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On the Fundamental Groups of Products of Topological Spaces

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Summary. In the paper we show that fundamental group of the product of two topological spaces is isomorphic to the product of fundamental groups of the spaces.

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The articles [15], [7], [14], [19], [5], [20], [6], [3], [4], [1], [2], [12], [17], [18], [10], [13], [16], [8], [9], and [11] provide the terminology and notation for this paper.

1. On the Product of Groups

The following proposition is true

(1) Let G, H be non empty groupoids and x be an element of $\prod\langle G, H\rangle$. Then there exists an element g of G and there exists an element h of H such that $x = \langle g, h\rangle$.

Let G_1, G_2, H_1, H_2 be non empty groupoids, let f be a map from G_1 into H_1, and let g be a map from G_2 into H_2. The functor $\mathrm{Gr2Iso}(f, g)$ yields a map from $\prod\langle G_1, G_2\rangle$ into $\prod\langle H_1, H_2\rangle$ and is defined by the condition (Def. 1).

(Def. 1) Let x be an element of $\prod\langle G_1, G_2\rangle$. Then there exists an element x_1 of G_1 and there exists an element x_2 of G_2 such that $x = \langle x_1, x_2\rangle$ and $(\mathrm{Gr2Iso}(f, g))(x) = \langle f(x_1), g(x_2)\rangle$.

The following proposition is true

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(2) Let G_1, G_2, H_1, H_2 be non empty groupoids, f be a map from G_1 into H_1, g be a map from G_2 into H_2, x_1 be an element of G_1, and x_2 be an element of G_2. Then $(\mathrm{Gr2Iso}(f,g))(\langle x_1, x_2\rangle) = \langle f(x_1), g(x_2)\rangle$.

Let G_1, G_2, H_1, H_2 be groups, let f be a homomorphism from G_1 to H_1, and let g be a homomorphism from G_2 to H_2. Then $\mathrm{Gr2Iso}(f,g)$ is a homomorphism from $\prod\langle G_1, G_2\rangle$ to $\prod\langle H_1, H_2\rangle$.

One can prove the following four propositions:

(3) Let G_1, G_2, H_1, H_2 be non empty groupoids, f be a map from G_1 into H_1, and g be a map from G_2 into H_2. If f is one-to-one and g is one-to-one, then $\mathrm{Gr2Iso}(f,g)$ is one-to-one.

(4) Let G_1, G_2, H_1, H_2 be non empty groupoids, f be a map from G_1 into H_1, and g be a map from G_2 into H_2. If f is onto and g is onto, then $\mathrm{Gr2Iso}(f,g)$ is onto.

(5) Let G_1, G_2, H_1, H_2 be groups, f be a homomorphism from G_1 to H_1, and g be a homomorphism from G_2 to H_2. If f is an isomorphism and g is an isomorphism, then $\mathrm{Gr2Iso}(f,g)$ is an isomorphism.

(6) Let G_1, G_2, H_1, H_2 be groups. Suppose G_1 and H_1 are isomorphic and G_2 and H_2 are isomorphic. Then $\prod\langle G_1, G_2\rangle$ and $\prod\langle H_1, H_2\rangle$ are isomorphic.

2. On the Fundamental Groups of Products of Topological Spaces

For simplicity, we adopt the following rules: S, T, Y denote non empty topological spaces, s, s_1, s_2, s_3 denote points of S, t, t_1, t_2, t_3 denote points of T, l_1, l_2 denote paths from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$, and H denotes a homotopy between l_1 and l_2.

We now state two propositions:

(7) For all functions f, g such that $\mathrm{dom}\, f = \mathrm{dom}\, g$ holds $\mathrm{pr1}(\langle f, g\rangle) = f$.

(8) For all functions f, g such that $\mathrm{dom}\, f = \mathrm{dom}\, g$ holds $\mathrm{pr2}(\langle f, g\rangle) = g$.

Let us consider S, T, Y, let f be a map from Y into S, and let g be a map from Y into T. Then $\langle f, g\rangle$ is a map from Y into $[\!:S, T:\!]$.

Let us consider S, T, Y and let f be a map from Y into $[\!:S, T:\!]$. Then $\mathrm{pr1}(f)$ is a map from Y into S. Then $\mathrm{pr2}(f)$ is a map from Y into T.

The following propositions are true:

(9) For every continuous map f from Y into $[\!:S, T:\!]$ holds $\mathrm{pr1}(f)$ is continuous.

(10) For every continuous map f from Y into $[\!:S, T:\!]$ holds $\mathrm{pr2}(f)$ is continuous.

(11) If $\langle s_1, t_1\rangle$, $\langle s_2, t_2\rangle$ are connected, then s_1, s_2 are connected.

(12) If $\langle s_1, t_1\rangle$, $\langle s_2, t_2\rangle$ are connected, then t_1, t_2 are connected.

