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Exponential Function on Complex Banach Algebra

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Summary. This article is an extension of [18].

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The papers [23], [24], [4], [5], [2], [20], [21], [9], [1], [22], [13], [15], [16], [12], [10], [11], [17], [14], [25], [3], [7], [6], [19], and [8] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: X denotes a complex Banach algebra, w, z, z_1, z_2 denote elements of X, k, l, m, n denote natural numbers, s_1, s_2, s_3, s, s' denote sequences of X, and r_1 denotes a sequence of real numbers.

Let X be a non empty normed complex algebra structure and let x, y be elements of X. We say that x, y are commutative if and only if:

(Def. 1) $x \cdot y = y \cdot x$.

Let us note that the predicate x, y are commutative is symmetric.

One can prove the following propositions:

(1) If s_2 is convergent and s_3 is convergent and $\lim(s_2 - s_3) = 0_X$, then $\lim s_2 = \lim s_3$.

(2) For every z such that for every natural number n holds $s(n) = z$ holds $\lim s = z$.

(3) If s is convergent and s' is convergent, then $s \cdot s'$ is convergent.

(4) If s is convergent, then $z \cdot s$ is convergent.

(5) If s is convergent, then $s \cdot z$ is convergent.

(6) If s is convergent, then $\lim(z \cdot s) = z \cdot \lim s$.

(7) If s is convergent, then $\lim(s \cdot z) = \lim s \cdot z$.

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(8) If s is convergent and s' is convergent, then $\lim(s \cdot s') = \lim s \cdot \lim s'$.

(9) $(\sum_{\alpha=0}^{\kappa}(z \cdot s_1)(\alpha))_{\kappa \in \mathbb{N}} = z \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}$ and $(\sum_{\alpha=0}^{\kappa}(s_1 \cdot z)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}} \cdot z$.

(10) $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}(k)\| \leqslant (\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa \in \mathbb{N}}(k)$.

(11) If for every n such that $n \leqslant m$ holds $s_2(n) = s_3(n)$, then $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa \in \mathbb{N}}(m) = (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa \in \mathbb{N}}(m)$.

(12) If for every n holds $\|s_1(n)\| \leqslant r_1(n)$ and r_1 is convergent and $\lim r_1 = 0$, then s_1 is convergent and $\lim s_1 = 0_X$.

Let us consider X, z. The functor $z\,\mathrm{ExpSeq}$ yields a sequence of X and is defined as follows:

(Def. 2) For every n holds $z\,\mathrm{ExpSeq}(n) = \frac{1_{\mathbb{C}}}{n!_{\mathbb{C}}} \cdot z_{\mathbb{N}}^{n}$.

The scheme *ExNormSpace CASE* deals with a non empty complex Banach algebra $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

For every k there exists a sequence s_1 of $\mathcal{A}$ such that for every n holds if $n \leqslant k$, then $s_1(n) = \mathcal{F}(k, n)$ and if $n > k$, then $s_1(n) = 0_{\mathcal{A}}$

for all values of the parameters.

Let us consider X, s_1. The functor $\mathrm{Shift}\,s_1$ yielding a sequence of X is defined by:

(Def. 3) $(\mathrm{Shift}\,s_1)(0) = 0_X$ and for every natural number k holds $(\mathrm{Shift}\,s_1)(k + 1) = s_1(k)$.

Let us consider n, X, z, w. The functor $\mathrm{Expan}(n, z, w)$ yielding a sequence of X is defined by:

(Def. 4) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Expan}(n, z, w))(k) = (\mathrm{Coef}\,n)(k) \cdot z_{\mathbb{N}}^{k} \cdot w_{\mathbb{N}}^{n -' k}$ and if $n < k$, then $(\mathrm{Expan}(n, z, w))(k) = 0_X$.

Let us consider n, X, z, w. The functor $\mathrm{Expan\_e}(n, z, w)$ yields a sequence of X and is defined as follows:

(Def. 5) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Expan\_e}(n, z, w))(k) = (\mathrm{Coef\_e}\,n)(k) \cdot z_{\mathbb{N}}^{k} \cdot w_{\mathbb{N}}^{n -' k}$ and if $n < k$, then $(\mathrm{Expan\_e}(n, z, w))(k) = 0_X$.

Let us consider n, X, z, w. The functor $\mathrm{Alfa}(n, z, w)$ yielding a sequence of X is defined by:

(Def. 6) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Alfa}(n, z, w))(k) = z\,\mathrm{ExpSeq}(k) \cdot (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n -' k)$ and if $n < k$, then $(\mathrm{Alfa}(n, z, w))(k) = 0_X$.

Let us consider X, z, w, n. The functor $\mathrm{Conj}(n, z, w)$ yields a sequence of X and is defined as follows:

(Def. 7) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Conj}(n, z, w))(k) = z\,\mathrm{ExpSeq}(k) \cdot ((\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n -' k))$ and if $n < k$, then $(\mathrm{Conj}(n, z, w))(k) = 0_X$.

Next we state several propositions:

(13) $z\,\mathrm{ExpSeq}(n+1) = \frac{1_{\mathbb{C}}}{(n+1)+0i} \cdot z \cdot z\,\mathrm{ExpSeq}(n)$ and $z\,\mathrm{ExpSeq}(0) = \mathbf{1}_X$ and $\|z\,\mathrm{ExpSeq}(n)\| \leqslant \|z\|\,\mathrm{ExpSeq}(n)$.

(14) If $0 < k$, then $(\mathrm{Shift}\, s_1)(k) = s_1(k -' 1)$.

