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Intuitionistic Propositional Calculus in the Extended Framework with Modal Operator. Part I

Takao Inoué
The Iida Technical High School
Nagano

Summary. In this paper, we develop intuitionistic propositional calculus IPC in the extended language with single modal operator. The formulation that we adopt in this paper is very useful not only to formalize the calculus but also to do a number of logics with essentially propositional character. In addition, it is much simpler than the past formalization for modal logic. In the first section, we give the mentioned formulation which the author heavily owes to the formalism of Adam Grabowski's [4]. After the theoretical development of the logic, we prove a number of valid formulas of IPC in the sections 2–4. The last two sections are devoted to present classical propositional calculus and modal calculus S4 in our framework, as a preparation for future study. In the forthcoming Part II of this paper, we shall prove, among others, a number of intuitionistically valid formulas with negation.

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The articles [6], [7], [5], [8], [3], [1], and [2] provide the notation and terminology for this paper.

1. Intuitionistic Propositional Calculus IPC in the Extended Language with Modal Operator

Let E be a set. We say that E has FALSUM if and only if:

(Def. 1) $\langle 0 \rangle \in E$.

Let E be a set. We say that E has intuitionistic implication if and only if:

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(Def. 2) For all finite sequences p, q such that $p \in E$ and $q \in E$ holds $\langle 1 \rangle \frown p \frown q \in E$.

Let E be a set. We say that E has intuitionistic conjunction if and only if:

(Def. 3) For all finite sequences p, q such that $p \in E$ and $q \in E$ holds $\langle 2 \rangle \frown p \frown q \in E$.

Let E be a set. We say that E has intuitionistic disjunction if and only if:

(Def. 4) For all finite sequences p, q such that $p \in E$ and $q \in E$ holds $\langle 3 \rangle \frown p \frown q \in E$.

Let E be a set. We say that E has intuitionistic propositional variables if and only if:

(Def. 5) For every natural number n holds $\langle 5 + 2 \cdot n \rangle \in E$.

Let E be a set. We say that E has intuitionistic modal operator if and only if:

(Def. 6) For every finite sequence p such that $p \in E$ holds $\langle 6 \rangle \frown p \in E$.

Let E be a set. We say that E is MC-closed if and only if the conditions (Def. 7) are satisfied.

(Def. 7)(i) $E \subseteq \mathbb{N}^*$, and

(ii) E has FALSUM, intuitionistic implication, intuitionistic conjunction, intuitionistic disjunction, intuitionistic propositional variables, and intuitionistic modal operator.

One can check that every set which is MC-closed is also non empty and has FALSUM, intuitionistic implication, intuitionistic conjunction, intuitionistic disjunction, intuitionistic propositional variables, and intuitionistic modal operator and every subset of $\mathbb{N}^*$ which has FALSUM, intuitionistic implication, intuitionistic conjunction, intuitionistic disjunction, intuitionistic propositional variables, and intuitionistic modal operator is also MC-closed.

The set MC-wff is defined by:

(Def. 8) MC-wff is MC-closed and for every set E such that E is MC-closed holds MC-wff $\subseteq E$.

One can verify that MC-wff is MC-closed.

Let us note that there exists a set which is MC-closed and non empty.

One can verify that every element of MC-wff is relation-like and function-like.

Let us note that every element of MC-wff is finite sequence-like.

A MC-formula is an element of MC-wff.

The MC-formula FALSUM is defined as follows:

(Def. 9) FALSUM $= \langle 0 \rangle$.

Let p, q be elements of MC-wff. The functor $p \Rightarrow q$ yields a MC-formula and is defined as follows:

(Def. 10) $p \Rightarrow q = \langle 1 \rangle \frown p \frown q$.

The functor $p \wedge q$ yields a MC-formula and is defined as follows:

(Def. 11) $p \wedge q = \langle 2 \rangle \frown p \frown q$.

The functor $p \vee q$ yielding a MC-formula is defined by:

(Def. 12) $p \vee q = \langle 3 \rangle \frown p \frown q$.

Let p be an element of MC-wff. The functor Nes(p) yielding a MC-formula is defined by:

(Def. 13) $\mathrm{Nes}(p) = \langle 6 \rangle \frown p$.

We use the following convention: T, X, Y denote subsets of MC-wff and p, q, r, s denote elements of MC-wff.

Let T be a subset of MC-wff. We say that T is IPC theory if and only if the condition (Def. 14) is satisfied.

