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Cross Products and Tripple Vector Products in 3-dimensional Euclidean Space

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Summary. First, we extend the basic theorems of 3-dimensional Euclidean space, and then define the cross product in the same space and relative vector relations using the above definition.

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The articles [14], [2], [12], [9], [6], [4], [3], [5], [13], [10], [11], [7], [8], and [1] provide the terminology and notation for this paper.

We adopt the following convention: x, y, z denote real numbers, x_3, y_3 denote elements of $\mathbb{R}$, and p denotes a point of $\mathcal{E}^3_{\mathrm{T}}$.

We now state the proposition

(1) There exist x, y, z such that $p = \langle x, y, z \rangle$.

Let us consider p. The functor $p_{\mathbf{1}}$ yielding a real number is defined as follows:

(Def. 1) For every finite sequence f such that $p = f$ holds $p_{\mathbf{1}} = f(1)$.

The functor $p_{\mathbf{2}}$ yields a real number and is defined by:

(Def. 2) For every finite sequence f such that $p = f$ holds $p_{\mathbf{2}} = f(2)$.

The functor $p_{\mathbf{3}}$ yields a real number and is defined by:

(Def. 3) For every finite sequence f such that $p = f$ holds $p_{\mathbf{3}} = f(3)$.

Let us consider x, y, z. The functor $[x, y, z]$ yields a point of $\mathcal{E}^3_{\mathrm{T}}$ and is defined as follows:

(Def. 4) $[x, y, z] = \langle x, y, z \rangle$.

One can prove the following three propositions:

(2) $[x, y, z]_{\mathbf{1}} = x$ and $[x, y, z]_{\mathbf{2}} = y$ and $[x, y, z]_{\mathbf{3}} = z$.

(3) $p = [p_{\mathbf{1}}, p_{\mathbf{2}}, p_{\mathbf{3}}]$.

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(4) $0_{\mathcal{E}^3_{\rm T}} = [0,0,0]$.

We adopt the following rules: p_1, p_2, p_3, p_4 are points of $\mathcal{E}^3_{\rm T}$ and x_1, x_2, y_1, y_2, z_1, z_2 are real numbers.

Next we state several propositions:

(5) $p_1 + p_2 = [(p_1)_{\mathbf{1}} + (p_2)_{\mathbf{1}}, (p_1)_{\mathbf{2}} + (p_2)_{\mathbf{2}}, (p_1)_{\mathbf{3}} + (p_2)_{\mathbf{3}}]$.

(6) $[x_1, y_1, z_1] + [x_2, y_2, z_2] = [x_1 + x_2, y_1 + y_2, z_1 + z_2]$.

(7) $x \cdot p = [x \cdot p_{\mathbf{1}}, x \cdot p_{\mathbf{2}}, x \cdot p_{\mathbf{3}}]$.

(8) $x \cdot [x_1, y_1, z_1] = [x \cdot x_1, x \cdot y_1, x \cdot z_1]$.

(9) $(x \cdot p)_{\mathbf{1}} = x \cdot p_{\mathbf{1}}$ and $(x \cdot p)_{\mathbf{2}} = x \cdot p_{\mathbf{2}}$ and $(x \cdot p)_{\mathbf{3}} = x \cdot p_{\mathbf{3}}$.

(10) $-p = [-p_{\mathbf{1}}, -p_{\mathbf{2}}, -p_{\mathbf{3}}]$.

(11) $-[x_1, y_1, z_1] = [-x_1, -y_1, -z_1]$.

(12) $p_1 - p_2 = [(p_1)_{\mathbf{1}} - (p_2)_{\mathbf{1}}, (p_1)_{\mathbf{2}} - (p_2)_{\mathbf{2}}, (p_1)_{\mathbf{3}} - (p_2)_{\mathbf{3}}]$.

(13) $[x_1, y_1, z_1] - [x_2, y_2, z_2] = [x_1 - x_2, y_1 - y_2, z_1 - z_2]$.

Let us consider p_1, p_2. The functor $p_1 \times p_2$ yielding a point of $\mathcal{E}^3_{\rm T}$ is defined by:

(Def. 5) $p_1 \times p_2 = [(p_1)_{\mathbf{2}} \cdot (p_2)_{\mathbf{3}} - (p_1)_{\mathbf{3}} \cdot (p_2)_{\mathbf{2}}, (p_1)_{\mathbf{3}} \cdot (p_2)_{\mathbf{1}} - (p_1)_{\mathbf{1}} \cdot (p_2)_{\mathbf{3}}, (p_1)_{\mathbf{1}} \cdot (p_2)_{\mathbf{2}} - (p_1)_{\mathbf{2}} \cdot (p_2)_{\mathbf{1}}]$.

