrm{and}_2\rangle, \langle\langle y, z\rangle, \mathrm{and}_2\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\rangle, \mathrm{or}_3\rangle \in$ the carrier of GFA0CarryStr(x, y, z).

(20) For all sets x, y, z holds $\langle\langle x, y\rangle, \mathrm{and}_2\rangle \in$ InnerVertices(GFA0CarryStr(x, y, z)) and $\langle\langle y, z\rangle, \mathrm{and}_2\rangle \in$ InnerVertices(GFA0CarryStr(x, y, z)) and $\langle\langle z, x\rangle, \mathrm{and}_2\rangle \in$ InnerVertices(GFA0CarryStr(x, y, z)) and GFA0CarryOutput (x, y, z) $\in$ InnerVertices(GFA0CarryStr(x, y, z)).

(21) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_2\rangle$ holds $x \in$ InputVertices(GFA0CarryStr(x, y, z)) and $y \in$ InputVertices(GFA0CarryStr(x, y, z)) and $z \in$ InputVertices(GFA0CarryStr(x, y, z)).

(22) For all non pair sets x, y, z holds InputVertices(GFA0CarryStr(x, y, z)) $= \{x, y, z\}$.

(23) Let x, y, z be sets, s be a state of GFA0CarryCirc(x, y, z), and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following(s))($\langle\langle x, y\rangle, \mathrm{and}_2\rangle$) $= a_1 \wedge a_2$ and (Following(s))($\langle\langle y, z\rangle, \mathrm{and}_2\rangle$) $= a_2 \wedge a_3$ and (Following(s))($\langle\langle z, x\rangle, \mathrm{and}_2\rangle$) $= a_3 \wedge a_1$.

(24) Let x, y, z be sets, s be a state of GFA0CarryCirc(x, y, z), and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(\langle\langle x, y\rangle, \mathrm{and}_2\rangle)$ and $a_2 = s(\langle\langle y, z\rangle, \mathrm{and}_2\rangle)$ and $a_3 = s(\langle\langle z, x\rangle, \mathrm{and}_2\rangle)$, then (Following($s$))(GFA0CarryOutput($x, y, z$)) $= a_1 \vee a_2 \vee a_3$.

(25) Let x, y, z be sets. Suppose $x \neq \langle\langle y, z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_2\rangle$. Let s be a state of GFA0CarryCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA0CarryOutput(x, y, z)) $= a_1 \wedge a_2 \vee a_2 \wedge a_3 \vee a_3 \wedge a_1$ and (Following$(s, 2)$)($\langle\langle x, y\rangle, \mathrm{and}_2\rangle$) $= a_1 \wedge a_2$ and (Following$(s, 2)$)($\langle\langle y, z\rangle, \mathrm{and}_2\rangle$) $= a_2 \wedge a_3$ and (Following$(s, 2)$)($\langle\langle z, x\rangle, \mathrm{and}_2\rangle$) $= a_3 \wedge a_1$.

(26) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_2\rangle$ and for every state s of GFA0CarryCirc(x, y, z) holds Following$(s, 2)$ is stable.

Let x, y, z be sets. The functor GFA0AdderStr(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined as follows:

(Def. 10) GFA0AdderStr$(x, y, z) =$ 2GatesCircStr$(x, y, z, \mathrm{xor}_2)$.

Let x, y, z be sets. The functor GFA0AdderCirc(x, y, z) yielding a strict Boolean circuit of GFA0AdderStr(x, y, z) with denotation held in gates is defined by:

(Def. 11) GFA0AdderCirc$(x, y, z) =$ 2GatesCircuit$(x, y, z, \mathrm{xor}_2)$.

Let x, y, z be sets. The functor GFA0AdderOutput(x, y, z) yielding an element of InnerVertices(GFA0AdderStr(x, y, z)) is defined by:

(Def. 12) GFA0AdderOutput$(x, y, z) =$ 2GatesCircOutput$(x, y, z, \mathrm{xor}_2)$.

Next we state a number of propositions:

(27) For all sets x, y, z holds InnerVertices(GFA0AdderStr(x, y, z)) $= \{\langle\langle x, y\rangle, \mathrm{xor}_2\rangle\} \cup \{$GFA0AdderOutput$(x, y, z)\}$.

(28) For all sets x, y, z holds InnerVertices(GFA0AdderStr(x, y, z)) is a binary relation.

(29) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ holds InputVertices(GFA0AdderStr(x, y, z)) $= \{x, y, z\}$.

(30) For all non pair sets x, y, z holds InputVertices(GFA0AdderStr(x, y, z)) has no pairs.

(31) Let x, y, z be sets. Then
(i) $x \in$ the carrier of GFA0AdderStr(x, y, z),
(ii) $y \in$ the carrier of GFA0AdderStr(x, y, z),
(iii) $z \in$ the carrier of GFA0AdderStr(x, y, z),
(iv) $\langle\langle x, y\rangle, \mathrm{xor}_2\rangle \in$ the carrier of GFA0AdderStr(x, y, z), and
(v) $\langle\langle\langle\langle x, y\rangle, \mathrm{xor}_2\rangle, z\rangle, \mathrm{xor}_2\rangle \in$ the carrier of GFA0AdderStr(x, y, z).

(32) For all sets x, y, z holds $\langle\langle x, y\rangle, \mathrm{xor}_2\rangle \in$ InnerVertices(GFA0AdderStr(x, y, z)) and GFA0AdderOutput$(x, y, z) \in$ InnerVertices(GFA0AdderStr(x, y, z)).

(33) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ holds $x \in$ InputVertices(GFA0AdderStr(x, y, z)) and

$y \in$ InputVertices(GFA0AdderStr(x, y, z)) and
$z \in$ InputVertices(GFA0AdderStr(x, y, z)).

(34) For all non pair sets x, y, z holds InputVertices(GFA0AdderStr(x, y, z)) $= \{x, y, z\}$.

(35) Let x, y, z be sets. Suppose $z \neq \langle\!\langle \langle x, y \rangle, \mathrm{xor}_2 \rangle\!\rangle$. Let s be a state of GFA0AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following(s))($\langle\!\langle \langle x, y \rangle$, $\mathrm{xor}_2 \rangle\!\rangle$) $= a_1 \oplus a_2$ and (Following(s))$(x) = a_1$ and (Following(s))$(y) = a_2$ and (Following(s))$(z) = a_3$.

(36) Let x, y, z be sets. Suppose $z \neq \langle\!\langle \langle x, y \rangle, \mathrm{xor}_2 \rangle\!\rangle$. Let s be a state of GFA0AdderCirc(x, y, z) and a_4, a_1, a_2, a_3 be elements of *Boolean*. If $a_4 = s(\langle\!\langle \langle x, y \rangle, \mathrm{xor}_2 \rangle\!\rangle)$ and $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$, then (Following(s))(GFA0AdderOutput(x, y, z)) $= a_4 \oplus a_3$.

(37) Let x, y, z be sets. Suppose $z \neq \langle\!\langle \langle x, y \rangle, \mathrm{xor}_2 \rangle\!\rangle$. Let s be a state of GFA0AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA0AdderOutput(x, y, z)) $= a_1 \oplus a_2 \oplus a_3$ and (Following$(s, 2)$)($\langle\!\langle \langle x, y \rangle$, $\mathrm{xor}_2 \rangle\!\rangle$) $= a_1 \oplus a_2$ and (Following$(s, 2)$)$(x) = a_1$ and (Following$(s, 2)$)$(y) = a_2$ and (Following$(s, 2)$)$(z) = a_3$.

(38) For all sets x, y, z such that $z \neq \langle\!\langle \langle x, y \rangle, \mathrm{xor}_2 \rangle\!\rangle$ and for every state s of GFA0AdderCirc(x, y, z) holds Following$(s, 2)$ is stable.

Let x, y, z be sets. The functor BitGFA0Str(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined as follows:

(Def. 13) BitGFA0Str(x, y, z) = GFA0AdderStr(x, y, z)+· GFA0CarryStr(x, y, z).

Let x, y, z be sets. The functor BitGFA0Circ(x, y, z) yielding a strict Boolean circuit of BitGFA0Str(x, y, z) with denotation held in gates is defined by:

(Def. 14) BitGFA0Circ(x, y, z) = GFA0AdderCirc(x, y, z)+· GFA0CarryCirc(x, y, z).

We now state several propositions:

(39) For all sets x, y, z holds InnerVertices(BitGFA0Str(x, y, z)) $= \{\langle\!\langle \langle x, y \rangle, \mathrm{xor}_2 \rangle\!\rangle\} \cup \{$GFA0AdderOutput$(x, y, z)\} \cup \{\langle\!\langle \langle x, y \rangle, \mathrm{and}_2 \rangle\!\rangle, \langle\!\langle \langle y, z \rangle, \mathrm{and}_2 \rangle\!\rangle, \langle\!\langle \langle z, x \rangle, \mathrm{and}_2 \rangle\!\rangle\} \cup \{$GFA0CarryOutput$(x, y, z)\}$.

(40) For all sets x, y, z holds InnerVertices(BitGFA0Str(x, y, z)) is a binary relation.

(41) For all sets x, y, z such that $z \neq \langle\!\langle \langle x, y \rangle, \mathrm{xor}_2 \rangle\!\rangle$ and $x \neq \langle\!\langle \langle y, z \rangle, \mathrm{and}_2 \rangle\!\rangle$ and $y \neq \langle\!\langle \langle z, x \rangle, \mathrm{and}_2 \rangle\!\rangle$ and $z \neq \langle\!\langle \langle x, y \rangle, \mathrm{and}_2 \rangle\!\rangle$ holds InputVertices(BitGFA0Str(x, y, z)) $= \{x, y, z\}$.

(42) For all non pair sets x, y, z holds InputVertices(BitGFA0Str(x, y, z)) $= \{x, y, z\}$.

(43) For all non pair sets x, y, z holds InputVertices(BitGFA0Str(x, y, z)) has no pairs.