(Def. 14) Let p, q, r be elements of MC-wff. Then $p \Rightarrow (q \Rightarrow p) \in T$ and $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in T$ and $p \wedge q \Rightarrow p \in T$ and $p \wedge q \Rightarrow q \in T$ and $p \Rightarrow (q \Rightarrow p \wedge q) \in T$ and $p \Rightarrow p \vee q \in T$ and $q \Rightarrow p \vee q \in T$ and $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)) \in T$ and FALSUM $\Rightarrow p \in T$ and if $p \in T$ and $p \Rightarrow q \in T$, then $q \in T$.

Let us consider X. The functor CnIPC(X) yielding a subset of MC-wff is defined as follows:

(Def. 15) $r \in \mathrm{CnIPC}(X)$ iff for every T such that T is IPC theory and $X \subseteq T$ holds $r \in T$.

The subset IPC-Taut of MC-wff is defined as follows:

(Def. 16) $\text{IPC-Taut} = \mathrm{CnIPC}(\emptyset_{\text{MC-wff}})$.

Let p be an element of MC-wff. The functor neg(p) yields a MC-formula and is defined as follows:

(Def. 17) $\mathrm{neg}(p) = p \Rightarrow \mathrm{FALSUM}$.

The MC-formula IVERUM is defined by:

(Def. 18) IVERUM = FALSUM $\Rightarrow$ FALSUM.

The following propositions are true:

(1) $p \Rightarrow (q \Rightarrow p) \in \mathrm{CnIPC}(X)$.

(2) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in \mathrm{CnIPC}(X)$.

(3) $p \wedge q \Rightarrow p \in \mathrm{CnIPC}(X)$.

(4) $p \wedge q \Rightarrow q \in \mathrm{CnIPC}(X)$.

(5) $p \Rightarrow (q \Rightarrow p \wedge q) \in \mathrm{CnIPC}(X)$.

(6) $p \Rightarrow p \vee q \in \mathrm{CnIPC}(X)$.

(7) $q \Rightarrow p \vee q \in \mathrm{CnIPC}(X)$.

(8) $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)) \in \mathrm{CnIPC}(X)$.

(9) FALSUM $\Rightarrow p \in \mathrm{CnIPC}(X)$.

(10) If $p \in \mathrm{CnIPC}(X)$ and $p \Rightarrow q \in \mathrm{CnIPC}(X)$, then $q \in \mathrm{CnIPC}(X)$.

(11) If T is IPC theory and $X \subseteq T$, then $\mathrm{CnIPC}(X) \subseteq T$.

(12) $X \subseteq \mathrm{CnIPC}(X)$.

(13) If $X \subseteq Y$, then $\mathrm{CnIPC}(X) \subseteq \mathrm{CnIPC}(Y)$.

(14) $\mathrm{CnIPC}(\mathrm{CnIPC}(X)) = \mathrm{CnIPC}(X)$.

Let X be a subset of MC-wff. Observe that $\mathrm{CnIPC}(X)$ is IPC theory. The following propositions are true:

(15) T is IPC theory iff $\mathrm{CnIPC}(T) = T$.

(16) If T is IPC theory, then IPC-Taut $\subseteq T$.

One can verify that IPC-Taut is IPC theory.

2. Formulas Provable in IPC: Implication

We now state a number of propositions:

(17) $p \Rightarrow p \in$ IPC-Taut.

(18) If $q \in$ IPC-Taut, then $p \Rightarrow q \in$ IPC-Taut.

(19) IVERUM $\in$ IPC-Taut.

(20) $(p \Rightarrow q) \Rightarrow (p \Rightarrow p) \in$ IPC-Taut.

(21) $(q \Rightarrow p) \Rightarrow (p \Rightarrow p) \in$ IPC-Taut.

(22) $(q \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in$ IPC-Taut.

(23) If $p \Rightarrow (q \Rightarrow r) \in$ IPC-Taut, then $q \Rightarrow (p \Rightarrow r) \in$ IPC-Taut.

(24) $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r)) \in$ IPC-Taut.

(25) If $p \Rightarrow q \in$ IPC-Taut, then $(q \Rightarrow r) \Rightarrow (p \Rightarrow r) \in$ IPC-Taut.

(26) If $p \Rightarrow q \in$ IPC-Taut and $q \Rightarrow r \in$ IPC-Taut, then $p \Rightarrow r \in$ IPC-Taut.

