FORMALIZED MATHEMATICS
Volume 11, Number **4**, 2003
University of Białystok

Lines in n-Dimensional Euclidean Spaces

Akihiro Kubo
Shinshu University
Nagano

Summary. In this paper, we define the line of n-dimensional Euclidean space and we introduce basic properties of affine space on this space. Next, we define the inner product of elements of this space. At the end, we introduce orthogonality of lines of this space.

MML Identifier: `EUCLID_4`.

The papers [13], [4], [15], [2], [12], [8], [5], [11], [10], [3], [6], [1], [14], [7], and [9] provide the terminology and notation for this paper.

We adopt the following rules: a, b, l_1 are real numbers, n is a natural number, and x, x_1, x_2, y_1, y_2 are elements of $\mathcal{R}^n$.

Next we state several propositions:

(1) $0 \cdot x + x = x$ and $x + \langle \underbrace{0, \ldots, 0}_{n} \rangle = x$.

(2) $a \cdot \langle \underbrace{0, \ldots, 0}_{n} \rangle = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(3) $1 \cdot x = x$ and $0 \cdot x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(4) $(a \cdot b) \cdot x = a \cdot (b \cdot x)$.

(5) If $a \cdot x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$, then $a = 0$ or $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(6) $a \cdot (x_1 + x_2) = a \cdot x_1 + a \cdot x_2$.

(7) $(a + b) \cdot x = a \cdot x + b \cdot x$.

(8) If $a \cdot x_1 = a \cdot x_2$, then $a = 0$ or $x_1 = x_2$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. The functor $\mathrm{Line}(x_1, x_2)$ yields a subset of $\mathcal{R}^n$ and is defined by:

ISSN 1426–2630

(Def. 1) $\mathrm{Line}(x_1, x_2) = \{(1 - l_1) \cdot x_1 + l_1 \cdot x_2\}$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. Observe that $\mathrm{Line}(x_1, x_2)$ is non empty.

The following proposition is true

(9) $\mathrm{Line}(x_1, x_2) = \mathrm{Line}(x_2, x_1)$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. Let us observe that the functor $\mathrm{Line}(x_1, x_2)$ is commutative.

One can prove the following propositions:

(10) $x_1 \in \mathrm{Line}(x_1, x_2)$ and $x_2 \in \mathrm{Line}(x_1, x_2)$.

(11) If $y_1 \in \mathrm{Line}(x_1, x_2)$ and $y_2 \in \mathrm{Line}(x_1, x_2)$, then $\mathrm{Line}(y_1, y_2) \subseteq \mathrm{Line}(x_1, x_2)$.

(12) If $y_1 \in \mathrm{Line}(x_1, x_2)$ and $y_2 \in \mathrm{Line}(x_1, x_2)$ and $y_1 \neq y_2$, then $\mathrm{Line}(x_1, x_2) \subseteq \mathrm{Line}(y_1, y_2)$.

Let us consider n and let A be a subset of $\mathcal{R}^n$. We say that A is line if and only if:

(Def. 2) There exist x_1, x_2 such that $x_1 \neq x_2$ and $A = \mathrm{Line}(x_1, x_2)$.

We introduce A is a line as a synonym of A is line.

Next we state three propositions:

(13) Let A, C be subsets of $\mathcal{R}^n$ and given x_1, x_2. Suppose A is a line and C is a line and $x_1 \in A$ and $x_2 \in A$ and $x_1 \in C$ and $x_2 \in C$. Then $x_1 = x_2$ or $A = C$.

(14) For every subset A of $\mathcal{R}^n$ such that A is a line there exist x_1, x_2 such that $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$.

(15) For every subset A of $\mathcal{R}^n$ such that A is a line there exists x_2 such that $x_1 \neq x_2$ and $x_2 \in A$.

Let us consider n and let x be an element of $\mathcal{R}^n$. The functor $\mathrm{Rn2Fin}(x)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by:

(Def. 3) $\mathrm{Rn2Fin}(x) = x$.

Let us consider n and let x be an element of $\mathcal{R}^n$. The functor $|x|$ yields a real number and is defined as follows:

(Def. 4) $|x| = |\,\mathrm{Rn2Fin}(x)|$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. The functor $|(x_1, x_2)|$ yielding a real number is defined by:

(Def. 5) $|(x_1, x_2)| = |(\mathrm{Rn2Fin}(x_1), \mathrm{Rn2Fin}(x_2))|$.

Let us observe that the functor $|(x_1, x_2)|$ is commutative.

We now state a number of propositions:

(16) For all elements x_1, x_2 of $\mathcal{R}^n$ holds $|(x_1, x_2)| = \frac{1}{4} \cdot (|x_1 + x_2|^{\mathbf{2}} - |x_1 - x_2|^{\mathbf{2}})$.