(44) Let x, y, z be sets. Then $x \in$ the carrier of BitGFA0Str(x, y, z) and $y \in$ the carrier of BitGFA0Str(x, y, z) and $z \in$ the carrier of BitGFA0Str(x, y, z) and $\langle\langle x, y\rangle, \mathrm{xor}_2\rangle \in$ the carrier of BitGFA0Str(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \mathrm{xor}_2\rangle, z\rangle, \mathrm{xor}_2\rangle \in$ the carrier of BitGFA0Str(x, y, z) and $\langle\langle x, y\rangle, \mathrm{and}_2\rangle \in$ the carrier of BitGFA0Str(x, y, z) and $\langle\langle y, z\rangle, \mathrm{and}_2\rangle \in$ the carrier of BitGFA0Str(x, y, z) and $\langle\langle z, x\rangle, \mathrm{and}_2\rangle \in$ the carrier of BitGFA0Str(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \mathrm{and}_2\rangle, \langle\langle y, z\rangle, \mathrm{and}_2\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\rangle, \mathrm{or}_3\rangle \in$ the carrier of BitGFA0Str(x, y, z).

(45) Let x, y, z be sets. Then $\langle\langle x, y\rangle, \mathrm{xor}_2\rangle \in$ InnerVertices(BitGFA0Str(x, y, z)) and GFA0AdderOutput$(x, y, z) \in$ InnerVertices(BitGFA0Str(x, y, z)) and $\langle\langle x, y\rangle, \mathrm{and}_2\rangle \in$ InnerVertices(BitGFA0Str(x, y, z)) and $\langle\langle y, z\rangle, \mathrm{and}_2\rangle \in$ InnerVertices(BitGFA0Str(x, y, z)) and $\langle\langle z, x\rangle, \mathrm{and}_2\rangle \in$ InnerVertices(BitGFA0Str(x, y, z)) and GFA0CarryOutput$(x, y, z) \in$ InnerVertices(BitGFA0Str(x, y, z)).

(46) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_2\rangle$. Then $x \in$ InputVertices(BitGFA0Str(x, y, z)) and $y \in$ InputVertices(BitGFA0Str(x, y, z)) and $z \in$ InputVertices(BitGFA0Str(x, y, z)).

Let x, y, z be sets. The functor BitGFA0CarryOutput(x, y, z) yielding an element of InnerVertices(BitGFA0Str(x, y, z)) is defined as follows:

(Def. 15) BitGFA0CarryOutput$(x, y, z) = \langle\langle\langle\langle x, y\rangle, \mathrm{and}_2\rangle, \langle\langle y, z\rangle, \mathrm{and}_2\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\rangle, \mathrm{or}_3\rangle$.

Let x, y, z be sets. The functor BitGFA0AdderOutput(x, y, z) yielding an element of InnerVertices(BitGFA0Str(x, y, z)) is defined as follows:

(Def. 16) BitGFA0AdderOutput$(x, y, z) =$ 2GatesCircOutput$(x, y, z, \mathrm{xor}_2)$.

One can prove the following two propositions:

(47) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_2\rangle$. Let s be a state of BitGFA0Circ(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA0AdderOutput(x, y, z)) $= a_1 \oplus a_2 \oplus a_3$ and (Following$(s, 2)$)(GFA0CarryOutput(x, y, z)) $= a_1 \wedge a_2 \vee a_2 \wedge a_3 \vee a_3 \wedge a_1$.

(48) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_2\rangle$. Let s be a state of BitGFA0Circ(x, y, z). Then Following$(s, 2)$ is stable.

3. Generalized Full Adder (GFA) Circuit (TYPE-1)

Let x, y, z be sets. The functor GFA1CarryIStr(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def. 17) GFA1CarryIStr$(x, y, z) =$ 1GateCircStr$(\langle x, y\rangle, \mathrm{and2c}) +\cdot$ 1GateCircStr$(\langle y, z\rangle, \mathrm{and}_{2a}) +\cdot$ 1GateCircStr$(\langle z, x\rangle, \mathrm{and}_2)$.

Let x, y, z be sets. The functor GFA1CarryICirc(x, y, z) yields a strict Boolean circuit of GFA1CarryIStr(x, y, z) with denotation held in gates and is defined as follows:

(Def. 18) GFA1CarryICirc$(x, y, z) =$ 1GateCircuit$(x, y, \mathrm{and2c}) +\cdot$ 1GateCircuit$(y, z, \mathrm{and}_{2a}) +\cdot$ 1GateCircuit(z, x, and_2).

Let x, y, z be sets. The functor GFA1CarryStr(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def. 19) GFA1CarryStr$(x, y, z) =$ GFA1CarryIStr$(x, y, z) +\cdot$ 1GateCircStr$(\langle\langle\langle x, y\rangle, \mathrm{and2c}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\rangle, \mathrm{or}_3)$.

Let x, y, z be sets. The functor GFA1CarryCirc(x, y, z) yielding a strict Boolean circuit of GFA1CarryStr(x, y, z) with denotation held in gates is defined by:

(Def. 20) GFA1CarryCirc$(x, y, z) =$ GFA1CarryICirc$(x, y, z) +\cdot$ 1GateCircuit$(\langle\langle x, y\rangle, \mathrm{and2c}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle, \mathrm{or}_3)$.

Let x, y, z be sets. The functor GFA1CarryOutput(x, y, z) yielding an element of InnerVertices(GFA1CarryStr(x, y, z)) is defined as follows:

(Def. 21) GFA1CarryOutput$(x, y, z) = \langle\langle\langle\langle x, y\rangle, \mathrm{and2c}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\rangle, \mathrm{or}_3\rangle$.

We now state a number of propositions:

(49) For all sets x, y, z holds InnerVertices(GFA1CarryIStr(x, y, z)) $= \{\langle\langle x, y\rangle, \mathrm{and2c}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\}$.

(50) For all sets x, y, z holds InnerVertices(GFA1CarryStr(x, y, z)) $= \{\langle\langle x, y\rangle, \mathrm{and2c}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\} \cup \{$GFA1CarryOutput$(x, y, z)\}$.

(51) For all sets x, y, z holds InnerVertices(GFA1CarryStr(x, y, z)) is a binary relation.

(52) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and2c}\rangle$ holds InputVertices(GFA1CarryIStr(x, y, z)) $= \{x, y, z\}$.

(53) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and2c}\rangle$ holds InputVertices(GFA1CarryStr(x, y, z)) $= \{x, y, z\}$.

(54) For all non pair sets x, y, z holds InputVertices(GFA1CarryStr(x,y,z)) has no pairs.

(55) Let x, y, z be sets. Then $x \in$ the carrier of GFA1CarryStr(x,y,z) and $y \in$ the carrier of GFA1CarryStr(x,y,z) and $z \in$ the carrier of GFA1CarryStr(x,y,z) and $\langle\langle x,y\rangle, \text{and2c}\rangle \in$ the carrier of GFA1CarryStr(x,y,z) and $\langle\langle y,z\rangle, \text{and}_{2a}\rangle \in$ the carrier of GFA1CarryStr(x,y,z) and $\langle\langle z,x\rangle, \text{and}_2\rangle \in$ the carrier of GFA1CarryStr(x,y,z) and $\langle\langle\langle\langle x,y\rangle, \text{and2c}\rangle, \langle\langle y,z\rangle, \text{and}_{2a}\rangle, \langle\langle z,x\rangle, \text{and}_2\rangle\rangle, \text{or}_3\rangle \in$ the carrier of GFA1CarryStr(x,y,z).

(56) For all sets x, y, z holds $\langle\langle x,y\rangle, \text{and2c}\rangle \in$ InnerVertices(GFA1CarryStr(x, y,z)) and $\langle\langle y,z\rangle, \text{and}_{2a}\rangle \in$ InnerVertices(GFA1CarryStr(x,y,z)) and $\langle\langle z, x\rangle, \text{and}_2\rangle \in$ InnerVertices(GFA1CarryStr(x,y,z)) and GFA1CarryOutput $(x,y,z) \in$ InnerVertices(GFA1CarryStr(x,y,z)).

(57) For all sets x, y, z such that $x \neq \langle\langle y,z\rangle, \text{and}_{2a}\rangle$ and $y \neq \langle\langle z,x\rangle, \text{and}_2\rangle$ and $z \neq \langle\langle x,y\rangle, \text{and2c}\rangle$ holds $x \in$ InputVertices(GFA1CarryStr(x,y,z)) and $y \in$ InputVertices(GFA1CarryStr(x,y,z)) and $z \in$ InputVertices(GFA1CarryStr(x,y,z)).

(58) For all non pair sets x, y, z holds InputVertices(GFA1CarryStr(x,y,z)) $= \{x,y,z\}$.

(59) Let x, y, z be sets, s be a state of GFA1CarryCirc(x,y,z), and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following(s))($\langle\langle x,y\rangle, \text{and2c}\rangle$) $= a_1 \wedge \neg a_2$ and (Following(s))($\langle\langle y, z\rangle, \text{and}_{2a}\rangle$) $= \neg a_2 \wedge a_3$ and (Following(s))($\langle\langle z,x\rangle, \text{and}_2\rangle$) $= a_3 \wedge a_1$.

(60) Let x, y, z be sets, s be a state of GFA1CarryCirc(x,y,z), and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(\langle\langle x,y\rangle, \text{and2c}\rangle)$ and $a_2 = s(\langle\langle y,z\rangle, \text{and}_{2a}\rangle)$ and $a_3 = s(\langle\langle z,x\rangle, \text{and}_2\rangle)$, then (Following$(s)$)(GFA1CarryOutput$(x,y,z)$) $= a_1 \vee a_2 \vee a_3$.

