FORMALIZED MATHEMATICS
Volume 11, Number 4, 2003
University of Białystok

On the Upper and Lower Approximations of the Curve[1]

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MML Identifier: JORDAN19.

The papers [28], [32], [2], [15], [1], [5], [6], [4], [31], [16], [29], [17], [27], [13], [3], [25], [26], [10], [11], [8], [30], [14], [20], [18], [12], [23], [22], [24], [7], [9], [19], and [21] provide the terminology and notation for this paper.

In this paper n denotes a natural number.

Let C be a simple closed curve. The functor $\mathrm{UpperAppr}(C)$ yields a sequence of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 1) For every natural number i holds $(\mathrm{UpperAppr}(C))(i) = \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, i)))$.

The functor $\mathrm{LowerAppr}(C)$ yielding a sequence of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 2) For every natural number i holds $(\mathrm{LowerAppr}(C))(i) = \mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, i)))$.

Let C be a simple closed curve. The functor $\mathrm{NorthArc}(C)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 3) $\mathrm{NorthArc}(C) = \mathrm{Li}\ \mathrm{UpperAppr}(C)$.

The functor $\mathrm{SouthArc}(C)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 4) $\mathrm{SouthArc}(C) = \mathrm{Li}\ \mathrm{LowerAppr}(C)$.

We now state a number of propositions:

(1) For all natural numbers n, m such that $n \leqslant m$ and $n \neq 0$ holds $\frac{n+1}{n} \geqslant \frac{m+1}{m}$.

[1]This work has been partially supported by the CALCULEMUS grant HPRN-CT-2000-00102.

ISSN 1426–2630

(2) Let E be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and m, j be natural numbers. Suppose $1 \leqslant m$ and $m \leqslant n$ and $1 \leqslant j$ and $j \leqslant$ width Gauge(E, n). Then $\mathcal{L}(\mathrm{Gauge}(E,n) \circ (\mathrm{Center\,Gauge}(E,n), \mathrm{width\,Gauge}(E,n)), \mathrm{Gauge}(E,n) \circ (\mathrm{Center\,Gauge}(E,n), j)) \subseteq \mathcal{L}(\mathrm{Gauge}(E,m) \circ (\mathrm{Center\,Gauge}(E,m), \mathrm{width\,Gauge}(E,m)), \mathrm{Gauge}(E,n) \circ (\mathrm{Center\,Gauge}(E,n), j))$.

(3) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and i, j be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant$ len Gauge(C, n) and $1 \leqslant j$ and $j \leqslant$ width Gauge(C, n) and Gauge$(C, n) \circ (i, j) \in \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i, \mathrm{width\,Gauge}(C,n)), \mathrm{Gauge}(C,n) \circ (i,j))$ meets $\widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$.

(4) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. Suppose $n > 0$. Let i, j be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant$ len Gauge(C, n) and $1 \leqslant j$ and $j \leqslant$ width Gauge(C, n) and Gauge$(C, n) \circ (i, j) \in \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i, \mathrm{width\,Gauge}(C,n)), \mathrm{Gauge}(C,n) \circ (i,j))$ meets UpperArc$(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(5) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and j be a natural number. Suppose Gauge$(C, n+1) \circ (\mathrm{Center\,Gauge}(C, n+1), j) \in$ LowerArc$(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n+1)))$ and $1 \leqslant j$ and $j \leqslant$ width Gauge$(C, n+1)$. Then $\mathcal{L}(\mathrm{Gauge}(C,1) \circ (\mathrm{Center\,Gauge}(C,1), \mathrm{width\,Gauge}(C,1)), \mathrm{Gauge}(C,n+1) \circ (\mathrm{Center\,Gauge}(C,n+1), j))$ meets UpperArc$(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n+1)))$.

(6) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$, f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, and k be a natural number. Suppose $1 \leqslant k$ and $k+1 \leqslant$ len f and f is a sequence which elements belong to Gauge(C, n). Then $\rho(f_k, f_{k+1}) = \frac{\mathrm{N\text{-}bound}(C) - \mathrm{S\text{-}bound}(C)}{2^n}$ or $\rho(f_k, f_{k+1}) = \frac{\mathrm{E\text{-}bound}(C) - \mathrm{W\text{-}bound}(C)}{2^n}$.

(7) Let M be a symmetric triangle metric structure, r be a real number, and p, q, x be elements of M. If $p \in \mathrm{Ball}(x, r)$ and $q \in \mathrm{Ball}(x, r)$, then $\rho(p, q) < 2 \cdot r$.