(13) If $\langle s_1, t_1\rangle$, $\langle s_2, t_2\rangle$ are connected, then for every path L from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$ holds $\mathrm{pr1}(L)$ is a path from s_1 to s_2.

(14) If $\langle s_1, t_1\rangle$, $\langle s_2, t_2\rangle$ are connected, then for every path L from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$ holds $\mathrm{pr2}(L)$ is a path from t_1 to t_2.

(15) If s_1, s_2 are connected and t_1, t_2 are connected, then $\langle s_1, t_1\rangle$, $\langle s_2, t_2\rangle$ are connected.

(16) Suppose s_1, s_2 are connected and t_1, t_2 are connected. Let L_1 be a path from s_1 to s_2 and L_2 be a path from t_1 to t_2. Then $\langle L_1, L_2\rangle$ is a path from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$.

Let S, T be non empty arcwise connected topological spaces, let s_1, s_2 be points of S, let t_1, t_2 be points of T, let L_1 be a path from s_1 to s_2, and let L_2 be a path from t_1 to t_2. Then $\langle L_1, L_2\rangle$ is a path from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$.

Let S, T be non empty topological spaces, let s be a point of S, let t be a point of T, let L_1 be a loop of s, and let L_2 be a loop of t. Then $\langle L_1, L_2\rangle$ is a loop of $\langle s, t\rangle$.

Let S, T be non empty arcwise connected topological spaces. One can verify that $[\!: S, T :\!]$ is arcwise connected.

Let S, T be non empty arcwise connected topological spaces, let s_1, s_2 be points of S, let t_1, t_2 be points of T, and let L be a path from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$. Then $\mathrm{pr1}(L)$ is a path from s_1 to s_2. Then $\mathrm{pr2}(L)$ is a path from t_1 to t_2.

Let S, T be non empty topological spaces, let s be a point of S, let t be a point of T, and let L be a loop of $\langle s, t\rangle$. Then $\mathrm{pr1}(L)$ is a loop of s. Then $\mathrm{pr2}(L)$ is a loop of t.

Next we state a number of propositions:

(17) Let p, q be paths from s_1 to s_2. Suppose $p = \mathrm{pr1}(l_1)$ and $q = \mathrm{pr1}(l_2)$ and l_1, l_2 are homotopic. Then $\mathrm{pr1}(H)$ is a homotopy between p and q.

(18) Let p, q be paths from t_1 to t_2. Suppose $p = \mathrm{pr2}(l_1)$ and $q = \mathrm{pr2}(l_2)$ and l_1, l_2 are homotopic. Then $\mathrm{pr2}(H)$ is a homotopy between p and q.

(19) For all paths p, q from s_1 to s_2 such that $p = \mathrm{pr1}(l_1)$ and $q = \mathrm{pr1}(l_2)$ and l_1, l_2 are homotopic holds p, q are homotopic.

(20) For all paths p, q from t_1 to t_2 such that $p = \mathrm{pr2}(l_1)$ and $q = \mathrm{pr2}(l_2)$ and l_1, l_2 are homotopic holds p, q are homotopic.

(21) Let p, q be paths from s_1 to s_2, x, y be paths from t_1 to t_2, f be a homotopy between p and q, and g be a homotopy between x and y. Suppose $p = \mathrm{pr1}(l_1)$ and $q = \mathrm{pr1}(l_2)$ and $x = \mathrm{pr2}(l_1)$ and $y = \mathrm{pr2}(l_2)$ and p, q are homotopic and x, y are homotopic. Then $\langle f, g\rangle$ is a homotopy between l_1 and l_2.

(22) Let p, q be paths from s_1 to s_2 and x, y be paths from t_1 to t_2. Suppose $p = \mathrm{pr1}(l_1)$ and $q = \mathrm{pr1}(l_2)$ and $x = \mathrm{pr2}(l_1)$ and $y = \mathrm{pr2}(l_2)$ and p, q are homotopic and x, y are homotopic. Then l_1, l_2 are homotopic.

(23) Let l_1 be a path from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$, l_2 be a path from $\langle s_2, t_2\rangle$ to $\langle s_3, t_3\rangle$, p_1 be a path from s_1 to s_2, and p_2 be a path from s_2 to s_3. Suppose $\langle s_1, t_1\rangle$, $\langle s_2, t_2\rangle$ are connected and $\langle s_2, t_2\rangle$, $\langle s_3, t_3\rangle$ are connected and $p_1 = \mathrm{pr1}(l_1)$ and $p_2 = \mathrm{pr1}(l_2)$. Then $\mathrm{pr1}(l_1 + l_2) = p_1 + p_2$.