(15) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Shift}\, s_1)(\alpha))_{\kappa\in\mathbb{N}}(k) + s_1(k)$.

(16) For all z, w such that z, w are commutative holds $(z+w)_{\mathbb{N}}^{n} = (\sum_{\alpha=0}^{\kappa}(\mathrm{Expan}(n,z,w))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(17) $\mathrm{Expan\_e}(n,z,w) = \frac{1_{\mathbb{C}}}{n!_{\mathbb{C}}} \cdot \mathrm{Expan}(n,z,w)$.

(18) For all z, w such that z, w are commutative holds $\frac{1_{\mathbb{C}}}{n!_{\mathbb{C}}} \cdot (z+w)_{\mathbb{N}}^{n} = (\sum_{\alpha=0}^{\kappa}(\mathrm{Expan\_e}(n,z,w))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(19) $0_X\,\mathrm{ExpSeq}$ is norm-summable and $\sum(0_X\,\mathrm{ExpSeq}) = \mathbf{1}_X$.

Let us consider X and let z be an element of X. One can check that $z\,\mathrm{ExpSeq}$ is norm-summable.

We now state a number of propositions:

(20) $z\,\mathrm{ExpSeq}(0) = \mathbf{1}_X$ and $(\mathrm{Expan}(0,z,w))(0) = \mathbf{1}_X$.

(21) If $l \leqslant k$, then $(\mathrm{Alfa}(k+1,z,w))(l) = (\mathrm{Alfa}(k,z,w))(l) + (\mathrm{Expan\_e}(k+1,z,w))(l)$.

(22) $(\sum_{\alpha=0}^{\kappa}(\mathrm{Alfa}(k+1,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Alfa}(k,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k) + (\sum_{\alpha=0}^{\kappa}(\mathrm{Expan\_e}(k+1,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k)$.

(23) $z\,\mathrm{ExpSeq}(k) = (\mathrm{Expan\_e}(k,z,w))(k)$.

(24) For all z, w such that z, w are commutative holds $(\sum_{\alpha=0}^{\kappa} z + w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Alfa}(n,z,w))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(25) For all z, w such that z, w are commutative holds $(\sum_{\alpha=0}^{\kappa} z\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) \cdot (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) - (\sum_{\alpha=0}^{\kappa} z + w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Conj}(k,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k)$.

(26) $0 \leqslant \|z\|\,\mathrm{ExpSeq}(n)$.

(27) $\|(\sum_{\alpha=0}^{\kappa} z\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)\| \leqslant (\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)$ and $(\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) \leqslant \sum(\|z\|\,\mathrm{ExpSeq})$ and $\|(\sum_{\alpha=0}^{\kappa} z\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)\| \leqslant \sum(\|z\|\,\mathrm{ExpSeq})$.

(28) $1 \leqslant \sum(\|z\|\,\mathrm{ExpSeq})$.

(29) $|(\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)$ and if $n \leqslant m$, then $|(\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}\|z\|\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(30) $|(\sum_{\alpha=0}^{\kappa}\|\mathrm{Conj}(k,z,w)\|(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}\|\mathrm{Conj}(k,z,w)\|(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(31) For every real number p such that $p > 0$ there exists n such that for every k such that $n \leqslant k$ holds $|(\sum_{\alpha=0}^{\kappa}\|\mathrm{Conj}(k,z,w)\|(\alpha))_{\kappa\in\mathbb{N}}(k)| < p$.

(32) For every s_1 such that for every k holds $s_1(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Conj}(k,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k)$ holds s_1 is convergent and $\lim s_1 = 0_X$.

Let us consider X. The functor $\exp X$ yields a function from the carrier of X into the carrier of X and is defined by:

(Def. 8) For every element z of the carrier of X holds $(\exp X)(z) = \sum(z \operatorname{ExpSeq})$.

Let us consider X, z. The functor $\exp z$ yielding an element of X is defined as follows:

(Def. 9) $\exp z = (\exp X)(z)$.

The following propositions are true:

(33) For every z holds $\exp z = \sum(z \operatorname{ExpSeq})$.

(34) Let given z_1, z_2. Suppose z_1, z_2 are commutative. Then $\exp(z_1 + z_2) = \exp z_1 \cdot \exp z_2$ and $\exp(z_2 + z_1) = \exp z_2 \cdot \exp z_1$ and $\exp(z_1 + z_2) = \exp(z_2 + z_1)$ and $\exp z_1$, $\exp z_2$ are commutative.

(35) For all z_1, z_2 such that z_1, z_2 are commutative holds $z_1 \cdot \exp z_2 = \exp z_2 \cdot z_1$.

(36) $\exp(0_X) = \mathbf{1}_X$.

(37) $\exp z \cdot \exp(-z) = \mathbf{1}_X$ and $\exp(-z) \cdot \exp z = \mathbf{1}_X$.

(38) $\exp z$ is invertible and $(\exp z)^{-1} = \exp(-z)$ and $\exp(-z)$ is invertible and $(\exp(-z))^{-1} = \exp z$.

(39) For every z and for all complex numbers s, t holds $s \cdot z$, $t \cdot z$ are commutative.

(40) Let given z and s, t be complex numbers. Then $\exp(s \cdot z) \cdot \exp(t \cdot z) = \exp((s+t) \cdot z)$ and $\exp(t \cdot z) \cdot \exp(s \cdot z) = \exp((t+s) \cdot z)$ and $\exp((s+t) \cdot z) = \exp((t+s) \cdot z)$ and $\exp(s \cdot z)$, $\exp(t \cdot z)$ are commutative.

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