(17) For every element x of $\mathcal{R}^n$ holds $|(x, x)| \geqslant 0$.

(18) For every element x of $\mathcal{R}^n$ holds $|x|^{\mathbf{2}} = |(x, x)|$.

(19) For every element x of $\mathcal{R}^n$ holds $0 \leqslant |x|$.

(20) For every element x of $\mathcal{R}^n$ holds $|x| = \sqrt{|(x,x)|}$.

(21) For every element x of $\mathcal{R}^n$ holds $|(x,x)| = 0$ iff $|x| = 0$.

(22) For every element x of $\mathcal{R}^n$ holds $|(x,x)| = 0$ iff $x = \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(23) For every element x of $\mathcal{R}^n$ holds $|(x, \langle\underbrace{0,\ldots,0}_{n}\rangle)| = 0$.

(24) For every element x of $\mathcal{R}^n$ holds $|(\langle\underbrace{0,\ldots,0}_{n}\rangle, x)| = 0$.

(25) For all elements x_1, x_2, x_3 of $\mathcal{R}^n$ holds $|(x_1 + x_2, x_3)| = |(x_1, x_3)| + |(x_2, x_3)|$.

(26) For all elements x_1, x_2 of $\mathcal{R}^n$ and for every real number a holds $|(a \cdot x_1, x_2)| = a \cdot |(x_1, x_2)|$.

(27) For all elements x_1, x_2 of $\mathcal{R}^n$ and for every real number a holds $|(x_1, a \cdot x_2)| = a \cdot |(x_1, x_2)|$.

(28) For all elements x_1, x_2 of $\mathcal{R}^n$ holds $|(-x_1, x_2)| = -|(x_1, x_2)|$.

(29) For all elements x_1, x_2 of $\mathcal{R}^n$ holds $|(x_1, -x_2)| = -|(x_1, x_2)|$.

(30) For all elements x_1, x_2 of $\mathcal{R}^n$ holds $|(-x_1, -x_2)| = |(x_1, x_2)|$.

(31) For all elements x_1, x_2, x_3 of $\mathcal{R}^n$ holds $|(x_1 - x_2, x_3)| = |(x_1, x_3)| - |(x_2, x_3)|$.

(32) For all real numbers a, b and for all elements x_1, x_2, x_3 of $\mathcal{R}^n$ holds $|(a \cdot x_1 + b \cdot x_2, x_3)| = a \cdot |(x_1, x_3)| + b \cdot |(x_2, x_3)|$.

(33) For all elements x_1, y_1, y_2 of $\mathcal{R}^n$ holds $|(x_1, y_1 + y_2)| = |(x_1, y_1)| + |(x_1, y_2)|$.

(34) For all elements x_1, y_1, y_2 of $\mathcal{R}^n$ holds $|(x_1, y_1 - y_2)| = |(x_1, y_1)| - |(x_1, y_2)|$.

(35) For all elements x_1, x_2, y_1, y_2 of $\mathcal{R}^n$ holds $|(x_1+x_2, y_1+y_2)| = |(x_1, y_1)| + |(x_1, y_2)| + |(x_2, y_1)| + |(x_2, y_2)|$.

(36) For all elements x_1, x_2, y_1, y_2 of $\mathcal{R}^n$ holds $|(x_1-x_2, y_1-y_2)| = (|(x_1, y_1)| - |(x_1, y_2)| - |(x_2, y_1)|) + |(x_2, y_2)|$.

(37) For all elements x, y of $\mathcal{R}^n$ holds $|(x+y, x+y)| = |(x,x)| + 2 \cdot |(x,y)| + |(y,y)|$.

(38) For all elements x, y of $\mathcal{R}^n$ holds $|(x-y, x-y)| = (|(x,x)| - 2 \cdot |(x,y)|) + |(y,y)|$.

(39) For all elements x, y of $\mathcal{R}^n$ holds $|x+y|^{\mathbf{2}} = |x|^{\mathbf{2}} + 2 \cdot |(x,y)| + |y|^{\mathbf{2}}$.

(40) For all elements x, y of $\mathcal{R}^n$ holds $|x-y|^{\mathbf{2}} = (|x|^{\mathbf{2}} - 2 \cdot |(x,y)|) + |y|^{\mathbf{2}}$.

(41) For all elements x, y of $\mathcal{R}^n$ holds $|x+y|^{\mathbf{2}} + |x-y|^{\mathbf{2}} = 2 \cdot (|x|^{\mathbf{2}} + |y|^{\mathbf{2}})$.

(42) For all elements x, y of $\mathcal{R}^n$ holds $|x+y|^{\mathbf{2}} - |x-y|^{\mathbf{2}} = 4 \cdot |(x,y)|$.

(43) For all elements x, y of $\mathcal{R}^n$ holds $||(x,y)|| \leqslant |x| \cdot |y|$.