(24) Let S, T be non empty arcwise connected topological spaces, s_1, s_2, s_3 be points of S, t_1, t_2, t_3 be points of T, l_1 be a path from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$, and l_2 be a path from $\langle s_2, t_2\rangle$ to $\langle s_3, t_3\rangle$. Then $\mathrm{pr1}(l_1 + l_2) = \mathrm{pr1}(l_1) + \mathrm{pr1}(l_2)$.

(25) Let l_1 be a path from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$, l_2 be a path from $\langle s_2, t_2\rangle$ to $\langle s_3, t_3\rangle$, p_1 be a path from t_1 to t_2, and p_2 be a path from t_2 to t_3. Suppose $\langle s_1, t_1\rangle$, $\langle s_2, t_2\rangle$ are connected and $\langle s_2, t_2\rangle$, $\langle s_3, t_3\rangle$ are connected and $p_1 = \mathrm{pr2}(l_1)$ and $p_2 = \mathrm{pr2}(l_2)$. Then $\mathrm{pr2}(l_1 + l_2) = p_1 + p_2$.

(26) Let S, T be non empty arcwise connected topological spaces, s_1, s_2, s_3 be points of S, t_1, t_2, t_3 be points of T, l_1 be a path from $\langle s_1, t_1\rangle$ to $\langle s_2, t_2\rangle$, and l_2 be a path from $\langle s_2, t_2\rangle$ to $\langle s_3, t_3\rangle$. Then $\mathrm{pr2}(l_1 + l_2) = \mathrm{pr2}(l_1) + \mathrm{pr2}(l_2)$.

Let S, T be non empty topological spaces, let s be a point of S, and let t be a point of T. The functor $\mathrm{FGPrIso}(s, t)$ yielding a map from $\pi_1([\!:S, T:\!], \langle s, t\rangle)$ into $\prod\langle\pi_1(S, s), \pi_1(T, t)\rangle$ is defined as follows:

(Def. 2) For every point x of $\pi_1([\!:S, T:\!], \langle s, t\rangle)$ there exists a loop l of $\langle s, t\rangle$ such that $x = [l]_{\mathrm{EqRel}([\!:S, T:\!], \langle s, t\rangle)}$ and $(\mathrm{FGPrIso}(s, t))(x) = \langle[\mathrm{pr1}(l)]_{\mathrm{EqRel}(S,s)}, [\mathrm{pr2}(l)]_{\mathrm{EqRel}(T,t)}\rangle$.

The following propositions are true:

(27) For every point x of $\pi_1([\!:S, T:\!], \langle s, t\rangle)$ and for every loop l of $\langle s, t\rangle$ such that $x = [l]_{\mathrm{EqRel}([\!:S, T:\!], \langle s, t\rangle)}$ holds $(\mathrm{FGPrIso}(s, t))(x) = \langle[\mathrm{pr1}(l)]_{\mathrm{EqRel}(S,s)}, [\mathrm{pr2}(l)]_{\mathrm{EqRel}(T,t)}\rangle$.

(28) For every loop l of $\langle s, t\rangle$ holds $(\mathrm{FGPrIso}(s, t))([l]_{\mathrm{EqRel}([\!:S, T:\!], \langle s, t\rangle)}) = \langle[\mathrm{pr1}(l)]_{\mathrm{EqRel}(S,s)}, [\mathrm{pr2}(l)]_{\mathrm{EqRel}(T,t)}\rangle$.

Let S, T be non empty topological spaces, let s be a point of S, and let t be a point of T. Observe that $\mathrm{FGPrIso}(s, t)$ is one-to-one and onto.

Let S, T be non empty topological spaces, let s be a point of S, and let t be a point of T. Then $\mathrm{FGPrIso}(s, t)$ is a homomorphism from $\pi_1([\!:S, T:\!], \langle s, t\rangle)$ to $\prod\langle\pi_1(S, s), \pi_1(T, t)\rangle$.

The following propositions are true:

(29) $\mathrm{FGPrIso}(s, t)$ is an isomorphism.

(30) $\pi_1([\!:S, T:\!], \langle s, t\rangle)$ and $\prod\langle\pi_1(S, s), \pi_1(T, t)\rangle$ are isomorphic.

(31) Let f be a homomorphism from $\pi_1(S, s_1)$ to $\pi_1(S, s_2)$ and g be a homomorphism from $\pi_1(T, t_1)$ to $\pi_1(T, t_2)$. Suppose f is an isomorphism and g is an isomorphism. Then $\mathrm{Gr2Iso}(f, g) \cdot \mathrm{FGPrIso}(s_1, t_1)$ is an isomorphism.

(32) Let S, T be non empty arcwise connected topological spaces, s_1, s_2 be points of S, and t_1, t_2 be points of T. Then $\pi_1([\!: S, T :\!], \langle s_1, t_1 \rangle)$ and $\prod \langle \pi_1(S, s_2), \pi_1(T, t_2) \rangle$ are isomorphic.

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