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Trigonometric Functions on Complex Space

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Summary. This article describes definitions of sine, cosine, hyperbolic sine and hyperbolic cosine. Some of their basic properties are discussed.

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The notation and terminology used here are introduced in the following papers: [9], [4], [10], [1], [8], [2], [3], [5], [7], [11], and [6].

1. Definitions of Trigonometric Functions

We adopt the following convention: x, y denote elements of $\mathbb{R}$, z, z_1, z_2 denote elements of $\mathbb{C}$, and n denotes a natural number.

The function $\sin_{\mathbb{C}}$ from $\mathbb{C}$ into $\mathbb{C}$ is defined by:

(Def. 1) $\sin_{\mathbb{C}}(z) = \frac{\exp(i \cdot z) - \exp(-i \cdot z)}{(2+0i) \cdot i}$.

The function $\cos_{\mathbb{C}}$ from $\mathbb{C}$ into $\mathbb{C}$ is defined by:

(Def. 2) $\cos_{\mathbb{C}}(z) = \frac{\exp(i \cdot z) + \exp(-i \cdot z)}{2+0i}$.

The function $\sinh_{\mathbb{C}}$ from $\mathbb{C}$ into $\mathbb{C}$ is defined by:

(Def. 3) $\sinh_{\mathbb{C}}(z) = \frac{\exp z - \exp(-z)}{2+0i}$.

The function $\cosh_{\mathbb{C}}$ from $\mathbb{C}$ into $\mathbb{C}$ is defined by:

(Def. 4) $\cosh_{\mathbb{C}}(z) = \frac{\exp z + \exp(-z)}{2+0i}$.

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2. Properties of Trigonometric Functions on Complex Space

We now state a number of propositions:

(1) For every element z of $\mathbb{C}$ holds $\sin_{\mathbb{C}z} \cdot \sin_{\mathbb{C}z} + \cos_{\mathbb{C}z} \cdot \cos_{\mathbb{C}z} = 1_{\mathbb{C}}$.

(2) $-\sin_{\mathbb{C}z} = \sin_{\mathbb{C}-z}$.

(3) $\cos_{\mathbb{C}z} = \cos_{\mathbb{C}-z}$.

(4) $\sin_{\mathbb{C}z_1+z_2} = \sin_{\mathbb{C}z_1} \cdot \cos_{\mathbb{C}z_2} + \cos_{\mathbb{C}z_1} \cdot \sin_{\mathbb{C}z_2}$.

(5) $\sin_{\mathbb{C}z_1-z_2} = \sin_{\mathbb{C}z_1} \cdot \cos_{\mathbb{C}z_2} - \cos_{\mathbb{C}z_1} \cdot \sin_{\mathbb{C}z_2}$.

(6) $\cos_{\mathbb{C}z_1+z_2} = \cos_{\mathbb{C}z_1} \cdot \cos_{\mathbb{C}z_2} - \sin_{\mathbb{C}z_1} \cdot \sin_{\mathbb{C}z_2}$.

(7) $\cos_{\mathbb{C}z_1-z_2} = \cos_{\mathbb{C}z_1} \cdot \cos_{\mathbb{C}z_2} + \sin_{\mathbb{C}z_1} \cdot \sin_{\mathbb{C}z_2}$.

(8) $\cosh_{\mathbb{C}z} \cdot \cosh_{\mathbb{C}z} - \sinh_{\mathbb{C}z} \cdot \sinh_{\mathbb{C}z} = 1_{\mathbb{C}}$.

(9) $-\sinh_{\mathbb{C}z} = \sinh_{\mathbb{C}-z}$.

(10) $\cosh_{\mathbb{C}z} = \cosh_{\mathbb{C}-z}$.

(11) $\sinh_{\mathbb{C}z_1+z_2} = \sinh_{\mathbb{C}z_1} \cdot \cosh_{\mathbb{C}z_2} + \cosh_{\mathbb{C}z_1} \cdot \sinh_{\mathbb{C}z_2}$.

(12) $\sinh_{\mathbb{C}z_1-z_2} = \sinh_{\mathbb{C}z_1} \cdot \cosh_{\mathbb{C}z_2} - \cosh_{\mathbb{C}z_1} \cdot \sinh_{\mathbb{C}z_2}$.