(61) Let x, y, z be sets. Suppose $x \neq \langle\langle y,z\rangle, \text{and}_{2a}\rangle$ and $y \neq \langle\langle z,x\rangle, \text{and}_2\rangle$ and $z \neq \langle\langle x,y\rangle, \text{and2c}\rangle$. Let s be a state of GFA1CarryCirc(x,y,z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s,2)$)(GFA1CarryOutput(x,y,z)) $= a_1 \wedge \neg a_2 \vee \neg a_2 \wedge a_3 \vee a_3 \wedge a_1$ and (Following$(s,2)$)($\langle\langle x,y\rangle, \text{and2c}\rangle$) $= a_1 \wedge \neg a_2$ and (Following$(s,2)$)($\langle\langle y,z\rangle, \text{and}_{2a}\rangle$) $= \neg a_2 \wedge a_3$ and (Following$(s,2)$)($\langle\langle z,x\rangle, \text{and}_2\rangle$) $= a_3 \wedge a_1$.

(62) For all sets x, y, z such that $x \neq \langle\langle y,z\rangle, \text{and}_{2a}\rangle$ and $y \neq \langle\langle z,x\rangle, \text{and}_2\rangle$ and $z \neq \langle\langle x,y\rangle, \text{and2c}\rangle$ and for every state s of GFA1CarryCirc(x,y,z) holds Following$(s,2)$ is stable.

Let x, y, z be sets. The functor GFA1AdderStr(x,y,z) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined as follows:

(Def. 22) $\mathrm{GFA1AdderStr}(x, y, z) = \mathrm{2GatesCircStr}(x, y, z, \mathrm{xor2c})$.

Let x, y, z be sets. The functor $\mathrm{GFA1AdderCirc}(x, y, z)$ yielding a strict Boolean circuit of $\mathrm{GFA1AdderStr}(x, y, z)$ with denotation held in gates is defined by:

(Def. 23) $\mathrm{GFA1AdderCirc}(x, y, z) = \mathrm{2GatesCircuit}(x, y, z, \mathrm{xor2c})$.

Let x, y, z be sets. The functor $\mathrm{GFA1AdderOutput}(x, y, z)$ yields an element of $\mathrm{InnerVertices}(\mathrm{GFA1AdderStr}(x, y, z))$ and is defined as follows:

(Def. 24) $\mathrm{GFA1AdderOutput}(x, y, z) = \mathrm{2GatesCircOutput}(x, y, z, \mathrm{xor2c})$.

We now state a number of propositions:

(63) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{GFA1AdderStr}(x, y, z)) = \{\langle\langle x, y\rangle, \mathrm{xor2c}\rangle\} \cup \{\mathrm{GFA1AdderOutput}(x, y, z)\}$.

(64) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{GFA1AdderStr}(x, y, z))$ is a binary relation.

(65) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ holds $\mathrm{InputVertices}(\mathrm{GFA1AdderStr}(x, y, z)) = \{x, y, z\}$.

(66) For all non pair sets x, y, z holds $\mathrm{InputVertices}(\mathrm{GFA1AdderStr}(x, y, z))$ has no pairs.

(67) Let x, y, z be sets. Then
  (i) $x \in$ the carrier of $\mathrm{GFA1AdderStr}(x, y, z)$,
  (ii) $y \in$ the carrier of $\mathrm{GFA1AdderStr}(x, y, z)$,
  (iii) $z \in$ the carrier of $\mathrm{GFA1AdderStr}(x, y, z)$,
  (iv) $\langle\langle x, y\rangle, \mathrm{xor2c}\rangle \in$ the carrier of $\mathrm{GFA1AdderStr}(x, y, z)$, and
  (v) $\langle\langle\langle\langle x, y\rangle, \mathrm{xor2c}\rangle, z\rangle, \mathrm{xor2c}\rangle \in$ the carrier of $\mathrm{GFA1AdderStr}(x, y, z)$.

(68) For all sets x, y, z holds $\langle\langle x, y\rangle, \mathrm{xor2c}\rangle \in \mathrm{InnerVertices}(\mathrm{GFA1AdderStr}(x, y, z))$ and $\mathrm{GFA1AdderOutput}(x, y, z) \in \mathrm{InnerVertices}(\mathrm{GFA1AdderStr}(x, y, z))$.

(69) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ holds $x \in \mathrm{InputVertices}(\mathrm{GFA1AdderStr}(x, y, z))$ and
$y \in \mathrm{InputVertices}(\mathrm{GFA1AdderStr}(x, y, z))$ and
$z \in \mathrm{InputVertices}(\mathrm{GFA1AdderStr}(x, y, z))$.

(70) For all non pair sets x, y, z holds $\mathrm{InputVertices}(\mathrm{GFA1AdderStr}(x, y, z)) = \{x, y, z\}$.

(71) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$. Let s be a state of $\mathrm{GFA1AdderCirc}(x, y, z)$ and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then $(\mathrm{Following}(s))(\langle\langle x, y\rangle, \mathrm{xor2c}\rangle) = a_1 \oplus \neg a_2$ and $(\mathrm{Following}(s))(x) = a_1$ and $(\mathrm{Following}(s))(y) = a_2$ and $(\mathrm{Following}(s))(z) = a_3$.

(72) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$. Let s be a state of $\mathrm{GFA1AdderCirc}(x, y, z)$ and a_4, a_1, a_2, a_3 be elements of *Boolean*. If

$a_4 = s(\langle\langle x, y\rangle, \mathrm{xor2c}\rangle)$ and $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$, then $(\mathrm{Following}(s))(\mathrm{GFA1AdderOutput}(x, y, z)) = a_4 \oplus \neg a_3$.

(73) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$. Let s be a state of GFA1AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then $(\mathrm{Following}(s, 2))(\mathrm{GFA1AdderOutput}(x, y, z)) = a_1 \oplus \neg a_2 \oplus \neg a_3$ and $(\mathrm{Following}(s, 2))(\langle\langle x, y\rangle, \mathrm{xor2c}\rangle) = a_1 \oplus \neg a_2$ and $(\mathrm{Following}(s, 2))(x) = a_1$ and $(\mathrm{Following}(s, 2))(y) = a_2$ and $(\mathrm{Following}(s, 2))(z) = a_3$.

(74) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$. Let s be a state of GFA1AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$, then $(\mathrm{Following}(s, 2))(\mathrm{GFA1AdderOutput}(x, y, z)) = \neg(a_1 \oplus \neg a_2 \oplus a_3)$.

(75) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and for every state s of GFA1AdderCirc(x, y, z) holds Following$(s, 2)$ is stable.

Let x, y, z be sets. The functor BitGFA1Str(x, y, z) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined as follows:

(Def. 25) $\mathrm{BitGFA1Str}(x, y, z) = \mathrm{GFA1AdderStr}(x, y, z) +\cdot \mathrm{GFA1CarryStr}(x, y, z)$.

Let x, y, z be sets. The functor BitGFA1Circ(x, y, z) yielding a strict Boolean circuit of BitGFA1Str(x, y, z) with denotation held in gates is defined by:

(Def. 26) $\mathrm{BitGFA1Circ}(x, y, z) = \mathrm{GFA1AdderCirc}(x, y, z) +\cdot \mathrm{GFA1CarryCirc}(x, y, z)$.

We now state several propositions:

(76) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{BitGFA1Str}(x, y, z)) = \{\langle\langle x, y\rangle, \mathrm{xor2c}\rangle\} \cup \{\mathrm{GFA1AdderOutput}(x, y, z)\} \cup \{\langle\langle x, y\rangle, \mathrm{and2c}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\} \cup \{\mathrm{GFA1CarryOutput}(x, y, z)\}$.

(77) For all sets x, y, z holds InnerVertices(BitGFA1Str(x, y, z)) is a binary relation.

(78) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and2c}\rangle$ holds $\mathrm{InputVertices}(\mathrm{BitGFA1Str}(x, y, z)) = \{x, y, z\}$.

(79) For all non pair sets x, y, z holds $\mathrm{InputVertices}(\mathrm{BitGFA1Str}(x, y, z)) = \{x, y, z\}$.

(80) For all non pair sets x, y, z holds InputVertices(BitGFA1Str(x, y, z)) has no pairs.

(81) Let x, y, z be sets. Then $x \in$ the carrier of BitGFA1Str(x, y, z) and $y \in$ the carrier of BitGFA1Str(x, y, z) and $z \in$ the carrier of BitGFA1Str(x, y, z) and $\langle\langle x, y\rangle, \mathrm{xor2c}\rangle \in$ the carrier of BitGFA1Str(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \mathrm{xor2c}\rangle, z\rangle, \mathrm{xor2c}\rangle \in$ the carrier of BitGFA1Str(x, y, z) and $\langle\langle x, y\rangle, \mathrm{and2c}\rangle \in$ the car-

rier of BitGFA1Str(x, y, z) and $\langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle \in$ the carrier of BitGFA1Str(x, y, z) and $\langle\langle z, x\rangle, \mathrm{and}_2\rangle \in$ the carrier of BitGFA1Str(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \mathrm{and2c}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\rangle, \mathrm{or}_3\rangle \in$ the carrier of BitGFA1Str(x, y, z).

(82) Let x, y, z be sets. Then $\langle\langle x, y\rangle, \mathrm{xor2c}\rangle \in$ InnerVertices(BitGFA1Str(x, y, z)) and GFA1AdderOutput$(x, y, z) \in$ InnerVertices(BitGFA1Str(x, y, z)) and $\langle\langle x, y\rangle, \mathrm{and2c}\rangle \in$ InnerVertices(BitGFA1Str(x, y, z)) and $\langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle \in$ InnerVertices(BitGFA1Str(x, y, z)) and $\langle\langle z, x\rangle, \mathrm{and}_2\rangle \in$ InnerVertices(BitGFA1Str(x, y, z)) and GFA1CarryOutput$(x, y, z) \in$ InnerVertices(BitGFA1Str(x, y, z)).

(83) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and2c}\rangle$. Then $x \in$ InputVertices(BitGFA1Str(x, y, z)) and $y \in$ InputVertices(BitGFA1Str(x, y, z)) and $z \in$ InputVertices(BitGFA1Str(x, y, z)).