The following propositions are true:

(14) If $p = [x, y, z]$, then $p_{\mathbf{1}} = x$ and $p_{\mathbf{2}} = y$ and $p_{\mathbf{3}} = z$.

(15) $[x_1, y_1, z_1] \times [x_2, y_2, z_2] = [y_1 \cdot z_2 - z_1 \cdot y_2, z_1 \cdot x_2 - x_1 \cdot z_2, x_1 \cdot y_2 - y_1 \cdot x_2]$.

(16) $(x \cdot p_1) \times p_2 = x \cdot (p_1 \times p_2)$ and $(x \cdot p_1) \times p_2 = p_1 \times (x \cdot p_2)$.

(17) $p_1 \times p_2 = -p_2 \times p_1$.

(18) $(-p_1) \times p_2 = p_1 \times -p_2$.

(19) $[0,0,0] \times [x, y, z] = 0_{\mathcal{E}^3_{\rm T}}$.

(20) $[x_1, 0, 0] \times [x_2, 0, 0] = 0_{\mathcal{E}^3_{\rm T}}$.

(21) $[0, y_1, 0] \times [0, y_2, 0] = 0_{\mathcal{E}^3_{\rm T}}$.

(22) $[0, 0, z_1] \times [0, 0, z_2] = 0_{\mathcal{E}^3_{\rm T}}$.

(23) $p_1 \times (p_2 + p_3) = p_1 \times p_2 + p_1 \times p_3$.

(24) $(p_1 + p_2) \times p_3 = p_1 \times p_3 + p_2 \times p_3$.

(25) $p_1 \times p_1 = 0_{\mathcal{E}^3_{\rm T}}$.

(26) $(p_1 + p_2) \times (p_3 + p_4) = p_1 \times p_3 + p_1 \times p_4 + p_2 \times p_3 + p_2 \times p_4$.

(27) $p = \langle p_{\mathbf{1}}, p_{\mathbf{2}}, p_{\mathbf{3}} \rangle$.

(28) For all finite sequences f_1, f_2 of elements of $\mathbb{R}$ such that $\operatorname{len} f_1 = 3$ and $\operatorname{len} f_2 = 3$ holds $f_1 \bullet f_2 = \langle f_1(1) \cdot f_2(1), f_1(2) \cdot f_2(2), f_1(3) \cdot f_2(3) \rangle$.

(29) $|(p_1, p_2)| = (p_1)_{\mathbf{1}} \cdot (p_2)_{\mathbf{1}} + (p_1)_{\mathbf{2}} \cdot (p_2)_{\mathbf{2}} + (p_1)_{\mathbf{3}} \cdot (p_2)_{\mathbf{3}}$.

(30) $|([x_1, x_2, x_3], [y_1, y_2, y_3])| = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$.

Let us consider p_1, p_2, p_3. The functor $\langle|p_1, p_2, p_3|\rangle$ yielding a real number is defined as follows:

(Def. 6) $\langle|p_1, p_2, p_3|\rangle = |(p_1, p_2 \times p_3)|$.

The following propositions are true:

(31) $\langle|p_1, p_1, p_2|\rangle = 0$ and $\langle|p_2, p_1, p_2|\rangle = 0$.

(32) $p_1 \times (p_2 \times p_3) = |(p_1, p_3)| \cdot p_2 - |(p_1, p_2)| \cdot p_3$.

(33) $\langle|p_1, p_2, p_3|\rangle = \langle|p_2, p_3, p_1|\rangle$.

(34) $\langle|p_1, p_2, p_3|\rangle = \langle|p_3, p_1, p_2|\rangle$.

(35) $\langle|p_1, p_2, p_3|\rangle = |(p_1 \times p_2, p_3)|$.

References

[1] Kanchun and Yatsuka Nakamura. The inner product of finite sequences and of points of n-dimensional topological space. *Formalized Mathematics*, 11(**2**):179–183, 2003.
[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[8] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[10] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[11] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[12] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[13] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[14] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

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