(27) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((s \Rightarrow q) \Rightarrow (p \Rightarrow (s \Rightarrow r))) \in$ IPC-Taut.

(28) $((p \Rightarrow q) \Rightarrow r) \Rightarrow (q \Rightarrow r) \in$ IPC-Taut.

(29) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow (q \Rightarrow (p \Rightarrow r)) \in$ IPC-Taut.

(30) $(p \Rightarrow (p \Rightarrow q)) \Rightarrow (p \Rightarrow q) \in$ IPC-Taut.

(31) $q \Rightarrow ((q \Rightarrow p) \Rightarrow p) \in$ IPC-Taut.

(32) If $s \Rightarrow (q \Rightarrow p) \in$ IPC-Taut and $q \in$ IPC-Taut, then $s \Rightarrow p \in$ IPC-Taut.

3. Formulas Provable in IPC: Conjunction

The following propositions are true:

(33) $p \Rightarrow p \wedge p \in$ IPC-Taut.

(34) $p \wedge q \in$ IPC-Taut iff $p \in$ IPC-Taut and $q \in$ IPC-Taut.

(35) $p \wedge q \in$ IPC-Taut iff $q \wedge p \in$ IPC-Taut.

(36) $(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r)) \in$ IPC-Taut .

(37) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow (p \wedge q \Rightarrow r) \in$ IPC-Taut .

(38) $(r \Rightarrow p) \Rightarrow ((r \Rightarrow q) \Rightarrow (r \Rightarrow p \wedge q)) \in$ IPC-Taut .

(39) $(p \Rightarrow q) \wedge p \Rightarrow q \in$ IPC-Taut .

(40) $(p \Rightarrow q) \wedge p \wedge s \Rightarrow q \in$ IPC-Taut .

(41) $(q \Rightarrow s) \Rightarrow (p \wedge q \Rightarrow s) \in$ IPC-Taut .

(42) $(q \Rightarrow s) \Rightarrow (q \wedge p \Rightarrow s) \in$ IPC-Taut .

(43) $(p \wedge s \Rightarrow q) \Rightarrow (p \wedge s \Rightarrow q \wedge s) \in$ IPC-Taut .

(44) $(p \Rightarrow q) \Rightarrow (p \wedge s \Rightarrow q \wedge s) \in$ IPC-Taut .

(45) $(p \Rightarrow q) \wedge (p \wedge s) \Rightarrow q \wedge s \in$ IPC-Taut .

(46) $p \wedge q \Rightarrow q \wedge p \in$ IPC-Taut .

(47) $(p \Rightarrow q) \wedge (p \wedge s) \Rightarrow s \wedge q \in$ IPC-Taut .

(48) $(p \Rightarrow q) \Rightarrow (p \wedge s \Rightarrow s \wedge q) \in$ IPC-Taut .

(49) $(p \Rightarrow q) \Rightarrow (s \wedge p \Rightarrow s \wedge q) \in$ IPC-Taut .

(50) $p \wedge (s \wedge q) \Rightarrow p \wedge (q \wedge s) \in$ IPC-Taut .

(51) $(p \Rightarrow q) \wedge (p \Rightarrow s) \Rightarrow (p \Rightarrow q \wedge s) \in$ IPC-Taut .

(52) $p \wedge q \wedge s \Rightarrow p \wedge (q \wedge s) \in$ IPC-Taut .

(53) $p \wedge (q \wedge s) \Rightarrow p \wedge q \wedge s \in$ IPC-Taut .

4. Formulas Provable in IPC: Disjunction

We now state a number of propositions:

(54) $p \vee p \Rightarrow p \in$ IPC-Taut .

(55) If $p \in$ IPC-Taut or $q \in$ IPC-Taut, then $p \vee q \in$ IPC-Taut .

(56) $p \vee q \Rightarrow q \vee p \in$ IPC-Taut .

(57) $p \vee q \in$ IPC-Taut iff $q \vee p \in$ IPC-Taut .

(58) $(p \Rightarrow q) \Rightarrow (p \Rightarrow q \vee s) \in$ IPC-Taut .

(59) $(p \Rightarrow q) \Rightarrow (p \Rightarrow s \vee q) \in$ IPC-Taut .

(60) $(p \Rightarrow q) \Rightarrow (p \vee s \Rightarrow q \vee s) \in$ IPC-Taut .

(61) If $p \Rightarrow q \in$ IPC-Taut, then $p \vee s \Rightarrow q \vee s \in$ IPC-Taut .