Let x, y, z be sets. The functor BitGFA1CarryOutput(x, y, z) yielding an element of InnerVertices(BitGFA1Str(x, y, z)) is defined as follows:

(Def. 27) BitGFA1CarryOutput$(x, y, z) = \langle\langle\langle\langle x, y\rangle, \mathrm{and2c}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\rangle, \mathrm{or}_3\rangle$.

Let x, y, z be sets. The functor BitGFA1AdderOutput(x, y, z) yielding an element of InnerVertices(BitGFA1Str(x, y, z)) is defined as follows:

(Def. 28) BitGFA1AdderOutput$(x, y, z) =$ 2GatesCircOutput$(x, y, z, \mathrm{xor2c})$.

The following two propositions are true:

(84) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and2c}\rangle$. Let s be a state of BitGFA1Circ(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA1AdderOutput(x, y, z)) $= \neg(a_1 \oplus \neg a_2 \oplus a_3)$ and (Following$(s, 2)$)(GFA1CarryOutput(x, y, z)) $= a_1 \wedge \neg a_2 \vee \neg a_2 \wedge a_3 \vee a_3 \wedge a_1$.

(85) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and2c}\rangle$. Let s be a state of BitGFA1Circ(x, y, z). Then Following$(s, 2)$ is stable.

4. Generalized Full Adder (GFA) Circuit (TYPE-2)

Let x, y, z be sets. The functor GFA2CarryIStr(x, y, z) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined by:

(Def. 29) GFA2CarryIStr$(x, y, z) =$ 1GateCircStr$(\langle x, y\rangle, \mathrm{and}_{2a})+\cdot$ 1GateCircStr$(\langle y, z\rangle, \mathrm{and2c})+\cdot$ 1GateCircStr$(\langle z, x\rangle, \mathrm{and}_{2b})$.

Let x, y, z be sets. The functor $\mathrm{GFA2CarryICirc}(x, y, z)$ yielding a strict Boolean circuit of $\mathrm{GFA2CarryIStr}(x, y, z)$ with denotation held in gates is defined as follows:

(Def. 30) $\mathrm{GFA2CarryICirc}(x, y, z) = \mathrm{1GateCircuit}(x, y, \mathrm{and}_{2a}) +\cdot \mathrm{1GateCircuit}(y, z, \mathrm{and2c}) +\cdot \mathrm{1GateCircuit}(z, x, \mathrm{and}_{2b})$.

Let x, y, z be sets. The functor $\mathrm{GFA2CarryStr}(x, y, z)$ yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined as follows:

(Def. 31) $\mathrm{GFA2CarryStr}(x, y, z) = \mathrm{GFA2CarryIStr}(x, y, z) +\cdot \mathrm{1GateCircStr}(\langle \langle \langle x, y \rangle, \mathrm{and}_{2a} \rangle, \langle \langle y, z \rangle, \mathrm{and2c} \rangle, \langle \langle z, x \rangle, \mathrm{and}_{2b} \rangle \rangle, \mathrm{nor}_3)$.

Let x, y, z be sets. The functor $\mathrm{GFA2CarryCirc}(x, y, z)$ yields a strict Boolean circuit of $\mathrm{GFA2CarryStr}(x, y, z)$ with denotation held in gates and is defined as follows:

(Def. 32) $\mathrm{GFA2CarryCirc}(x, y, z) = \mathrm{GFA2CarryICirc}(x, y, z) +\cdot \mathrm{1GateCircuit}(\langle \langle x, y \rangle, \mathrm{and}_{2a} \rangle, \langle \langle y, z \rangle, \mathrm{and2c} \rangle, \langle \langle z, x \rangle, \mathrm{and}_{2b} \rangle, \mathrm{nor}_3)$.

Let x, y, z be sets. The functor $\mathrm{GFA2CarryOutput}(x, y, z)$ yields an element of $\mathrm{InnerVertices}(\mathrm{GFA2CarryStr}(x, y, z))$ and is defined by:

(Def. 33) $\mathrm{GFA2CarryOutput}(x, y, z) = \langle \langle \langle \langle x, y \rangle, \mathrm{and}_{2a} \rangle, \langle \langle y, z \rangle, \mathrm{and2c} \rangle, \langle \langle z, x \rangle, \mathrm{and}_{2b} \rangle \rangle, \mathrm{nor}_3 \rangle$.

We now state a number of propositions:

(86) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{GFA2CarryIStr}(x, y, z)) = \{\langle \langle x, y \rangle, \mathrm{and}_{2a} \rangle, \langle \langle y, z \rangle, \mathrm{and2c} \rangle, \langle \langle z, x \rangle, \mathrm{and}_{2b} \rangle\}$.

(87) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{GFA2CarryStr}(x, y, z)) = \{\langle \langle x, y \rangle, \mathrm{and}_{2a} \rangle, \langle \langle y, z \rangle, \mathrm{and2c} \rangle, \langle \langle z, x \rangle, \mathrm{and}_{2b} \rangle\} \cup \{\mathrm{GFA2CarryOutput}(x, y, z)\}$.

(88) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{GFA2CarryStr}(x, y, z))$ is a binary relation.

(89) For all sets x, y, z such that $x \neq \langle \langle y, z \rangle, \mathrm{and2c} \rangle$ and $y \neq \langle \langle z, x \rangle, \mathrm{and}_{2b} \rangle$ and $z \neq \langle \langle x, y \rangle, \mathrm{and}_{2a} \rangle$ holds $\mathrm{InputVertices}(\mathrm{GFA2CarryIStr}(x, y, z)) = \{x, y, z\}$.

(90) For all sets x, y, z such that $x \neq \langle \langle y, z \rangle, \mathrm{and2c} \rangle$ and $y \neq \langle \langle z, x \rangle, \mathrm{and}_{2b} \rangle$ and $z \neq \langle \langle x, y \rangle, \mathrm{and}_{2a} \rangle$ holds $\mathrm{InputVertices}(\mathrm{GFA2CarryStr}(x, y, z)) = \{x, y, z\}$.

(91) For all non pair sets x, y, z holds $\mathrm{InputVertices}(\mathrm{GFA2CarryStr}(x, y, z))$ has no pairs.

(92) Let x, y, z be sets. Then $x \in$ the carrier of $\mathrm{GFA2CarryStr}(x, y, z)$ and $y \in$ the carrier of $\mathrm{GFA2CarryStr}(x, y, z)$ and $z \in$ the carrier of $\mathrm{GFA2CarryStr}(x, y, z)$ and $\langle \langle x, y \rangle, \mathrm{and}_{2a} \rangle \in$ the carrier of $\mathrm{GFA2CarryStr}(x, y, z)$ and $\langle \langle y, z \rangle, \mathrm{and2c} \rangle \in$ the carrier of $\mathrm{GFA2CarryStr}(x, y, z)$ and $\langle \langle z, x \rangle, \mathrm{and}_{2b} \rangle \in$ the carrier

of GFA2CarryStr(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, z\rangle, \mathrm{and2c}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\rangle, \mathrm{nor}_3\rangle \in$ the carrier of GFA2CarryStr(x, y, z).

(93) For all sets x, y, z holds $\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle \in$ InnerVertices(GFA2CarryStr(x, y, z)) and $\langle\langle y, z\rangle, \mathrm{and2c}\rangle \in$ InnerVertices(GFA2CarryStr(x, y, z)) and $\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle \in$ InnerVertices(GFA2CarryStr(x, y, z)) and GFA2CarryOutput $(x, y, z) \in$ InnerVertices(GFA2CarryStr(x, y, z)).

(94) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ holds $x \in$ InputVertices(GFA2CarryStr(x, y, z)) and $y \in$ InputVertices(GFA2CarryStr(x, y, z)) and $z \in$ InputVertices(GFA2CarryStr(x, y, z)).

(95) For all non pair sets x, y, z holds InputVertices(GFA2CarryStr(x, y, z)) $= \{x, y, z\}$.

(96) Let x, y, z be sets, s be a state of GFA2CarryCirc(x, y, z), and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following(s))$(\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle) = \neg a_1 \wedge a_2$ and (Following(s))$(\langle\langle y, z\rangle, \mathrm{and2c}\rangle) = a_2 \wedge \neg a_3$ and (Following(s))$(\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle) = \neg a_3 \wedge \neg a_1$.

(97) Let x, y, z be sets, s be a state of GFA2CarryCirc(x, y, z), and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle)$ and $a_2 = s(\langle\langle y, z\rangle, \mathrm{and2c}\rangle)$ and $a_3 = s(\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle)$, then (Following$(s)$)(GFA2CarryOutput$(x, y, z)$) $= \neg(a_1 \vee a_2 \vee a_3)$.

(98) Let x, y, z be sets. Suppose $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$. Let s be a state of GFA2CarryCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA2CarryOutput(x, y, z)) $= \neg(\neg a_1 \wedge a_2 \vee a_2 \wedge \neg a_3 \vee \neg a_3 \wedge \neg a_1)$ and (Following$(s, 2)$)$(\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle) = \neg a_1 \wedge a_2$ and (Following$(s, 2)$)$(\langle\langle y, z\rangle, \mathrm{and2c}\rangle) = a_2 \wedge \neg a_3$ and (Following$(s, 2)$)$(\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle) = \neg a_3 \wedge \neg a_1$.

(99) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ and for every state s of GFA2CarryCirc(x, y, z) holds Following$(s, 2)$ is stable.

Let x, y, z be sets. The functor GFA2AdderStr(x, y, z) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined as follows:

(Def. 34) GFA2AdderStr(x, y, z) = 2GatesCircStr$(x, y, z, \mathrm{xor2c})$.

Let x, y, z be sets. The functor GFA2AdderCirc(x, y, z) yielding a strict Boolean circuit of GFA2AdderStr(x, y, z) with denotation held in gates is defined as follows:

(Def. 35) GFA2AdderCirc(x, y, z) = 2GatesCircuit$(x, y, z, \mathrm{xor2c})$.

