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The Uniform Continuity of Functions on Normed Linear Spaces

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Summary. In this article, the basic properties of uniform continuity of functions on normed linear spaces are described.

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The notation and terminology used in this paper are introduced in the following articles: [15], [18], [19], [1], [20], [3], [2], [7], [14], [16], [9], [13], [4], [17], [6], [5], [11], [21], [10], [12], and [8].

1. The Uniform Continuity of Functions on Normed Linear Spaces

For simplicity, we follow the rules: X, X_1 are sets, s, r, p are real numbers, S, T are real normed spaces, f, f_1, f_2 are partial functions from S to T, x_1, x_2 are points of S, and Y is a subset of S.

Let us consider X, S, T and let us consider f. We say that f is uniformly continuous on X if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $X \subseteq \mathrm{dom}\, f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$.

Let us consider X, S and let f be a partial function from the carrier of S to $\mathbb{R}$. We say that f is uniformly continuous on X if and only if the conditions (Def. 2) are satisfied.

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(Def. 2)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ and $\|x_1 - x_2\| < s$ holds $|f_{x_1} - f_{x_2}| < r$.

The following propositions are true:

(1) If f is uniformly continuous on X and $X_1 \subseteq X$, then f is uniformly continuous on X_1.

(2) If f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1, then $f_1 + f_2$ is uniformly continuous on $X \cap X_1$.

(3) If f_1 is uniformly continuous on X and f_2 is uniformly continuous on X_1, then $f_1 - f_2$ is uniformly continuous on $X \cap X_1$.

(4) If f is uniformly continuous on X, then $p\, f$ is uniformly continuous on X.

(5) If f is uniformly continuous on X, then $-f$ is uniformly continuous on X.

(6) If f is uniformly continuous on X, then $\|f\|$ is uniformly continuous on X.

(7) If f is uniformly continuous on X, then f is continuous on X.

(8) Let f be a partial function from the carrier of S to $\mathbb{R}$. If f is uniformly continuous on X, then f is continuous on X.

(9) If f is Lipschitzian on X, then f is uniformly continuous on X.

(10) For all f, Y such that Y is compact and f is continuous on Y holds f is uniformly continuous on Y.

(11) If $Y \subseteq \operatorname{dom} f$ and Y is compact and f is uniformly continuous on Y, then $f^\circ Y$ is compact.

(12) Let f be a partial function from the carrier of S to $\mathbb{R}$ and given Y. Suppose $Y \neq \emptyset$ and $Y \subseteq \operatorname{dom} f$ and Y is compact and f is uniformly continuous on Y. Then there exist x_1, x_2 such that $x_1 \in Y$ and $x_2 \in Y$ and $f_{x_1} = \sup(f^\circ Y)$ and $f_{x_2} = \inf(f^\circ Y)$.

(13) If $X \subseteq \operatorname{dom} f$ and f is a constant on X, then f is uniformly continuous on X.

2. The Contraction Mapping Principle on Normed Linear Spaces

Let M be a real Banach space. A function from the carrier of M into the carrier of M is said to be a contraction of M if:

(Def. 3) There exists a real number L such that $0 < L$ and $L < 1$ and for all points x, y of M holds $\|\mathrm{it}(x) - \mathrm{it}(y)\| \leqslant L \cdot \|x - y\|$.

The following two propositions are true:

(14) Let X be a real Banach space and f be a function from X into X. Suppose f is a contraction of X. Then there exists a point x_3 of X such that $f(x_3) = x_3$ and for every point x of X such that $f(x) = x$ holds $x_3 = x$.

(15) Let X be a real Banach space and f be a function from X into X. Given a natural number n_0 such that f^{n_0} is a contraction of X. Then there exists a point x_3 of X such that $f(x_3) = x_3$ and for every point x of X such that $f(x) = x$ holds $x_3 = x$.

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