FORMALIZED MATHEMATICS
Volume 11, Number 2, 2003
University of Białystok

On the Hausdorff Distance Between Compact Subsets[1]

Adam Grabowski
University of Białystok

Summary. In [1] the pseudo-metric $\mathrm{dist}_{\min}^{\max}$ on compact subsets A and B of a topological space generated from arbitrary metric space is defined. Using this notion we define the Hausdorff distance (see e.g. [5]) of A and B as a maximum of the two pseudo-distances: from A to B and from B to A. We justify its distance properties. At the end we define some special notions which enable to apply the Hausdorff distance operator "HausDist" to the subsets of the Euclidean topological space $\mathcal{E}_T^n$.

MML Identifier: `HAUSDORF`.

The papers [16], [18], [15], [10], [17], [19], [3], [14], [6], [9], [8], [11], [2], [7], [4], [1], [13], and [12] provide the terminology and notation for this paper.

1. Preliminaries

Let r be a real number. Then $\{r\}$ is a subset of $\mathbb{R}$.

Let M be a non empty metric space. One can verify that M_{top} is T_2.

Next we state a number of propositions:

(1) For all real numbers x, y such that $x \geqslant 0$ and $y \geqslant 0$ and $\max(x, y) = 0$ holds $x = 0$.

(2) For every non empty metric space M and for every point x of M holds $(\mathrm{dist}(x))(x) = 0$.

(3) For every non empty metric space M and for every subset P of M_{top} and for every point x of M such that $x \in P$ holds $0 \in (\mathrm{dist}(x))^{\circ}P$.

[1]This work has been partially supported by CALCULEMUS grant HPRN-CT-2000-00102 and TYPES grant IST-1999-29001.

ISSN 1426–2630

(4) Let M be a non empty metric space, P be a subset of M_{top}, x be a point of M, and y be a real number. If $y \in (\mathrm{dist}(x))^{\circ}P$, then $y \geqslant 0$.

(5) For every non empty metric space M and for every subset P of M_{top} and for every set x such that $x \in P$ holds $(\mathrm{dist}_{\mathrm{min}}(P))(x) = 0$.

(6) Let M be a non empty metric space, p be a point of M, q be a point of M_{top}, and r be a real number. If $p = q$ and $r > 0$, then $\mathrm{Ball}(p, r)$ is a neighbourhood of q.

(7) Let M be a non empty metric space, A be a subset of M_{top}, and p be a point of M. Then $p \in \overline{A}$ if and only if for every real number r such that $r > 0$ holds $\mathrm{Ball}(p, r)$ meets A.

(8) Let M be a non empty metric space, p be a point of M, and A be a subset of M_{top}. Then $p \in \overline{A}$ if and only if for every real number r such that $r > 0$ there exists a point q of M such that $q \in A$ and $\rho(p, q) < r$.

(9) Let M be a non empty metric space, P be a non empty subset of M_{top}, and x be a point of M. Then $(\mathrm{dist}_{\mathrm{min}}(P))(x) = 0$ if and only if for every real number r such that $r > 0$ there exists a point p of M such that $p \in P$ and $\rho(x, p) < r$.

(10) Let M be a non empty metric space, P be a non empty subset of M_{top}, and x be a point of M. Then $x \in \overline{P}$ if and only if $(\mathrm{dist}_{\mathrm{min}}(P))(x) = 0$.

(11) Let M be a non empty metric space, P be a non empty closed subset of M_{top}, and x be a point of M. Then $x \in P$ if and only if $(\mathrm{dist}_{\mathrm{min}}(P))(x) = 0$.

(12) For every non empty subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ there exists a non empty subset X of $\mathbb{R}$ such that $A = X$ and $\inf A = \inf X$.

(13) For every non empty subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ there exists a non empty subset X of $\mathbb{R}$ such that $A = X$ and $\sup A = \sup X$.

(14) Let M be a non empty metric space, P be a non empty subset of M_{top}, x be a point of M, and X be a subset of $\mathbb{R}$. If $X = (\mathrm{dist}(x))^{\circ}P$, then X is lower bounded.

(15) Let M be a non empty metric space, P be a non empty subset of M_{top}, and x, y be points of M. If $y \in P$, then $(\mathrm{dist}_{\mathrm{min}}(P))(x) \leqslant \rho(x, y)$.

(16) Let M be a non empty metric space, P be a non empty subset of M_{top}, r be a real number, and x be a point of M. If for every point y of M such that $y \in P$ holds $\rho(x, y) \geqslant r$, then $(\mathrm{dist}_{\mathrm{min}}(P))(x) \geqslant r$.