(13) $\cosh_{\mathbb{C}z_1-z_2} = \cosh_{\mathbb{C}z_1} \cdot \cosh_{\mathbb{C}z_2} - \sinh_{\mathbb{C}z_1} \cdot \sinh_{\mathbb{C}z_2}$.

(14) $\cosh_{\mathbb{C}z_1+z_2} = \cosh_{\mathbb{C}z_1} \cdot \cosh_{\mathbb{C}z_2} + \sinh_{\mathbb{C}z_1} \cdot \sinh_{\mathbb{C}z_2}$.

(15) $\sin_{\mathbb{C}i \cdot z} = i \cdot \sinh_{\mathbb{C}z}$.

(16) $\cos_{\mathbb{C}i \cdot z} = \cosh_{\mathbb{C}z}$.

(17) $\sinh_{\mathbb{C}i \cdot z} = i \cdot \sin_{\mathbb{C}z}$.

(18) $\cosh_{\mathbb{C}i \cdot z} = \cos_{\mathbb{C}z}$.

(19) For all elements x, y of $\mathbb{R}$ holds $\exp(x + yi) = \exp(x) \cdot \cos(y) + (\exp(x) \cdot \sin(y))i$.

(20) $\exp(0_{\mathbb{C}}) = 1 + 0i$.

(21) $\sin_{\mathbb{C}0_{\mathbb{C}}} = 0_{\mathbb{C}}$.

(22) $\sinh_{\mathbb{C}0_{\mathbb{C}}} = 0_{\mathbb{C}}$.

(23) $\cos_{\mathbb{C}0_{\mathbb{C}}} = 1 + 0i$.

(24) $\cosh_{\mathbb{C}0_{\mathbb{C}}} = 1 + 0i$.

(25) $\exp z = \cosh_{\mathbb{C}z} + \sinh_{\mathbb{C}z}$.

(26) $\exp(-z) = \cosh_{\mathbb{C}z} - \sinh_{\mathbb{C}z}$.

(27) $\exp(z + (2 \cdot \pi + 0i) \cdot i) = \exp z$ and $\exp(z + (0 + (2 \cdot \pi)i)) = \exp z$.

(28) $\exp(0 + (2 \cdot \pi \cdot n)i) = 1 + 0i$ and $\exp((2 \cdot \pi \cdot n + 0i) \cdot i) = 1 + 0i$.

(29) $\exp(0 + (-2 \cdot \pi \cdot n)i) = 1 + 0i$ and $\exp((-2 \cdot \pi \cdot n + 0i) \cdot i) = 1 + 0i$.

(30) $\exp(0 + ((2 \cdot n + 1) \cdot \pi)i) = -1 + 0i$ and $\exp(((2 \cdot n + 1) \cdot \pi + 0i) \cdot i) = -1 + 0i$.

(31) $\exp(0 + (-(2 \cdot n + 1) \cdot \pi)i) = -1 + 0i$ and $\exp((-(2 \cdot n + 1) \cdot \pi + 0i) \cdot i) = -1 + 0i$.

(32) $\exp(0 + ((2 \cdot n + \frac{1}{2}) \cdot \pi)i) = 0 + 1i$ and $\exp(((2 \cdot n + \frac{1}{2}) \cdot \pi + 0i) \cdot i) = 0 + 1i$.

(33) $\exp(0+(-(2\cdot n+\frac{1}{2})\cdot\pi)i) = 0+(-1)i$ and $\exp((-(2\cdot n+\frac{1}{2})\cdot\pi+0i)\cdot i) = 0+(-1)i$.

(34) $\sin_{\mathbb{C}z+(2\cdot n\cdot\pi+0i)} = \sin_{\mathbb{C}z}$.

(35) $\cos_{\mathbb{C}z+(2\cdot n\cdot\pi+0i)} = \cos_{\mathbb{C}z}$.

(36) $\exp(i\cdot z) = \cos_{\mathbb{C}z} + i\cdot\sin_{\mathbb{C}z}$.

(37) $\exp(-i\cdot z) = \cos_{\mathbb{C}z} - i\cdot\sin_{\mathbb{C}z}$.

