FORMALIZED MATHEMATICS
Volume 12, Number 3, 2004
University of Białystok

Algebra of Complex Vector Valued Functions

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Summary. This article is an extension of [17].

MML Identifier: `VFUNCT_2`.

The notation and terminology used here have been introduced in the following papers: [12], [15], [2], [11], [4], [16], [5], [7], [14], [9], [8], [3], [1], [13], [10], and [6].

For simplicity, we follow the rules: M denotes a non empty set, V denotes a complex normed space, f, f_1, f_2, f_3 denote partial functions from M to the carrier of V, and z, z_1, z_2 denote complex numbers.

Let M be a non empty set, let V be a complex normed space, and let f_1, f_2 be partial functions from M to the carrier of V. The functor $f_1 + f_2$ yields a partial function from M to the carrier of V and is defined by:

(Def. 1) $\mathrm{dom}(f_1 + f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every element c of M such that $c \in \mathrm{dom}(f_1 + f_2)$ holds $(f_1 + f_2)_c = (f_1)_c + (f_2)_c$.

The functor $f_1 - f_2$ yields a partial function from M to the carrier of V and is defined as follows:

(Def. 2) $\mathrm{dom}(f_1 - f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every element c of M such that $c \in \mathrm{dom}(f_1 - f_2)$ holds $(f_1 - f_2)_c = (f_1)_c - (f_2)_c$.

Let M be a non empty set, let V be a complex normed space, let f_1 be a partial function from M to $\mathbb{C}$, and let f_2 be a partial function from M to the carrier of V. The functor $f_1\, f_2$ yielding a partial function from M to the carrier of V is defined by:

(Def. 3) $\mathrm{dom}(f_1\, f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every element c of M such that $c \in \mathrm{dom}(f_1\, f_2)$ holds $(f_1\, f_2)_c = (f_1)_c \cdot (f_2)_c$.

Let X be a non empty set, let V be a complex normed space, let f be a partial function from X to the carrier of V, and let z be a complex number. The

ISSN 1426–2630

functor $z\,f$ yields a partial function from X to the carrier of V and is defined as follows:

(Def. 4) $\mathrm{dom}(z\,f) = \mathrm{dom}\,f$ and for every element x of X such that $x \in \mathrm{dom}(z\,f)$ holds $(z\,f)_x = z \cdot f_x$.

Let X be a non empty set, let V be a complex normed space, and let f be a partial function from X to the carrier of V. The functor $\|f\|$ yielding a partial function from X to $\mathbb{R}$ is defined as follows:

(Def. 5) $\mathrm{dom}\|f\| = \mathrm{dom}\,f$ and for every element x of X such that $x \in \mathrm{dom}\|f\|$ holds $\|f\|(x) = \|f_x\|$.

The functor $-f$ yields a partial function from X to the carrier of V and is defined by:

(Def. 6) $\mathrm{dom}(-f) = \mathrm{dom}\,f$ and for every element x of X such that $x \in \mathrm{dom}(-f)$ holds $(-f)_x = -f_x$.

The following propositions are true:

(1) Let f_1 be a partial function from M to $\mathbb{C}$ and f_2 be a partial function from M to the carrier of V. Then $\mathrm{dom}(f_1\,f_2) \setminus (f_1\,f_2)^{-1}(\{0_V\}) = (\mathrm{dom}\,f_1 \setminus {f_1}^{-1}(\{0\})) \cap (\mathrm{dom}\,f_2 \setminus {f_2}^{-1}(\{0_V\}))$.

(2) $\|f\|^{-1}(\{0\}) = f^{-1}(\{0_V\})$ and $(-f)^{-1}(\{0_V\}) = f^{-1}(\{0_V\})$.

(3) If $z \neq 0_{\mathbb{C}}$, then $(z\,f)^{-1}(\{0_V\}) = f^{-1}(\{0_V\})$.

(4) $f_1 + f_2 = f_2 + f_1$.

(5) $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$.

(6) Let f_1, f_2 be partial functions from M to $\mathbb{C}$ and f_3 be a partial function from M to the carrier of V. Then $(f_1\,f_2)\,f_3 = f_1\,(f_2\,f_3)$.

(7) For all partial functions f_1, f_2 from M to $\mathbb{C}$ holds $(f_1 + f_2)\,f_3 = f_1\,f_3 + f_2\,f_3$.

(8) For every partial function f_3 from M to $\mathbb{C}$ holds $f_3\,(f_1 + f_2) = f_3\,f_1 + f_3\,f_2$.

(9) For every partial function f_1 from M to $\mathbb{C}$ holds $z\,(f_1\,f_2) = (z\,f_1)\,f_2$.

(10) For every partial function f_1 from M to $\mathbb{C}$ holds $z\,(f_1\,f_2) = f_1\,(z\,f_2)$.

(11) For all partial functions f_1, f_2 from M to $\mathbb{C}$ holds $(f_1 - f_2)\,f_3 = f_1\,f_3 - f_2\,f_3$.

(12) For every partial function f_3 from M to $\mathbb{C}$ holds $f_3\,f_1 - f_3\,f_2 = f_3\,(f_1 - f_2)$.

(13) $z\,(f_1 + f_2) = z\,f_1 + z\,f_2$.

(14) $(z_1 \cdot z_2)\,f = z_1\,(z_2\,f)$.