Let x, y, z be sets. The functor GFA2AdderOutput(x, y, z) yields an element of InnerVertices(GFA2AdderStr(x, y, z)) and is defined by:

(Def. 36) $\mathrm{GFA2AdderOutput}(x, y, z) = \mathrm{2GatesCircOutput}(x, y, z, \mathrm{xor2c})$.

One can prove the following propositions:

(100) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{GFA2AdderStr}(x, y, z)) = \{\langle\langle x, y\rangle, \mathrm{xor2c}\rangle\} \cup \{\mathrm{GFA2AdderOutput}(x, y, z)\}$.

(101) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{GFA2AdderStr}(x, y, z))$ is a binary relation.

(102) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ holds $\mathrm{InputVertices}(\mathrm{GFA2AdderStr}(x, y, z)) = \{x, y, z\}$.

(103) For all non pair sets x, y, z holds $\mathrm{InputVertices}(\mathrm{GFA2AdderStr}(x, y, z))$ has no pairs.

(104) Let x, y, z be sets. Then

(i) $x \in$ the carrier of $\mathrm{GFA2AdderStr}(x, y, z)$,

(ii) $y \in$ the carrier of $\mathrm{GFA2AdderStr}(x, y, z)$,

(iii) $z \in$ the carrier of $\mathrm{GFA2AdderStr}(x, y, z)$,

(iv) $\langle\langle x, y\rangle, \mathrm{xor2c}\rangle \in$ the carrier of $\mathrm{GFA2AdderStr}(x, y, z)$, and

(v) $\langle\langle\langle\langle x, y\rangle, \mathrm{xor2c}\rangle, z\rangle, \mathrm{xor2c}\rangle \in$ the carrier of $\mathrm{GFA2AdderStr}(x, y, z)$.

(105) For all sets x, y, z holds $\langle\langle x, y\rangle, \mathrm{xor2c}\rangle \in \mathrm{InnerVertices}(\mathrm{GFA2AdderStr}(x, y, z))$ and $\mathrm{GFA2AdderOutput}(x, y, z) \in \mathrm{InnerVertices}(\mathrm{GFA2AdderStr}(x, y, z))$.

(106) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ holds $x \in \mathrm{InputVertices}(\mathrm{GFA2AdderStr}(x, y, z))$ and
$y \in \mathrm{InputVertices}(\mathrm{GFA2AdderStr}(x, y, z))$ and
$z \in \mathrm{InputVertices}(\mathrm{GFA2AdderStr}(x, y, z))$.

(107) For all non pair sets x, y, z holds $\mathrm{InputVertices}(\mathrm{GFA2AdderStr}(x, y, z)) = \{x, y, z\}$.

(108) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$. Let s be a state of $\mathrm{GFA2AdderCirc}(x, y, z)$ and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then $(\mathrm{Following}(s))(\langle\langle x, y\rangle, \mathrm{xor2c}\rangle) = a_1 \oplus \neg a_2$ and $(\mathrm{Following}(s))(x) = a_1$ and $(\mathrm{Following}(s))(y) = a_2$ and $(\mathrm{Following}(s))(z) = a_3$.

(109) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$. Let s be a state of $\mathrm{GFA2AdderCirc}(x, y, z)$ and a_4, a_1, a_2, a_3 be elements of *Boolean*. If $a_4 = s(\langle\langle x, y\rangle, \mathrm{xor2c}\rangle)$ and $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$, then $(\mathrm{Following}(s))(\mathrm{GFA2AdderOutput}(x, y, z)) = a_4 \oplus \neg a_3$.

(110) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$. Let s be a state of $\mathrm{GFA2AdderCirc}(x, y, z)$ and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then $(\mathrm{Following}(s, 2))(\mathrm{GFA2AdderOutput}(x, y, z)) = a_1 \oplus \neg a_2 \oplus \neg a_3$ and $(\mathrm{Following}(s, 2))(\langle\langle x, y\rangle, \mathrm{xor2c}\rangle) = a_1 \oplus \neg a_2$ and $(\mathrm{Following}(s, 2))(x) = a_1$ and $(\mathrm{Following}(s, 2))(y) = a_2$ and $(\mathrm{Following}(s, 2))(z) = a_3$.

(111) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$. Let s be a state of GFA2AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$, then (Following$(s, 2)$)(GFA2AdderOutput$(x, y, z)) = \neg a_1 \oplus a_2 \oplus \neg a_3$.

(112) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and for every state s of GFA2AdderCirc(x, y, z) holds Following$(s, 2)$ is stable.

Let x, y, z be sets. The functor BitGFA2Str(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined as follows:

(Def. 37) BitGFA2Str(x, y, z) = GFA2AdderStr(x, y, z)+· GFA2CarryStr(x, y, z).

Let x, y, z be sets. The functor BitGFA2Circ(x, y, z) yields a strict Boolean circuit of BitGFA2Str(x, y, z) with denotation held in gates and is defined by:

(Def. 38) BitGFA2Circ(x, y, z) = GFA2AdderCirc(x, y, z)+· GFA2CarryCirc(x, y, z).

Next we state several propositions:

(113) For all sets x, y, z holds InnerVertices(BitGFA2Str(x, y, z)) = $\{\langle\langle x, y\rangle, \mathrm{xor2c}\rangle\} \cup \{$GFA2AdderOutput$(x, y, z)\} \cup \{\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, z\rangle, \mathrm{and2c}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\} \cup \{$GFA2CarryOutput$(x, y, z)\}$.

(114) For all sets x, y, z holds InnerVertices(BitGFA2Str(x, y, z)) is a binary relation.

(115) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ holds InputVertices(BitGFA2Str$(x, y, z)) = \{x, y, z\}$.

(116) For all non pair sets x, y, z holds InputVertices(BitGFA2Str$(x, y, z)) = \{x, y, z\}$.

(117) For all non pair sets x, y, z holds InputVertices(BitGFA2Str(x, y, z)) has no pairs.

(118) Let x, y, z be sets. Then $x \in$ the carrier of BitGFA2Str(x, y, z) and $y \in$ the carrier of BitGFA2Str(x, y, z) and $z \in$ the carrier of BitGFA2Str(x, y, z) and $\langle\langle x, y\rangle, \mathrm{xor2c}\rangle \in$ the carrier of BitGFA2Str(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \mathrm{xor2c}\rangle, z\rangle, \mathrm{xor2c}\rangle \in$ the carrier of BitGFA2Str(x, y, z) and $\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle \in$ the carrier of BitGFA2Str(x, y, z) and $\langle\langle y, z\rangle, \mathrm{and2c}\rangle \in$ the carrier of BitGFA2Str(x, y, z) and $\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier of BitGFA2Str(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, z\rangle, \mathrm{and2c}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\rangle, \mathrm{nor}_3\rangle \in$ the carrier of BitGFA2Str(x, y, z).

(119) Let x, y, z be sets. Then $\langle\langle x, y\rangle, \mathrm{xor2c}\rangle \in$ InnerVertices(BitGFA2Str(x, y, z)) and GFA2AdderOutput$(x, y, z) \in$ InnerVertices(BitGFA2Str(x, y, z)) and $\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle \in$ InnerVertices(BitGFA2Str(x, y, z)) and $\langle\langle y, z\rangle, \mathrm{and2c}\rangle \in$ InnerVertices(BitGFA2Str(x, y, z)) and $\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle \in$ InnerVertices(BitGFA2Str(x, y, z)) and GFA2CarryOutput$(x, y, z) \in$

InnerVertices(BitGFA2Str(x, y, z)).

(120) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$. Then $x \in$ InputVertices(BitGFA2Str(x, y, z)) and $y \in$ InputVertices(BitGFA2Str(x, y, z)) and $z \in$ InputVertices(BitGFA2Str(x, y, z)).

Let x, y, z be sets. The functor BitGFA2CarryOutput(x, y, z) yields an element of InnerVertices(BitGFA2Str(x, y, z)) and is defined by:

(Def. 39) BitGFA2CarryOutput$(x, y, z) = \langle\langle\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, z\rangle, \mathrm{and2c}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\rangle, \mathrm{nor}_3\rangle$.

Let x, y, z be sets. The functor BitGFA2AdderOutput(x, y, z) yielding an element of InnerVertices(BitGFA2Str(x, y, z)) is defined by:

(Def. 40) BitGFA2AdderOutput$(x, y, z) =$ 2GatesCircOutput$(x, y, z, \mathrm{xor2c})$.

Next we state two propositions:

(121) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$. Let s be a state of BitGFA2Circ(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA2AdderOutput(x, y, z)) $= \neg a_1 \oplus a_2 \oplus \neg a_3$ and (Following$(s, 2)$)(GFA2CarryOutput(x, y, z)) $= \neg(\neg a_1 \wedge a_2 \vee a_2 \wedge \neg a_3 \vee \neg a_3 \wedge \neg a_1)$.

(122) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$. Let s be a state of BitGFA2Circ(x, y, z). Then Following$(s, 2)$ is stable.

5. Generalized Full Adder (GFA) Circuit (TYPE-3)

Let x, y, z be sets. The functor GFA3CarryIStr(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def. 41) GFA3CarryIStr$(x, y, z) =$ 1GateCircStr$(\langle x, y\rangle, \mathrm{and}_{2b}) +\cdot$ 1GateCircStr$(\langle y, z\rangle, \mathrm{and}_{2b}) +\cdot$ 1GateCircStr$(\langle z, x\rangle, \mathrm{and}_{2b})$.

Let x, y, z be sets. The functor GFA3CarryICirc(x, y, z) yielding a strict Boolean circuit of GFA3CarryIStr(x, y, z) with denotation held in gates is defined by:

(Def. 42) GFA3CarryICirc$(x, y, z) =$ 1GateCircuit$(x, y, \mathrm{and}_{2b}) +\cdot$ 1GateCircuit$(y, z, \mathrm{and}_{2b}) +\cdot$ 1GateCircuit$(z, x, \mathrm{and}_{2b})$.