(44) For all elements x, y of $\mathcal{R}^n$ holds $|x+y| \leqslant |x| + |y|$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. We say that x_1, x_2 are orthogonal if and only if:

(Def. 6) $|(x_1, x_2)| = 0$.

Let us note that the predicate x_1, x_2 are orthogonal is symmetric.

We now state the proposition

(45) Let R be a subset of $\mathbb{R}$ and x_1, x_2, y_1 be elements of $\mathcal{R}^n$. Suppose $R = \{|y_1 - x|; x$ ranges over elements of $\mathcal{R}^n$: $x \in \mathrm{Line}(x_1, x_2)\}$. Then there exists an element y_2 of $\mathcal{R}^n$ such that $y_2 \in \mathrm{Line}(x_1, x_2)$ and $|y_1 - y_2| = \inf R$ and $x_1 - x_2$, $y_1 - y_2$ are orthogonal.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{Line}(p_1, p_2)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def. 7) $\mathrm{Line}(p_1, p_2) = \{(1 - l_1) \cdot p_1 + l_1 \cdot p_2\}$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Observe that $\mathrm{Line}(p_1, p_2)$ is non empty.

In the sequel p_1, p_2, q_1, q_2 are points of $\mathcal{E}^n_{\mathrm{T}}$.

The following proposition is true

(46) $\mathrm{Line}(p_1, p_2) = \mathrm{Line}(p_2, p_1)$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Let us observe that the functor $\mathrm{Line}(p_1, p_2)$ is commutative.

One can prove the following three propositions:

(47) $p_1 \in \mathrm{Line}(p_1, p_2)$ and $p_2 \in \mathrm{Line}(p_1, p_2)$.

(48) If $q_1 \in \mathrm{Line}(p_1, p_2)$ and $q_2 \in \mathrm{Line}(p_1, p_2)$, then $\mathrm{Line}(q_1, q_2) \subseteq \mathrm{Line}(p_1, p_2)$.

(49) If $q_1 \in \mathrm{Line}(p_1, p_2)$ and $q_2 \in \mathrm{Line}(p_1, p_2)$ and $q_1 \neq q_2$, then $\mathrm{Line}(p_1, p_2) \subseteq \mathrm{Line}(q_1, q_2)$.

Let us consider n and let A be a subset of $\mathcal{E}^n_{\mathrm{T}}$. We say that A is line if and only if:

(Def. 8) There exist p_1, p_2 such that $p_1 \neq p_2$ and $A = \mathrm{Line}(p_1, p_2)$.

We introduce A is a line as a synonym of A is line.

We now state three propositions:

(50) For all subsets A, C of $\mathcal{E}^n_{\mathrm{T}}$ such that A is a line and C is a line and $p_1 \in A$ and $p_2 \in A$ and $p_1 \in C$ and $p_2 \in C$ holds $p_1 = p_2$ or $A = C$.

(51) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ such that A is a line there exist p_1, p_2 such that $p_1 \in A$ and $p_2 \in A$ and $p_1 \neq p_2$.

(52) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ such that A is a line there exists p_2 such that $p_1 \neq p_2$ and $p_2 \in A$.

Let us consider n and let p be a point of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{TPn2Rn}(p)$ yields an element of $\mathcal{R}^n$ and is defined as follows:

(Def. 9) $\mathrm{TPn2Rn}(p) = p$.

Let us consider n and let p be a point of $\mathcal{E}^n_{\mathrm{T}}$. The functor $|p|$ yields a real number and is defined as follows:

(Def. 10) $|p| = |\,\mathrm{TPn2Rn}(p)|$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $|(p_1, p_2)|$ yields a real number and is defined as follows:

(Def. 11) $|(p_1, p_2)| = |(\mathrm{TPn2Rn}(p_1), \mathrm{TPn2Rn}(p_2))|$.

Let us observe that the functor $|(p_1, p_2)|$ is commutative.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. We say that p_1, p_2 are orthogonal if and only if:

(Def. 12) $|(p_1, p_2)| = 0$.

Let us note that the predicate p_1, p_2 are orthogonal is symmetric.

Next we state the proposition

(53) Let R be a subset of $\mathbb{R}$ and p_1, p_2, q_1 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $R = \{|q_1 - p|; p$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $p \in \mathrm{Line}(p_1, p_2)\}$. Then there exists a point q_2 of $\mathcal{E}^n_{\mathrm{T}}$ such that $q_2 \in \mathrm{Line}(p_1, p_2)$ and $|q_1 - q_2| = \inf R$ and $p_1 - p_2$, $q_1 - q_2$ are orthogonal.

References

[1] Kanchun and Yatsuka Nakamura. The inner product of finite sequences and of points of n-dimensional topological space. *Formalized Mathematics*, 11(**2**):179–183, 2003.
[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[7] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[9] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[10] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[11] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[12] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[14] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received August 8, 2003