(15) $z\,(f_1 - f_2) = z\,f_1 - z\,f_2$.

(16) $f_1 - f_2 = (-1_{\mathbb{C}})\,(f_2 - f_1)$.

(17) $f_1 - (f_2 + f_3) = f_1 - f_2 - f_3$.

(18) $1_{\mathbb{C}}\, f = f$.

(19) $f_1 - (f_2 - f_3) = (f_1 - f_2) + f_3$.

(20) $f_1 + (f_2 - f_3) = (f_1 + f_2) - f_3$.

(21) For every partial function f_1 from M to $\mathbb{C}$ holds $\|f_1\, f_2\| = |f_1|\, \|f_2\|$.

(22) $\|z\, f\| = |z|\, \|f\|$.

(23) $-f = (-1_{\mathbb{C}})\, f$.

(24) $--f = f$.

(25) $f_1 - f_2 = f_1 + -f_2$.

(26) $f_1 - -f_2 = f_1 + f_2$.

In the sequel X, Y denote sets.

We now state a number of propositions:

(27) $(f_1+f_2)\restriction X = f_1\restriction X+f_2\restriction X$ and $(f_1+f_2)\restriction X = f_1\restriction X+f_2$ and $(f_1+f_2)\restriction X = f_1 + f_2\restriction X$.

(28) For every partial function f_1 from M to $\mathbb{C}$ holds $(f_1\, f_2)\restriction X = (f_1\restriction X)\,(f_2\restriction X)$ and $(f_1\, f_2)\restriction X = (f_1\restriction X)\, f_2$ and $(f_1\, f_2)\restriction X = f_1\,(f_2\restriction X)$.

(29) $(-f)\restriction X = -f\restriction X$ and $\|f\|\restriction X = \|f\restriction X\|$.

(30) $(f_1-f_2)\restriction X = f_1\restriction X-f_2\restriction X$ and $(f_1-f_2)\restriction X = f_1\restriction X-f_2$ and $(f_1-f_2)\restriction X = f_1 - f_2\restriction X$.

(31) $(z\, f)\restriction X = z\,(f\restriction X)$.

(32) f_1 is total and f_2 is total iff $f_1 + f_2$ is total and f_1 is total and f_2 is total iff $f_1 - f_2$ is total.

(33) For every partial function f_1 from M to $\mathbb{C}$ holds f_1 is total and f_2 is total iff $f_1\, f_2$ is total.

(34) f is total iff $z\, f$ is total.

(35) f is total iff $-f$ is total.

(36) f is total iff $\|f\|$ is total.

(37) For every element x of M such that f_1 is total and f_2 is total holds $(f_1 + f_2)_x = (f_1)_x + (f_2)_x$ and $(f_1 - f_2)_x = (f_1)_x - (f_2)_x$.

(38) Let f_1 be a partial function from M to $\mathbb{C}$ and x be an element of M. If f_1 is total and f_2 is total, then $(f_1\, f_2)_x = (f_1)_x \cdot (f_2)_x$.

(39) For every element x of M such that f is total holds $(z\, f)_x = z \cdot f_x$.

(40) For every element x of M such that f is total holds $(-f)_x = -f_x$ and $\|f\|(x) = \|f_x\|$.

Let us consider M, let us consider V, and let us consider f, Y. We say that f is bounded on Y if and only if:

(Def. 7) There exists a real number r such that for every element x of M such that $x \in Y \cap \operatorname{dom} f$ holds $\|f_x\| \leqslant r$.

One can prove the following propositions:

(41) If $Y \subseteq X$ and f is bounded on X, then f is bounded on Y.

(42) If X misses $\operatorname{dom} f$, then f is bounded on X.

(43) $0_{\mathbb{C}}\, f$ is bounded on Y.

(44) If f is bounded on Y, then $z\, f$ is bounded on Y.

(45) If f is bounded on Y, then $\|f\|$ is bounded on Y and $-f$ is bounded on Y.

(46) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(47) For every partial function f_1 from M to $\mathbb{C}$ such that f_1 is bounded on X and f_2 is bounded on Y holds $f_1\, f_2$ is bounded on $X \cap Y$.

(48) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 - f_2$ is bounded on $X \cap Y$.

(49) If f is bounded on X and bounded on Y, then f is bounded on $X \cup Y$.

(50) If f_1 is a constant on X and f_2 is a constant on Y, then $f_1 + f_2$ is a constant on $X \cap Y$ and $f_1 - f_2$ is a constant on $X \cap Y$.

(51) Let f_1 be a partial function from M to $\mathbb{C}$. Suppose f_1 is a constant on X and f_2 is a constant on Y. Then $f_1\, f_2$ is a constant on $X \cap Y$.

(52) If f is a constant on Y, then $z\, f$ is a constant on Y.

(53) If f is a constant on Y, then $\|f\|$ is a constant on Y and $-f$ is a constant on Y.

(54) If f is a constant on Y, then f is bounded on Y.

(55) If f is a constant on Y, then for every z holds $z\, f$ is bounded on Y and $-f$ is bounded on Y and $\|f\|$ is bounded on Y.

(56) If f_1 is bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(57) If f_1 is bounded on X and f_2 is a constant on Y, then $f_1 - f_2$ is bounded on $X \cap Y$ and $f_2 - f_1$ is bounded on $X \cap Y$.

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Received August 20, 2004