(38) For every element x of $\mathbb{R}$ holds $\sin_{\mathbb{C}x+0i} = \sin(x)+0i$.

(39) For every element x of $\mathbb{R}$ holds $\cos_{\mathbb{C}x+0i} = \cos(x)+0i$.

(40) For every element x of $\mathbb{R}$ holds $\sinh_{\mathbb{C}x+0i} = \sinh(x)+0i$.

(41) For every element x of $\mathbb{R}$ holds $\cosh_{\mathbb{C}x+0i} = \cosh(x)+0i$.

(42) For all elements x, y of $\mathbb{R}$ holds $x+yi = (x+0i)+i\cdot(y+0i)$.

(43) $\sin_{\mathbb{C}x+yi} = \sin(x)\cdot\cosh(y)+(\cos(x)\cdot\sinh(y))i$.

(44) $\sin_{\mathbb{C}x+(-y)i} = \sin(x)\cdot\cosh(y)+(-\cos(x)\cdot\sinh(y))i$.

(45) $\cos_{\mathbb{C}x+yi} = \cos(x)\cdot\cosh(y)+(-\sin(x)\cdot\sinh(y))i$.

(46) $\cos_{\mathbb{C}x+(-y)i} = \cos(x)\cdot\cosh(y)+(\sin(x)\cdot\sinh(y))i$.

(47) $\sinh_{\mathbb{C}x+yi} = \sinh(x)\cdot\cos(y)+(\cosh(x)\cdot\sin(y))i$.

(48) $\sinh_{\mathbb{C}x+(-y)i} = \sinh(x)\cdot\cos(y)+(-\cosh(x)\cdot\sin(y))i$.

(49) $\cosh_{\mathbb{C}x+yi} = \cosh(x)\cdot\cos(y)+(\sinh(x)\cdot\sin(y))i$.

(50) $\cosh_{\mathbb{C}x+(-y)i} = \cosh(x)\cdot\cos(y)+(-\sinh(x)\cdot\sin(y))i$.

(51) For every natural number n and for every element z of $\mathbb{C}$ holds $(\cos_{\mathbb{C}z} + i\cdot\sin_{\mathbb{C}z})^{n}_{\mathbb{N}} = \cos_{\mathbb{C}(n+0i)\cdot z} + i\cdot\sin_{\mathbb{C}(n+0i)\cdot z}$.

(52) For every natural number n and for every element z of $\mathbb{C}$ holds $(\cos_{\mathbb{C}z} - i\cdot\sin_{\mathbb{C}z})^{n}_{\mathbb{N}} = \cos_{\mathbb{C}(n+0i)\cdot z} - i\cdot\sin_{\mathbb{C}(n+0i)\cdot z}$.

(53) For every natural number n and for every element z of $\mathbb{C}$ holds $\exp(i\cdot(n+0i)\cdot z) = (\cos_{\mathbb{C}z} + i\cdot\sin_{\mathbb{C}z})^{n}_{\mathbb{N}}$.

(54) For every natural number n and for every element z of $\mathbb{C}$ holds $\exp(-i\cdot(n+0i)\cdot z) = (\cos_{\mathbb{C}z} - i\cdot\sin_{\mathbb{C}z})^{n}_{\mathbb{N}}$.

(55) For all elements x, y of $\mathbb{R}$ holds $\frac{1+(-1)i}{2+0i}\cdot\sinh_{\mathbb{C}x+yi} + \frac{1+1i}{2+0i}\cdot\sinh_{\mathbb{C}x+(-y)i} = (\sinh(x)\cdot\cos(y)+\cosh(x)\cdot\sin(y))+0i$.

(56) For all elements x, y of $\mathbb{R}$ holds $\frac{1+(-1)i}{2+0i}\cdot\cosh_{\mathbb{C}x+yi} + \frac{1+1i}{2+0i}\cdot\cosh_{\mathbb{C}x+(-y)i} = (\sinh(x)\cdot\sin(y)+\cosh(x)\cdot\cos(y))+0i$.

(57) $\sinh_{\mathbb{C}z}\cdot\sinh_{\mathbb{C}z} = \frac{\cosh_{\mathbb{C}(2+0i)\cdot z} - (1+0i)}{2+0i}$.