(62) $(p \Rightarrow q) \Rightarrow (s \vee p \Rightarrow s \vee q) \in$ IPC-Taut .

(63) If $p \Rightarrow q \in$ IPC-Taut, then $s \vee p \Rightarrow s \vee q \in$ IPC-Taut .

(64) $p \vee (q \vee s) \Rightarrow q \vee (p \vee s) \in$ IPC-Taut .

(65) $p \vee (q \vee s) \Rightarrow p \vee q \vee s \in$ IPC-Taut .

(66) $p \vee q \vee s \Rightarrow p \vee (q \vee s) \in$ IPC-Taut .

5. Classical Propositional Calculus CPC

We use the following convention: T, X, Y are subsets of MC-wff and p, q, r are elements of MC-wff.

Let T be a subset of MC-wff. We say that T is CPC theory if and only if the condition (Def. 19) is satisfied.

(Def. 19) Let p, q, r be elements of MC-wff. Then $p \Rightarrow (q \Rightarrow p) \in T$ and $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in T$ and $p \wedge q \Rightarrow p \in T$ and $p \wedge q \Rightarrow q \in T$ and $p \Rightarrow (q \Rightarrow p \wedge q) \in T$ and $p \Rightarrow p \vee q \in T$ and $q \Rightarrow p \vee q \in T$ and $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)) \in T$ and FALSUM $\Rightarrow p \in T$ and $p \vee (p \Rightarrow \text{FALSUM}) \in T$ and if $p \in T$ and $p \Rightarrow q \in T$, then $q \in T$.

One can prove the following proposition

(67) If T is CPC theory, then T is IPC theory.

Let us consider X. The functor CnCPC(X) yielding a subset of MC-wff is defined by:

(Def. 20) $r \in \text{CnCPC}(X)$ iff for every T such that T is CPC theory and $X \subseteq T$ holds $r \in T$.

The subset CPC-Taut of MC-wff is defined by:

(Def. 21) CPC-Taut $= \text{CnCPC}(\emptyset_{\text{MC-wff}})$.

Next we state several propositions:

(68) $\text{CnIPC}(X) \subseteq \text{CnCPC}(X)$.

(69) $p \Rightarrow (q \Rightarrow p) \in \text{CnCPC}(X)$ and $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in \text{CnCPC}(X)$ and $p \wedge q \Rightarrow p \in \text{CnCPC}(X)$ and $p \wedge q \Rightarrow q \in \text{CnCPC}(X)$ and $p \Rightarrow (q \Rightarrow p \wedge q) \in \text{CnCPC}(X)$ and $p \Rightarrow p \vee q \in \text{CnCPC}(X)$ and $q \Rightarrow p \vee q \in \text{CnCPC}(X)$ and $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)) \in \text{CnCPC}(X)$ and FALSUM $\Rightarrow p \in \text{CnCPC}(X)$ and $p \vee (p \Rightarrow \text{FALSUM}) \in \text{CnCPC}(X)$.

(70) If $p \in \text{CnCPC}(X)$ and $p \Rightarrow q \in \text{CnCPC}(X)$, then $q \in \text{CnCPC}(X)$.

(71) If T is CPC theory and $X \subseteq T$, then $\text{CnCPC}(X) \subseteq T$.

(72) $X \subseteq \text{CnCPC}(X)$.

(73) If $X \subseteq Y$, then $\text{CnCPC}(X) \subseteq \text{CnCPC}(Y)$.

(74) $\text{CnCPC}(\text{CnCPC}(X)) = \text{CnCPC}(X)$.

Let X be a subset of MC-wff. Note that CnCPC(X) is CPC theory.

Next we state two propositions:

(75) T is CPC theory iff $\text{CnCPC}(T) = T$.

(76) If T is CPC theory, then CPC-Taut $\subseteq T$.

Let us note that CPC-Taut is CPC theory.

The following proposition is true

(77) IPC-Taut $\subseteq$ CPC-Taut .

6. Modal Calculus S4

We use the following convention: T, X, Y are subsets of MC-wff and p, q, r are elements of MC-wff.

Let T be a subset of MC-wff. We say that T is S4 theory if and only if the condition (Def. 22) is satisfied.