(8) Let A be a subset of $\mathcal{E}^n_{\rm T}$, p be a point of $\mathcal{E}^n_{\rm T}$, and p' be a point of $\mathcal{E}^n$. Suppose $p = p'$. Let s be a real number. Suppose $s > 0$. Then $p \in \overline{A}$ if and only if for every real number r such that $0 < r$ and $r < s$ holds Ball(p', r) meets A.

(9) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds N-bound$(C) <$ N-bound$(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n)))$.

(10) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds E-bound$(C) <$ E-bound$(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n)))$.

(11) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$

holds S-bound($\widetilde{\mathcal{L}}$(Cage(C, n))) < S-bound(C).

(12) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds W-bound($\widetilde{\mathcal{L}}$(Cage(C, n))) < W-bound(C).

(13) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i <$ len Gauge(C, n) and $1 \leqslant k$ and $k \leqslant j$ and $j \leqslant$ width Gauge(C, n) and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, k), Gauge(C, n) $\circ$ (i, j)) $\cap$ $\widetilde{\mathcal{L}}$(UpperSeq(C, n)) = {Gauge(C, n) $\circ$ (i, k)} and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, k), Gauge(C, n) $\circ$ (i, j)) $\cap$ $\widetilde{\mathcal{L}}$(LowerSeq(C, n)) = {Gauge(C, n) $\circ$ (i, j)}. Then $\mathcal{L}$(Gauge(C, n) $\circ$ (i, k), Gauge(C, n) $\circ$ (i, j)) meets UpperArc(C).

(14) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i <$ len Gauge(C, n) and $1 \leqslant k$ and $k \leqslant j$ and $j \leqslant$ width Gauge(C, n) and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, k), Gauge(C, n) $\circ$ (i, j)) $\cap$ $\widetilde{\mathcal{L}}$(UpperSeq(C, n)) = {Gauge(C, n) $\circ$ (i, k)} and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, k), Gauge(C, n) $\circ$ (i, j)) $\cap$ $\widetilde{\mathcal{L}}$(LowerSeq(C, n)) = {Gauge(C, n) $\circ$ (i, j)}. Then $\mathcal{L}$(Gauge(C, n) $\circ$ (i, k), Gauge(C, n) $\circ$ (i, j)) meets LowerArc(C).

(15) Let C be a simple closed curve and i, j, k be natural numbers. Suppose that $1 < i$ and $i <$ len Gauge(C, n) and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant$ width Gauge(C, n) and $n > 0$ and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) $\cap$ LowerArc($\widetilde{\mathcal{L}}$(Cage(C, n))) = {Gauge(C, n) $\circ$ (i, k)} and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) $\cap$ UpperArc($\widetilde{\mathcal{L}}$(Cage(C, n))) = {Gauge(C, n) $\circ$ (i, j)}. Then $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) meets UpperArc(C).

(16) Let C be a simple closed curve and i, j, k be natural numbers. Suppose that $1 < i$ and $i <$ len Gauge(C, n) and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant$ width Gauge(C, n) and $n > 0$ and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) $\cap$ LowerArc($\widetilde{\mathcal{L}}$(Cage(C, n))) = {Gauge(C, n) $\circ$ (i, k)} and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) $\cap$ UpperArc($\widetilde{\mathcal{L}}$(Cage(C, n))) = {Gauge(C, n) $\circ$ (i, j)}. Then $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) meets LowerArc(C).

(17) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i <$ len Gauge(C, n) and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant$ width Gauge(C, n) and Gauge(C, n) $\circ$ (i, k) $\in$ $\widetilde{\mathcal{L}}$(LowerSeq(C, n)) and Gauge(C, n) $\circ$ (i, j) $\in$ $\widetilde{\mathcal{L}}$(UpperSeq(C, n)). Then $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) meets UpperArc(C).

(18) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i <$ len Gauge(C, n) and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant$ width Gauge(C, n) and Gauge(C, n) $\circ$ (i, k) $\in$ $\widetilde{\mathcal{L}}$(LowerSeq(C, n)) and Gauge(C, n) $\circ$ (i, j) $\in$ $\widetilde{\mathcal{L}}$(UpperSeq(C, n)). Then $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) meets LowerArc(C).