(58) $\cosh_{\mathbb{C}z}\cdot\cosh_{\mathbb{C}z} = \frac{\cosh_{\mathbb{C}(2+0i)\cdot z} + (1+0i)}{2+0i}$.

(59) $\sinh_{\mathbb{C}(2+0i)\cdot z} = (2+0i)\cdot\sinh_{\mathbb{C}z}\cdot\cosh_{\mathbb{C}z}$ and $\cosh_{\mathbb{C}(2+0i)\cdot z} = (2+0i)\cdot\cosh_{\mathbb{C}z}\cdot\cosh_{\mathbb{C}z} - (1+0i)$.

(60) $\sinh_{\mathbb{C}z_1}\cdot\sinh_{\mathbb{C}z_1} - \sinh_{\mathbb{C}z_2}\cdot\sinh_{\mathbb{C}z_2} = \sinh_{\mathbb{C}z_1+z_2}\cdot\sinh_{\mathbb{C}z_1-z_2}$ and $\cosh_{\mathbb{C}z_1}\cdot\cosh_{\mathbb{C}z_1} - \cosh_{\mathbb{C}z_2}\cdot\cosh_{\mathbb{C}z_2} = \sinh_{\mathbb{C}z_1+z_2}\cdot\sinh_{\mathbb{C}z_1-z_2}$ and

$\sinh_{\mathbb{C}z_1} \cdot \sinh_{\mathbb{C}z_1} - \sinh_{\mathbb{C}z_2} \cdot \sinh_{\mathbb{C}z_2} = \cosh_{\mathbb{C}z_1} \cdot \cosh_{\mathbb{C}z_1} - \cosh_{\mathbb{C}z_2} \cdot \cosh_{\mathbb{C}z_2}$.

(61) $\cosh_{\mathbb{C}z_1+z_2} \cdot \cosh_{\mathbb{C}z_1-z_2} = \sinh_{\mathbb{C}z_1} \cdot \sinh_{\mathbb{C}z_1} + \cosh_{\mathbb{C}z_2} \cdot \cosh_{\mathbb{C}z_2}$ and $\cosh_{\mathbb{C}z_1+z_2} \cdot \cosh_{\mathbb{C}z_1-z_2} = \cosh_{\mathbb{C}z_1} \cdot \cosh_{\mathbb{C}z_1} + \sinh_{\mathbb{C}z_2} \cdot \sinh_{\mathbb{C}z_2}$ and $\sinh_{\mathbb{C}z_1} \cdot \sinh_{\mathbb{C}z_1} + \cosh_{\mathbb{C}z_2} \cdot \cosh_{\mathbb{C}z_2} = \cosh_{\mathbb{C}z_1} \cdot \cosh_{\mathbb{C}z_1} + \sinh_{\mathbb{C}z_2} \cdot \sinh_{\mathbb{C}z_2}$.

(62) $\sinh_{\mathbb{C}(2+0i)\cdot z_1} + \sinh_{\mathbb{C}(2+0i)\cdot z_2} = (2+0i) \cdot \sinh_{\mathbb{C}z_1+z_2} \cdot \cosh_{\mathbb{C}z_1-z_2}$ and $\sinh_{\mathbb{C}(2+0i)\cdot z_1} - \sinh_{\mathbb{C}(2+0i)\cdot z_2} = (2+0i) \cdot \sinh_{\mathbb{C}z_1-z_2} \cdot \cosh_{\mathbb{C}z_1+z_2}$.

(63) $\cosh_{\mathbb{C}(2+0i)\cdot z_1} + \cosh_{\mathbb{C}(2+0i)\cdot z_2} = (2+0i) \cdot \cosh_{\mathbb{C}z_1+z_2} \cdot \cosh_{\mathbb{C}z_1-z_2}$ and $\cosh_{\mathbb{C}(2+0i)\cdot z_1} - \cosh_{\mathbb{C}(2+0i)\cdot z_2} = (2+0i) \cdot \sinh_{\mathbb{C}z_1+z_2} \cdot \sinh_{\mathbb{C}z_1-z_2}$.

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