Let x, y, z be sets. The functor GFA3CarryStr(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def. 43) $\mathrm{GFA3CarryStr}(x, y, z) = \mathrm{GFA3CarryIStr}(x, y, z) +\cdot \mathrm{1GateCircStr}(\langle\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\rangle, \mathrm{nor}_3)$.

Let x, y, z be sets. The functor $\mathrm{GFA3CarryCirc}(x, y, z)$ yielding a strict Boolean circuit of $\mathrm{GFA3CarryStr}(x, y, z)$ with denotation held in gates is defined by:

(Def. 44) $\mathrm{GFA3CarryCirc}(x, y, z) = \mathrm{GFA3CarryICirc}(x, y, z) +\cdot \mathrm{1GateCircuit}(\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle, \mathrm{nor}_3)$.

Let x, y, z be sets. The functor $\mathrm{GFA3CarryOutput}(x, y, z)$ yields an element of $\mathrm{InnerVertices}(\mathrm{GFA3CarryStr}(x, y, z))$ and is defined as follows:

(Def. 45) $\mathrm{GFA3CarryOutput}(x, y, z) = \langle\langle\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\rangle, \mathrm{nor}_3\rangle$.

The following propositions are true:

(123) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{GFA3CarryIStr}(x, y, z)) = \{\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\}$.

(124) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{GFA3CarryStr}(x, y, z)) = \{\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\} \cup \{\mathrm{GFA3CarryOutput}(x, y, z)\}$.

(125) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{GFA3CarryStr}(x, y, z))$ is a binary relation.

(126) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle$ holds $\mathrm{InputVertices}(\mathrm{GFA3CarryIStr}(x, y, z)) = \{x, y, z\}$.

(127) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle$ holds $\mathrm{InputVertices}(\mathrm{GFA3CarryStr}(x, y, z)) = \{x, y, z\}$.

(128) For all non pair sets x, y, z holds $\mathrm{InputVertices}(\mathrm{GFA3CarryStr}(x, y, z))$ has no pairs.

(129) Let x, y, z be sets. Then $x \in$ the carrier of $\mathrm{GFA3CarryStr}(x, y, z)$ and $y \in$ the carrier of $\mathrm{GFA3CarryStr}(x, y, z)$ and $z \in$ the carrier of $\mathrm{GFA3CarryStr}(x, y, z)$ and $\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier of $\mathrm{GFA3CarryStr}(x, y, z)$ and $\langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier of $\mathrm{GFA3CarryStr}(x, y, z)$ and $\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier of $\mathrm{GFA3CarryStr}(x, y, z)$ and $\langle\langle\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\rangle, \mathrm{nor}_3\rangle \in$ the carrier of $\mathrm{GFA3CarryStr}(x, y, z)$.

(130) For all sets x, y, z holds $\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle \in \mathrm{InnerVertices}(\mathrm{GFA3CarryStr}(x, y, z))$ and $\langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle \in \mathrm{InnerVertices}(\mathrm{GFA3CarryStr}(x, y, z))$ and $\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle \in \mathrm{InnerVertices}(\mathrm{GFA3CarryStr}(x, y, z))$ and $\mathrm{GFA3CarryOutput}(x, y, z) \in \mathrm{InnerVertices}(\mathrm{GFA3CarryStr}(x, y, z))$.

(131) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle$ holds $x \in \mathrm{InputVertices}(\mathrm{GFA3CarryStr}(x, y, z))$ and $y \in \mathrm{InputVertices}(\mathrm{GFA3CarryStr}(x, y, z))$ and

$z \in$ InputVertices(GFA3CarryStr(x, y, z)).

(132) For all non pair sets x, y, z holds InputVertices(GFA3CarryStr(x, y, z)) $= \{x, y, z\}$.

(133) Let x, y, z be sets, s be a state of GFA3CarryCirc(x, y, z), and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following(s))($\langle\!\langle \langle x, y\rangle, \mathrm{and}_{2b} \rangle\!\rangle$) $= \neg a_1 \wedge \neg a_2$ and (Following(s))($\langle\!\langle \langle y, z\rangle, \mathrm{and}_{2b} \rangle\!\rangle$) $= \neg a_2 \wedge \neg a_3$ and (Following(s))($\langle\!\langle \langle z, x\rangle, \mathrm{and}_{2b} \rangle\!\rangle$) $= \neg a_3 \wedge \neg a_1$.

(134) Let x, y, z be sets, s be a state of GFA3CarryCirc(x, y, z), and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(\langle\!\langle \langle x, y\rangle, \mathrm{and}_{2b} \rangle\!\rangle)$ and $a_2 = s(\langle\!\langle \langle y, z\rangle, \mathrm{and}_{2b} \rangle\!\rangle)$ and $a_3 = s(\langle\!\langle \langle z, x\rangle, \mathrm{and}_{2b} \rangle\!\rangle)$, then (Following$(s)$)(GFA3CarryOutput$(x, y, z)$) $= \neg(a_1 \vee a_2 \vee a_3)$.

(135) Let x, y, z be sets. Suppose $x \neq \langle\!\langle \langle y, z\rangle, \mathrm{and}_{2b} \rangle\!\rangle$ and $y \neq \langle\!\langle \langle z, x\rangle, \mathrm{and}_{2b} \rangle\!\rangle$ and $z \neq \langle\!\langle \langle x, y\rangle, \mathrm{and}_{2b} \rangle\!\rangle$. Let s be a state of GFA3CarryCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA3CarryOutput(x, y, z)) $= \neg(\neg a_1 \wedge \neg a_2 \vee \neg a_2 \wedge \neg a_3 \vee \neg a_3 \wedge \neg a_1)$ and (Following$(s, 2)$)($\langle\!\langle \langle x, y\rangle, \mathrm{and}_{2b} \rangle\!\rangle$) $= \neg a_1 \wedge \neg a_2$ and (Following$(s, 2)$)($\langle\!\langle \langle y, z\rangle, \mathrm{and}_{2b} \rangle\!\rangle$) $= \neg a_2 \wedge \neg a_3$ and (Following$(s, 2)$)($\langle\!\langle \langle z, x\rangle, \mathrm{and}_{2b} \rangle\!\rangle$) $= \neg a_3 \wedge \neg a_1$.

(136) For all sets x, y, z such that $x \neq \langle\!\langle \langle y, z\rangle, \mathrm{and}_{2b} \rangle\!\rangle$ and $y \neq \langle\!\langle \langle z, x\rangle, \mathrm{and}_{2b} \rangle\!\rangle$ and $z \neq \langle\!\langle \langle x, y\rangle, \mathrm{and}_{2b} \rangle\!\rangle$ and for every state s of GFA3CarryCirc(x, y, z) holds Following$(s, 2)$ is stable.

Let x, y, z be sets. The functor GFA3AdderStr(x, y, z) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined by:

(Def. 46) GFA3AdderStr$(x, y, z) =$ 2GatesCircStr$(x, y, z, \mathrm{xor}_2)$.

Let x, y, z be sets. The functor GFA3AdderCirc(x, y, z) yielding a strict Boolean circuit of GFA3AdderStr(x, y, z) with denotation held in gates is defined by:

(Def. 47) GFA3AdderCirc$(x, y, z) =$ 2GatesCircuit$(x, y, z, \mathrm{xor}_2)$.

Let x, y, z be sets. The functor GFA3AdderOutput(x, y, z) yielding an element of InnerVertices(GFA3AdderStr(x, y, z)) is defined by:

(Def. 48) GFA3AdderOutput$(x, y, z) =$ 2GatesCircOutput$(x, y, z, \mathrm{xor}_2)$.

One can prove the following propositions:

(137) For all sets x, y, z holds InnerVertices(GFA3AdderStr(x, y, z)) $= \{\langle\!\langle \langle x, y\rangle, \mathrm{xor}_2 \rangle\!\rangle\} \cup \{$GFA3AdderOutput$(x, y, z)\}$.

(138) For all sets x, y, z holds InnerVertices(GFA3AdderStr(x, y, z)) is a binary relation.

(139) For all sets x, y, z such that $z \neq \langle\!\langle \langle x, y\rangle, \mathrm{xor}_2 \rangle\!\rangle$ holds InputVertices(GFA3AdderStr(x, y, z)) $= \{x, y, z\}$.

(140) For all non pair sets x, y, z holds InputVertices(GFA3AdderStr(x, y, z)) has no pairs.

(141) Let x, y, z be sets. Then

(i) $x \in$ the carrier of GFA3AdderStr(x, y, z),

(ii) $y \in$ the carrier of GFA3AdderStr(x, y, z),

(iii) $z \in$ the carrier of GFA3AdderStr(x, y, z),

(iv) $\langle\langle x, y\rangle, \mathrm{xor}_2\rangle \in$ the carrier of GFA3AdderStr(x, y, z), and

(v) $\langle\langle\langle\langle x, y\rangle, \mathrm{xor}_2\rangle, z\rangle, \mathrm{xor}_2\rangle \in$ the carrier of GFA3AdderStr(x, y, z).

(142) For all sets x, y, z holds $\langle\langle x, y\rangle, \mathrm{xor}_2\rangle \in$ InnerVertices(GFA3AdderStr(x, y, z)) and GFA3AdderOutput$(x, y, z) \in$ InnerVertices(GFA3AdderStr(x, y, z)).

(143) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ holds $x \in$ InputVertices(GFA3AdderStr(x, y, z)) and
$y \in$ InputVertices(GFA3AdderStr(x, y, z)) and
$z \in$ InputVertices(GFA3AdderStr(x, y, z)).

(144) For all non pair sets x, y, z holds InputVertices(GFA3AdderStr(x, y, z)) $= \{x, y, z\}$.

(145) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$. Let s be a state of GFA3AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following(s))$(\langle\langle x, y\rangle, \mathrm{xor}_2\rangle) = a_1 \oplus a_2$ and (Following(s))$(x) = a_1$ and (Following(s))$(y) = a_2$ and (Following(s))$(z) = a_3$.