(17) Let M be a non empty metric space, P be a non empty subset of M_{top}, and x, y be points of M. Then $(\mathrm{dist}_{\mathrm{min}}(P))(x) \leqslant \rho(x, y) + (\mathrm{dist}_{\mathrm{min}}(P))(y)$.

(18) Let M be a non empty metric space, P be a subset of the carrier of M_{top}, and Q be a non empty subset of the carrier of M. If $P = Q$, then $M_{\mathrm{top}} \restriction P = (M \restriction Q)_{\mathrm{top}}$.

(19) Let M be a non empty metric space, A be a subset of M, B be a non empty subset of the carrier of M, and C be a subset of $M \restriction B$. If $A \subseteq B$

and $A = C$ and C is bounded, then A is bounded.

(20) Let M be a non empty metric space, B be a subset of M, and A be a subset of $M_{\rm top}$. If $A = B$ and A is compact, then B is bounded.

(21) Let M be a non empty metric space, P be a non empty subset of $M_{\rm top}$, and z be a point of M. Then there exists a point w of M such that $w \in P$ and $({\rm dist}_{\rm min}(P))(z) \leqslant \rho(w, z)$.

Let M be a non empty metric space and let x be a point of M. Note that ${\rm dist}(x)$ is continuous.

Let M be a non empty metric space and let X be a compact non empty subset of $M_{\rm top}$. One can check that ${\rm dist}_{\rm max}(X)$ is continuous and ${\rm dist}_{\rm min}(X)$ is continuous.

One can prove the following propositions:

(22) Let M be a non empty metric space, P be a non empty subset of $M_{\rm top}$, and x, y be points of M. If $y \in P$ and P is compact, then $({\rm dist}_{\rm max}(P))(x) \geqslant \rho(x, y)$.

(23) Let M be a non empty metric space, P be a non empty subset of $M_{\rm top}$, and z be a point of M. If P is compact, then there exists a point w of M such that $w \in P$ and $({\rm dist}_{\rm max}(P))(z) \geqslant \rho(w, z)$.

(24) Let M be a non empty metric space, P, Q be non empty subsets of $M_{\rm top}$, and z be a point of M. If P is compact and Q is compact and $z \in Q$, then $({\rm dist}_{\rm min}(P))(z) \leqslant {\rm dist}^{\rm max}_{\rm max}(P, Q)$.

(25) Let M be a non empty metric space, P, Q be non empty subsets of $M_{\rm top}$, and z be a point of M. If P is compact and Q is compact and $z \in Q$, then $({\rm dist}_{\rm max}(P))(z) \leqslant {\rm dist}^{\rm max}_{\rm max}(P, Q)$.

(26) Let M be a non empty metric space, P, Q be non empty subsets of $M_{\rm top}$, and X be a subset of $\mathbb{R}$. If $X = ({\rm dist}_{\rm max}(P))^\circ Q$ and P is compact and Q is compact, then X is upper bounded.

(27) Let M be a non empty metric space, P, Q be non empty subsets of $M_{\rm top}$, and X be a subset of $\mathbb{R}$. If $X = ({\rm dist}_{\rm min}(P))^\circ Q$ and P is compact and Q is compact, then X is upper bounded.

(28) Let M be a non empty metric space, P be a non empty subset of $M_{\rm top}$, and z be a point of M. If P is compact, then $({\rm dist}_{\rm min}(P))(z) \leqslant ({\rm dist}_{\rm max}(P))(z)$.

(29) For every non empty metric space M and for every non empty subset P of $M_{\rm top}$ holds $({\rm dist}_{\rm min}(P))^\circ P = \{0\}$.

(30) Let M be a non empty metric space and P, Q be non empty subsets of $M_{\rm top}$. If P is compact and Q is compact, then ${\rm dist}^{\rm max}_{\rm min}(P, Q) \geqslant 0$.

(31) For every non empty metric space M and for every non empty subset P of $M_{\rm top}$ holds ${\rm dist}^{\rm max}_{\rm min}(P, P) = 0$.

(32) Let M be a non empty metric space and P, Q be non empty subsets of $M_{\rm top}$. If P is compact and Q is compact, then $\mathrm{dist}^{\min}_{\max}(P,Q) \geqslant 0$.

(33) Let M be a non empty metric space, Q, R be non empty subsets of $M_{\rm top}$, and y be a point of M. If Q is compact and R is compact and $y \in Q$, then $(\mathrm{dist}_{\min}(R))(y) \leqslant \mathrm{dist}^{\max}_{\min}(R,Q)$.

2. The Hausdorff Distance

Let M be a non empty metric space and let P, Q be subsets of $M_{\rm top}$. The functor $\mathrm{HausDist}(P,Q)$ yields a real number and is defined by:

(Def. 1) $\mathrm{HausDist}(P,Q) = \max(\mathrm{dist}^{\max}_{\min}(P,Q), \mathrm{dist}^{\max}_{\min}(Q,P))$.