(19) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i <$ len Gauge(C, n) and $1 \leqslant j$

and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $n > 0$ and $\mathrm{Gauge}(C,n) \circ (i,k) \in \mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$ and $\mathrm{Gauge}(C,n) \circ (i,j) \in \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j), \mathrm{Gauge}(C,n) \circ (i,k))$ meets $\mathrm{UpperArc}(C)$.

(20) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $n > 0$ and $\mathrm{Gauge}(C,n) \circ (i,k) \in \mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$ and $\mathrm{Gauge}(C,n) \circ (i,j) \in \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j), \mathrm{Gauge}(C,n) \circ (i,k))$ meets $\mathrm{LowerArc}(C)$.

(21) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 \leqslant i_2$ and $i_2 < \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i_1,j)\}$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i_2,k)\}$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))$ meets $\mathrm{UpperArc}(C)$.

(22) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 \leqslant i_2$ and $i_2 < \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i_1,j)\}$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i_2,k)\}$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))$ meets $\mathrm{LowerArc}(C)$.

(23) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_2$ and $i_2 \leqslant i_1$ and $i_1 < \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i_1,j)\}$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i_2,k)\}$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))$ meets $\mathrm{UpperArc}(C)$.

(24) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_2$ and $i_2 \leqslant i_1$ and $i_1 < \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ$

$(i_1, k), \mathrm{Gauge}(C, n) \circ (i_2, k))) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C, n)) = \{\mathrm{Gauge}(C, n) \circ (i_1, j)\}$ and $(\mathcal{L}(\mathrm{Gauge}(C, n) \circ (i_1, j), \mathrm{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\mathrm{Gauge}(C, n) \circ (i_1, k), \mathrm{Gauge}(C, n) \circ (i_2, k))) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C, n)) = \{\mathrm{Gauge}(C, n) \circ (i_2, k)\}$. Then $\mathcal{L}(\mathrm{Gauge}(C, n) \circ (i_1, j), \mathrm{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\mathrm{Gauge}(C, n) \circ (i_1, k), \mathrm{Gauge}(C, n) \circ (i_2, k))$ meets $\mathrm{LowerArc}(C)$.

(25) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 < \mathrm{len}\, \mathrm{Gauge}(C, n+1)$ and $1 < i_2$ and $i_2 < \mathrm{len}\, \mathrm{Gauge}(C, n+1)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\, \mathrm{Gauge}(C, n+1)$ and $\mathrm{Gauge}(C, n+1) \circ (i_1, k) \in \mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n+1)))$ and $\mathrm{Gauge}(C, n+1) \circ (i_2, j) \in \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n+1)))$. Then $\mathcal{L}(\mathrm{Gauge}(C, n+1) \circ (i_2, j), \mathrm{Gauge}(C, n+1) \circ (i_2, k)) \cup \mathcal{L}(\mathrm{Gauge}(C, n+1) \circ (i_2, k), \mathrm{Gauge}(C, n+1) \circ (i_1, k))$ meets $\mathrm{LowerArc}(C)$.

(26) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 < \mathrm{len}\, \mathrm{Gauge}(C, n+1)$ and $1 < i_2$ and $i_2 < \mathrm{len}\, \mathrm{Gauge}(C, n+1)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\, \mathrm{Gauge}(C, n+1)$ and $\mathrm{Gauge}(C, n+1) \circ (i_1, k) \in \mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n+1)))$ and $\mathrm{Gauge}(C, n+1) \circ (i_2, j) \in \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n+1)))$. Then $\mathcal{L}(\mathrm{Gauge}(C, n+1) \circ (i_2, j), \mathrm{Gauge}(C, n+1) \circ (i_2, k)) \cup \mathcal{L}(\mathrm{Gauge}(C, n+1) \circ (i_2, k), \mathrm{Gauge}(C, n+1) \circ (i_1, k))$ meets $\mathrm{UpperArc}(C)$.

(27) For every simple closed curve C and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{W\text{-}bound}(C) < p_{\mathbf{1}}$ and $p_{\mathbf{1}} < \mathrm{E\text{-}bound}(C)$ holds $p \notin \mathrm{NorthArc}(C)$ or $p \notin \mathrm{SouthArc}(C)$.

(28) For every simple closed curve C and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_{\mathbf{1}} = \frac{\mathrm{W\text{-}bound}(C) + \mathrm{E\text{-}bound}(C)}{2}$ holds $p \notin \mathrm{NorthArc}(C)$ or $p \notin \mathrm{SouthArc}(C)$.

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Received September 27, 2003