(146) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$. Let s be a state of GFA3AdderCirc(x, y, z) and a_4, a_1, a_2, a_3 be elements of *Boolean*. If $a_4 = s(\langle\langle x, y\rangle, \mathrm{xor}_2\rangle)$ and $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$, then (Following(s))(GFA3AdderOutput(x, y, z)) $= a_4 \oplus a_3$.

(147) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$. Let s be a state of GFA3AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA3AdderOutput(x, y, z)) $= a_1 \oplus a_2 \oplus a_3$ and (Following$(s, 2)$)$(\langle\langle x, y\rangle, \mathrm{xor}_2\rangle) = a_1 \oplus a_2$ and (Following$(s, 2)$)$(x) = a_1$ and (Following$(s, 2)$)$(y) = a_2$ and (Following$(s, 2)$)$(z) = a_3$.

(148) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$. Let s be a state of GFA3AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$, then (Following$(s, 2)$)(GFA3AdderOutput(x, y, z)) $= \neg(\neg a_1 \oplus \neg a_2 \oplus \neg a_3)$.

(149) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ and for every state s of GFA3AdderCirc(x, y, z) holds Following$(s, 2)$ is stable.

Let x, y, z be sets. The functor BitGFA3Str(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and

Boolean denotation held in gates is defined by:

(Def. 49) $\mathrm{BitGFA3Str}(x, y, z) = \mathrm{GFA3AdderStr}(x, y, z) {+\cdot}\, \mathrm{GFA3CarryStr}(x, y, z)$.

Let x, y, z be sets. The functor $\mathrm{BitGFA3Circ}(x, y, z)$ yields a strict Boolean circuit of $\mathrm{BitGFA3Str}(x, y, z)$ with denotation held in gates and is defined as follows:

(Def. 50) $\mathrm{BitGFA3Circ}(x, y, z) = \mathrm{GFA3AdderCirc}(x, y, z) {+\cdot}\, \mathrm{GFA3CarryCirc}(x, y, z)$.

One can prove the following propositions:

(150) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{BitGFA3Str}(x, y, z)) = \{\langle\langle x, y\rangle, \mathrm{xor}_2\rangle\} \cup \{\mathrm{GFA3AdderOutput}(x, y, z)\} \cup \{\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\} \cup \{\mathrm{GFA3CarryOutput}(x, y, z)\}$.

(151) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{BitGFA3Str}(x, y, z))$ is a binary relation.

(152) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle$ holds $\mathrm{InputVertices}(\mathrm{BitGFA3Str}(x, y, z)) = \{x, y, z\}$.

(153) For all non pair sets x, y, z holds $\mathrm{InputVertices}(\mathrm{BitGFA3Str}(x, y, z)) = \{x, y, z\}$.

(154) For all non pair sets x, y, z holds $\mathrm{InputVertices}(\mathrm{BitGFA3Str}(x, y, z))$ has no pairs.

(155) Let x, y, z be sets. Then $x \in$ the carrier of $\mathrm{BitGFA3Str}(x, y, z)$ and $y \in$ the carrier of $\mathrm{BitGFA3Str}(x, y, z)$ and $z \in$ the carrier of $\mathrm{BitGFA3Str}(x, y, z)$ and $\langle\langle x, y\rangle, \mathrm{xor}_2\rangle \in$ the carrier of $\mathrm{BitGFA3Str}(x, y, z)$ and $\langle\langle\langle\langle x, y\rangle, \mathrm{xor}_2\rangle, z\rangle, \mathrm{xor}_2\rangle \in$ the carrier of $\mathrm{BitGFA3Str}(x, y, z)$ and $\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier of $\mathrm{BitGFA3Str}(x, y, z)$ and $\langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier of $\mathrm{BitGFA3Str}(x, y, z)$ and $\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier of $\mathrm{BitGFA3Str}(x, y, z)$ and $\langle\langle\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\rangle, \mathrm{nor}_3\rangle \in$ the carrier of $\mathrm{BitGFA3Str}(x, y, z)$.

(156) Let x, y, z be sets. Then $\langle\langle x, y\rangle, \mathrm{xor}_2\rangle \in \mathrm{InnerVertices}(\mathrm{BitGFA3Str}(x, y, z))$ and $\mathrm{GFA3AdderOutput}(x, y, z) \in \mathrm{InnerVertices}(\mathrm{BitGFA3Str}(x, y, z))$ and $\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle \in \mathrm{InnerVertices}(\mathrm{BitGFA3Str}(x, y, z))$ and $\langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle \in \mathrm{InnerVertices}(\mathrm{BitGFA3Str}(x, y, z))$ and $\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle \in \mathrm{InnerVertices}(\mathrm{BitGFA3Str}(x, y, z))$ and $\mathrm{GFA3CarryOutput}(x, y, z) \in \mathrm{InnerVertices}(\mathrm{BitGFA3Str}(x, y, z))$.

(157) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle$. Then $x \in \mathrm{InputVertices}(\mathrm{BitGFA3Str}(x, y, z))$ and $y \in \mathrm{InputVertices}(\mathrm{BitGFA3Str}(x, y, z))$ and $z \in \mathrm{InputVertices}(\mathrm{BitGFA3Str}(x, y, z))$.

Let x, y, z be sets. The functor $\mathrm{BitGFA3CarryOutput}(x, y, z)$ yields an element of $\mathrm{InnerVertices}(\mathrm{BitGFA3Str}(x, y, z))$ and is defined by:

(Def. 51) $\mathrm{BitGFA3CarryOutput}(x,y,z) = \langle\langle\langle\langle x,y\rangle, \mathrm{and}_{2b}\rangle, \langle\langle y,z\rangle, \mathrm{and}_{2b}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\rangle, \mathrm{nor}_3\rangle$.

Let x, y, z be sets. The functor $\mathrm{BitGFA3AdderOutput}(x,y,z)$ yielding an element of $\mathrm{InnerVertices}(\mathrm{BitGFA3Str}(x,y,z))$ is defined by:

(Def. 52) $\mathrm{BitGFA3AdderOutput}(x,y,z) = \mathrm{2GatesCircOutput}(x,y,z,\mathrm{xor}_2)$.

Next we state two propositions:

(158) Let x, y, z be sets. Suppose $z \neq \langle\langle x,y\rangle, \mathrm{xor}_2\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle$ and $y \neq \langle\langle z,x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x,y\rangle, \mathrm{and}_{2b}\rangle$. Let s be a state of $\mathrm{BitGFA3Circ}(x,y,z)$ and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then $(\mathrm{Following}(s,2))(\mathrm{GFA3AdderOutput}(x,y,z)) = \neg(\neg a_1 \oplus \neg a_2 \oplus \neg a_3)$ and $(\mathrm{Following}(s,2))(\mathrm{GFA3CarryOutput}(x,y,z)) = \neg(\neg a_1 \wedge \neg a_2 \vee \neg a_2 \wedge \neg a_3 \vee \neg a_3 \wedge \neg a_1)$.

(159) Let x, y, z be sets. Suppose $z \neq \langle\langle x,y\rangle, \mathrm{xor}_2\rangle$ and $x \neq \langle\langle y,z\rangle, \mathrm{and}_{2b}\rangle$ and $y \neq \langle\langle z,x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x,y\rangle, \mathrm{and}_{2b}\rangle$. Let s be a state of $\mathrm{BitGFA3Circ}(x,y,z)$. Then $\mathrm{Following}(s,2)$ is stable.

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Quotient Rings

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Summary. The notions of prime ideals and maximal ideals of a ring are introduced. Quotient rings are defined. Characterisation of prime and maximal ideals using quotient rings are proved.

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The articles [18], [10], [22], [17], [2], [19], [6], [23], [24], [7], [9], [8], [25], [15], [3], [4], [5], [14], [20], [16], [13], [21], [11], [12], and [1] provide the terminology and notation for this paper.

1. Preliminaries

Let S be a non empty 1-sorted structure. Note that Ω_S is non proper.

The following propositions are true:

(1) Let L be an add-associative right zeroed right complementable non empty loop structure and a, b be elements of L. Then $(a-b)+b=a$.

(2) Let L be an add-associative right zeroed right complementable Abelian non empty loop structure and b, c be elements of L. Then $c=b-(b-c)$.

(3) Let L be an add-associative right zeroed right complementable Abelian non empty loop structure and a, b, c be elements of L. Then $a-b-(c-b)=a-c$.

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2. Ideals

Let K be a non empty groupoid and let S be a subset of K. We say that S is quasi-prime if and only if:

(Def. 1) For all elements a, b of K such that $a \cdot b \in S$ holds $a \in S$ or $b \in S$.

Let K be a non empty multiplicative loop structure and let S be a subset of K. We say that S is prime if and only if:

(Def. 2) S is proper and quasi-prime.

Let R be a non empty double loop structure and let I be a subset of R. We say that I is quasi-maximal if and only if:

(Def. 3) For every ideal J of R such that $I \subseteq J$ holds $J = I$ or J is non proper.

Let R be a non empty double loop structure and let I be a subset of R. We say that I is maximal if and only if:

(Def. 4) I is proper and quasi-maximal.

Let K be a non empty multiplicative loop structure. Note that every subset of K which is prime is also proper and quasi-prime and every subset of K which is proper and quasi-prime is also prime.

Let R be a non empty double loop structure. One can verify that every subset of R which is maximal is also proper and quasi-maximal and every subset of R which is proper and quasi-maximal is also maximal.

Let R be a non empty loop structure. One can verify that Ω_R is add closed.

Let R be a non empty groupoid. Observe that Ω_R is left ideal and right ideal.

We now state the proposition

(4) For every integral domain R holds $\{0_R\}$ is prime.

3. Equivalence Relation

In the sequel R denotes a ring, I denotes an ideal of R, and a, b denote elements of R.

Let R be a ring and let I be an ideal of R. The functor $\approx_I$ yielding a binary relation on R is defined by:

(Def. 5) For all elements a, b of R holds $\langle a, b\rangle \in \approx_I$ iff $a - b \in I$.