Let us notice that the functor $\mathrm{HausDist}(P,Q)$ is commutative.

The following propositions are true:

(34) Let M be a non empty metric space, Q, R be non empty subsets of $M_{\rm top}$, and y be a point of M. If Q is compact and R is compact and $y \in Q$, then $(\mathrm{dist}_{\min}(R))(y) \leqslant \mathrm{HausDist}(Q,R)$.

(35) Let M be a non empty metric space and P, Q, R be non empty subsets of $M_{\rm top}$. If P is compact and Q is compact and R is compact, then $\mathrm{dist}^{\max}_{\min}(P,R) \leqslant \mathrm{HausDist}(P,Q) + \mathrm{HausDist}(Q,R)$.

(36) Let M be a non empty metric space and P, Q, R be non empty subsets of $M_{\rm top}$. If P is compact and Q is compact and R is compact, then $\mathrm{dist}^{\max}_{\min}(R,P) \leqslant \mathrm{HausDist}(P,Q) + \mathrm{HausDist}(Q,R)$.

(37) Let M be a non empty metric space and P, Q be non empty subsets of $M_{\rm top}$. If P is compact and Q is compact, then $\mathrm{HausDist}(P,Q) \geqslant 0$.

(38) For every non empty metric space M and for every non empty subset P of $M_{\rm top}$ holds $\mathrm{HausDist}(P,P) = 0$.

(39) Let M be a non empty metric space and P, Q be non empty subsets of $M_{\rm top}$. If P is compact and Q is compact and $\mathrm{HausDist}(P,Q) = 0$, then $P = Q$.

(40) Let M be a non empty metric space and P, Q, R be non empty subsets of $M_{\rm top}$. If P is compact and Q is compact and R is compact, then $\mathrm{HausDist}(P,R) \leqslant \mathrm{HausDist}(P,Q) + \mathrm{HausDist}(Q,R)$.

Let n be a natural number and let P, Q be subsets of the carrier of $\mathcal{E}^n_{\rm T}$. The functor $\mathrm{dist}^{\max}_{\min}(P,Q)$ yields a real number and is defined by:

(Def. 2) There exist subsets P', Q' of $(\mathcal{E}^n)_{\rm top}$ such that $P = P'$ and $Q = Q'$ and $\mathrm{dist}^{\max}_{\min}(P,Q) = \mathrm{dist}^{\max}_{\min}(P',Q')$.

Let n be a natural number and let P, Q be subsets of the carrier of $\mathcal{E}^n_{\rm T}$. The functor $\mathrm{HausDist}(P,Q)$ yields a real number and is defined by:

(Def. 3) There exist subsets P', Q' of $(\mathcal{E}^n)_{\rm top}$ such that $P = P'$ and $Q = Q'$ and $\mathrm{HausDist}(P,Q) = \mathrm{HausDist}(P',Q')$.

Let us note that the functor $\mathrm{HausDist}(P, Q)$ is commutative.

In the sequel n denotes a natural number.

Next we state four propositions:

(41) For all non empty subsets P, Q of $\mathcal{E}^n_{\mathrm{T}}$ such that P is compact and Q is compact holds $\mathrm{HausDist}(P, Q) \geqslant 0$.

(42) For every non empty subset P of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{HausDist}(P, P) = 0$.

(43) For all non empty subsets P, Q of $\mathcal{E}^n_{\mathrm{T}}$ such that P is compact and Q is compact and $\mathrm{HausDist}(P, Q) = 0$ holds $P = Q$.

(44) For all non empty subsets P, Q, R of $\mathcal{E}^n_{\mathrm{T}}$ such that P is compact and Q is compact and R is compact holds $\mathrm{HausDist}(P, R) \leqslant \mathrm{HausDist}(P, Q) + \mathrm{HausDist}(Q, R)$.

References

[1] Józef Białas and Yatsuka Nakamura. The theorem of Weierstrass. *Formalized Mathematics*, 5(**3**):353–359, 1996.

[2] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.

[5] Akos Csaszar. *General Topology.* Akademiai Kiado, Budapest, 1978.

[6] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[7] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[8] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces - fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.

[9] Alicia de la Cruz. Totally bounded metric spaces. *Formalized Mathematics*, 2(**4**):559–562, 1991.

[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[11] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.

[12] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.

[13] Beata Padlewska. Locally connected spaces. *Formalized Mathematics*, 2(**1**):93–96, 1991.

[14] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[15] Andrzej Trybulec. Introduction to arithmetics. *To appear in Formalized Mathematics.*

[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[17] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[18] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[19] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received January 27, 2003