(Def. 22) Let p, q, r be elements of MC-wff. Then $p \Rightarrow (q \Rightarrow p) \in T$ and $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in T$ and $p \wedge q \Rightarrow p \in T$ and $p \wedge q \Rightarrow q \in T$ and $p \Rightarrow (q \Rightarrow p \wedge q) \in T$ and $p \Rightarrow p \vee q \in T$ and $q \Rightarrow p \vee q \in T$ and $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)) \in T$ and $\mathrm{FALSUM} \Rightarrow p \in T$ and $p \vee (p \Rightarrow \mathrm{FALSUM}) \in T$ and $\mathrm{Nes}(p \Rightarrow q) \Rightarrow (\mathrm{Nes}(p) \Rightarrow \mathrm{Nes}(q)) \in T$ and $\mathrm{Nes}(p) \Rightarrow p \in T$ and $\mathrm{Nes}(p) \Rightarrow \mathrm{Nes}(\mathrm{Nes}(p)) \in T$ and if $p \in T$ and $p \Rightarrow q \in T$, then $q \in T$ and if $p \in T$, then $\mathrm{Nes}(p) \in T$.

Next we state two propositions:

(78) If T is S4 theory, then T is CPC theory.

(79) If T is S4 theory, then T is IPC theory.

Let us consider X. The functor $\mathrm{CnS4}(X)$ yielding a subset of MC-wff is defined by:

(Def. 23) $r \in \mathrm{CnS4}(X)$ iff for every T such that T is S4 theory and $X \subseteq T$ holds $r \in T$.

The subset S4-Taut of MC-wff is defined by:

(Def. 24) $\text{S4-Taut} = \mathrm{CnS4}(\emptyset_{\text{MC-wff}})$.

Next we state a number of propositions:

(80) $\mathrm{CnCPC}(X) \subseteq \mathrm{CnS4}(X)$.

(81) $\mathrm{CnIPC}(X) \subseteq \mathrm{CnS4}(X)$.

(82) $p \Rightarrow (q \Rightarrow p) \in \mathrm{CnS4}(X)$ and $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in \mathrm{CnS4}(X)$ and $p \wedge q \Rightarrow p \in \mathrm{CnS4}(X)$ and $p \wedge q \Rightarrow q \in \mathrm{CnS4}(X)$ and $p \Rightarrow (q \Rightarrow p \wedge q) \in \mathrm{CnS4}(X)$ and $p \Rightarrow p \vee q \in \mathrm{CnS4}(X)$ and $q \Rightarrow p \vee q \in \mathrm{CnS4}(X)$ and $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)) \in \mathrm{CnS4}(X)$ and $\mathrm{FALSUM} \Rightarrow p \in \mathrm{CnS4}(X)$ and $p \vee (p \Rightarrow \mathrm{FALSUM}) \in \mathrm{CnS4}(X)$.

(83) If $p \in \mathrm{CnS4}(X)$ and $p \Rightarrow q \in \mathrm{CnS4}(X)$, then $q \in \mathrm{CnS4}(X)$.

(84) $\mathrm{Nes}(p \Rightarrow q) \Rightarrow (\mathrm{Nes}(p) \Rightarrow \mathrm{Nes}(q)) \in \mathrm{CnS4}(X)$.

(85) $\mathrm{Nes}(p) \Rightarrow p \in \mathrm{CnS4}(X)$.

(86) $\mathrm{Nes}(p) \Rightarrow \mathrm{Nes}(\mathrm{Nes}(p)) \in \mathrm{CnS4}(X)$.

(87) If $p \in \mathrm{CnS4}(X)$, then $\mathrm{Nes}(p) \in \mathrm{CnS4}(X)$.

(88) If T is S4 theory and $X \subseteq T$, then $\mathrm{CnS4}(X) \subseteq T$.

(89) $X \subseteq \mathrm{CnS4}(X)$.

(90) If $X \subseteq Y$, then $\mathrm{CnS4}(X) \subseteq \mathrm{CnS4}(Y)$.

(91) $\mathrm{CnS4}(\mathrm{CnS4}(X)) = \mathrm{CnS4}(X)$.

Let X be a subset of MC-wff. One can verify that CnS4(X) is S4 theory.

Next we state two propositions:

(92) T is S4 theory iff $\mathrm{CnS4}(T) = T$.

(93) If T is S4 theory, then $\text{S4-Taut} \subseteq T$.

Let us note that S4-Taut is S4 theory.

The following propositions are true:

(94) $\text{CPC-Taut} \subseteq \text{S4-Taut}$.

(95) $\text{IPC-Taut} \subseteq \text{S4-Taut}$.

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