Let R be a ring and let I be an ideal of R. One can verify that $\approx_I$ is non empty, total, symmetric, and transitive.

We now state several propositions:

(5) $a \in [b]_{\approx_I}$ iff $a - b \in I$.

(6) $[a]_{\approx_I} = [b]_{\approx_I}$ iff $a - b \in I$.

(7) $[a]_{\approx_{\Omega_R}} =$ the carrier of R.

(8) $\approx_{\Omega_R} = \{\text{the carrier of } R\}$.

(9) $[a]_{\approx_{\{0_R\}}} = \{a\}$.

(10) $\approx_{\{0_R\}} = \text{rng}(\text{singleton}_{\text{the carrier of } R})$.

4. Quotient Ring

Let R be a ring and let I be an ideal of R. The functor ${}^{R}/_{I}$ yields a strict double loop structure and is defined by the conditions (Def. 6).

(Def. 6)(i) The carrier of ${}^{R}/_{I} = \text{Classes}(\approx_I)$,

(ii) the unity of ${}^{R}/_{I} = [1_R]_{\approx_I}$,

(iii) the zero of ${}^{R}/_{I} = [0_R]_{\approx_I}$,

(iv) for all elements x, y of ${}^{R}/_{I}$ there exist elements a, b of R such that $x = [a]_{\approx_I}$ and $y = [b]_{\approx_I}$ and (the addition of ${}^{R}/_{I}$)$(x, y) = [a + b]_{\approx_I}$, and

(v) for all elements x, y of ${}^{R}/_{I}$ there exist elements a, b of R such that $x = [a]_{\approx_I}$ and $y = [b]_{\approx_I}$ and (the multiplication of ${}^{R}/_{I}$)$(x, y) = [a \cdot b]_{\approx_I}$.

Let R be a ring and let I be an ideal of R. Note that ${}^{R}/_{I}$ is non empty.

In the sequel x, y denote elements of ${}^{R}/_{I}$.

We now state several propositions:

(11) There exists an element a of R such that $x = [a]_{\approx_I}$.

(12) $[a]_{\approx_I}$ is an element of ${}^{R}/_{I}$.

(13) If $x = [a]_{\approx_I}$ and $y = [b]_{\approx_I}$, then $x + y = [a + b]_{\approx_I}$.

(14) If $x = [a]_{\approx_I}$ and $y = [b]_{\approx_I}$, then $x \cdot y = [a \cdot b]_{\approx_I}$.

(15) $[1_R]_{\approx_I} = 1_{{}^{R}/_{I}}$.

Let R be a ring and let I be an ideal of R. Observe that ${}^{R}/_{I}$ is Abelian, add-associative, and right zeroed.

Let R be a commutative ring and let I be an ideal of R. Note that ${}^{R}/_{I}$ is commutative.

The following propositions are true:

(16) I is proper iff ${}^{R}/_{I}$ is non degenerated.

(17) I is quasi-prime iff ${}^{R}/_{I}$ is integral domain-like.

(18) For every commutative ring R and for every ideal I of R holds I is prime iff ${}^{R}/_{I}$ is an integral domain.

(19) If R is commutative and I is quasi-maximal, then ${}^{R}/_{I}$ is field-like.

(20) If ${}^{R}/_{I}$ is field-like, then I is quasi-maximal.

(21) For every commutative ring R and for every ideal I of R holds I is maximal iff ${}^{R}/_{I}$ is a skew field.

Let R be a non degenerated commutative ring. One can check that every ideal of R which is maximal is also prime.

Let R be a non degenerated ring. Note that there exists an ideal of R which is maximal.

Let R be a non degenerated commutative ring and let I be a quasi-prime ideal of R. Observe that ${}^{R}/_{I}$ is integral domain-like.

Let R be a non degenerated commutative ring and let I be a quasi-maximal ideal of R. Observe that ${}^{R}/_{I}$ is field-like.

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Completeness of the Real Euclidean Space

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The terminology and notation used here are introduced in the following articles: [21], [8], [24], [25], [6], [26], [7], [3], [14], [2], [5], [1], [20], [22], [4], [23], [15], [16], [13], [12], [11], [9], [18], [10], [19], and [17].

1. The Real Euclidean Space as a Real Linear Space

In this paper n is a natural number.

Let n be a natural number. The functor $\langle \mathcal{E}^n, \|\cdot\| \rangle$ yields a strict non empty normed structure and is defined by the conditions (Def. 1).

(Def. 1)(i) The carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle = \mathcal{R}^n$,

(ii) the zero of $\langle \mathcal{E}^n, \|\cdot\| \rangle = \langle \underbrace{0, \ldots, 0}_{n} \rangle$,

(iii) for all elements a, b of $\mathcal{R}^n$ holds (the addition of $\langle \mathcal{E}^n, \|\cdot\| \rangle)(a, b) = a+b$,

(iv) for every element r of $\mathbb{R}$ and for every element x of $\mathcal{R}^n$ holds (the external multiplication of $\langle \mathcal{E}^n, \|\cdot\| \rangle)(r, x) = r \cdot x$, and

(v) for every element x of $\mathcal{R}^n$ holds (the norm of $\langle \mathcal{E}^n, \|\cdot\| \rangle)(x) = |x|$.

Let n be a natural number. Note that the addition of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is commutative and associative.

Let n be a non empty natural number. Note that $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is non trivial.

One can prove the following propositions:

(1) For every vector x of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and for every element y of $\mathcal{R}^n$ such that $x = y$ holds $\|x\| = |y|$.

(2) Let n be a natural number, x, y be vectors of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a, b be elements of $\mathcal{R}^n$. If $x = a$ and $y = b$, then $x + y = a + b$.

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(3) For every vector x of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and for every element y of $\mathcal{R}^n$ and for every real number a such that $x = y$ holds $a \cdot x = a \cdot y$.

Let n be a natural number. Note that $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

One can prove the following propositions:

(4) For every vector x of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and for every element a of $\mathcal{R}^n$ such that $x = a$ holds $-x = -a$.

(5) For all vectors x, y of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and for all elements a, b of $\mathcal{R}^n$ such that $x = a$ and $y = b$ holds $x - y = a - b$.

(6) For every finite sequence f of elements of $\mathbb{R}$ such that $\operatorname{dom} f = \operatorname{Seg} n$ holds f is an element of $\mathcal{R}^n$.

(7) Let n be a natural number and x be an element of $\mathcal{R}^n$. Suppose that for every natural number i such that $i \in \operatorname{Seg} n$ holds $0 \leq x(i)$. Then $0 \leq \sum x$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $x(i) \leq \sum x$.

(8) For every element x of $\mathcal{R}^n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $|x(i)| \leq |x|$.

(9) Let x be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and y be an element of $\mathcal{R}^n$. If $x = y$, then for every natural number i such that $i \in \operatorname{Seg} n$ holds $|y(i)| \leq \|x\|$.

(10) For every element x of $\mathcal{R}^{n+1}$ holds $|x|^{\mathbf{2}} = |x{\restriction}n|^{\mathbf{2}} + x(n+1)^{\mathbf{2}}$.

Let n be a natural number, let f be a function from $\mathbb{N}$ into $\mathcal{R}^n$, and let k be a natural number. Then $f(k)$ is an element of $\mathcal{R}^n$.

We now state two propositions:

(11) Let n be a natural number, x be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x_2 be an element of $\mathcal{R}^n$, s_1 be a sequence of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and x_1 be a function from $\mathbb{N}$ into $\mathcal{R}^n$. Suppose $x_2 = x$ and $x_1 = s_1$. Then s_1 is convergent and $\lim s_1 = x$ if and only if for every natural number i such that $i \in \operatorname{Seg} n$ there exists a sequence r_1 of real numbers such that for every natural number k holds $r_1(k) = x_1(k)(i)$ and r_1 is convergent and $x_2(i) = \lim r_1$.

(12) For every sequence f of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that f is Cauchy sequence by norm holds f is convergent.

Let us consider n. Note that $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is complete.

2. The Real Euclidean Space as a Real Normed Space

Let n be a natural number. The functor $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ yields a strict non empty unitary space structure and is defined by the conditions (Def. 2).

(Def. 2)(i) The RLS structure of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ = the RLS structure of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and

(ii) for all elements x, y of $\mathcal{R}^n$ holds (the scalar product of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$)$(x, y) = \sum(x \bullet y)$.

Let n be a non empty natural number. One can verify that $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is non trivial.

Let n be a natural number. Observe that $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is real unitary space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

The following propositions are true:

(13) Let n be a natural number, a be a real number, x_3, y_1 be points of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and x_4, y_2 be points of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $x_3 = x_4$ and $y_1 = y_2$, then $x_3 + y_1 = x_4 + y_2$ and $-x_3 = -x_4$ and $a \cdot x_3 = a \cdot x_4$.

(14) For every natural number n and for every point x_3 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ and for every point x_4 of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ such that $x_3 = x_4$ holds $\|x_3\|^{\mathbf{2}} = (x_4|x_4)$.

(15) Let n be a natural number and f be a set. Then f is a sequence of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ if and only if f is a sequence of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$.

(16) Let n be a natural number, s_2 be a sequence of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and s_3 be a sequence of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ such that $s_2 = s_3$. Then

(i) if s_2 is convergent, then s_3 is convergent and $\lim s_2 = \lim s_3$, and

(ii) if s_3 is convergent, then s_2 is convergent and $\lim s_2 = \lim s_3$.

(17) Let n be a natural number, s_2 be a sequence of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and s_3 be a sequence of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $s_2 = s_3$ and s_2 is Cauchy sequence by norm, then s_3 is Cauchy.

(18) Let n be a natural number, s_2 be a sequence of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and s_3 be a sequence of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $s_2 = s_3$ and s_3 is Cauchy, then s_2 is Cauchy sequence by norm.

Let us consider n. Note that $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is Hilbert.

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