FORMALIZED MATHEMATICS
Volume 11, Number 1, 2003
University of Białystok

Convex Sets and Convex Combinations

Noboru Endou
Gifu National College of Technology

Takashi Mitsuishi
Miyagi University

Yasunari Shidama
Shinshu University
Nagano

Summary. Convexity is one of the most important concepts in a study of analysis. Especially, it has been applied around the optimization problem widely. Our purpose is to define the concept of convexity of a set on Mizar, and to develop the generalities of convex analysis. The construction of this article is as follows: Convexity of the set is defined in the section 1. The section 2 gives the definition of convex combination which is a kind of the linear combination and related theorems are proved there. In section 3, we define the convex hull which is an intersection of all convex sets including a given set. The last section is some theorems which are necessary to compose this article.

MML Identifier: CONVEX1.

The notation and terminology used in this paper are introduced in the following articles: [13], [12], [17], [9], [10], [3], [1], [8], [4], [2], [16], [15], [14], [5], [11], [6], and [7].

1. Convex Sets

Let V be a non empty RLS structure, let M be a subset of V, and let r be a real number. The functor $r \cdot M$ yielding a subset of V is defined by:

(Def. 1) $r \cdot M = \{r \cdot v; v$ ranges over elements of the carrier of $V\colon v \in M\}$.

Let V be a non empty RLS structure and let M be a subset of V. We say that M is convex if and only if:

(Def. 2) For all vectors u, v of V and for every real number r such that $0 < r$ and $r < 1$ and $u \in M$ and $v \in M$ holds $r \cdot u + (1 - r) \cdot v \in M$.

ISSN 1426–2630

We now state a number of propositions:

(1) Let V be a real linear space-like non empty RLS structure, M be a subset of V, and r be a real number. If M is convex, then $r \cdot M$ is convex.

(2) Let V be an Abelian add-associative real linear space-like non empty RLS structure and M, N be subsets of V. If M is convex and N is convex, then $M + N$ is convex.

(3) For every real linear space V and for all subsets M, N of V such that M is convex and N is convex holds $M - N$ is convex.

(4) Let V be a non empty RLS structure and M be a subset of V. Then M is convex if and only if for every real number r such that $0 < r$ and $r < 1$ holds $r \cdot M + (1 - r) \cdot M \subseteq M$.

(5) Let V be an Abelian non empty RLS structure and M be a subset of V. Suppose M is convex. Let r be a real number. If $0 < r$ and $r < 1$, then $(1 - r) \cdot M + r \cdot M \subseteq M$.

(6) Let V be an Abelian add-associative real linear space-like non empty RLS structure and M, N be subsets of V. Suppose M is convex and N is convex. Let r be a real number. Then $r \cdot M + (1 - r) \cdot N$ is convex.

(7) Let V be a real linear space, M be a subset of V, and v be a vector of V. Then M is convex if and only if $v + M$ is convex.

(8) For every real linear space V holds $\mathrm{Up}(\mathbf{0}_V)$ is convex.

(9) For every real linear space V holds $\mathrm{Up}(\Omega_V)$ is convex.

(10) For every non empty RLS structure V and for every subset M of V such that $M = \emptyset$ holds M is convex.

(11) Let V be an Abelian add-associative real linear space-like non empty RLS structure, M_1, M_2 be subsets of V, and r_1, r_2 be real numbers. If M_1 is convex and M_2 is convex, then $r_1 \cdot M_1 + r_2 \cdot M_2$ is convex.

(12) Let V be a real linear space-like non empty RLS structure, M be a subset of V, and r_1, r_2 be real numbers. Then $(r_1 + r_2) \cdot M \subseteq r_1 \cdot M + r_2 \cdot M$.

(13) Let V be a real linear space, M be a subset of V, and r_1, r_2 be real numbers. If $r_1 \geqslant 0$ and $r_2 \geqslant 0$ and M is convex, then $r_1 \cdot M + r_2 \cdot M \subseteq (r_1 + r_2) \cdot M$.

(14) Let V be an Abelian add-associative real linear space-like non empty RLS structure, M_1, M_2, M_3 be subsets of V, and r_1, r_2, r_3 be real numbers. If M_1 is convex and M_2 is convex and M_3 is convex, then $r_1 \cdot M_1 + r_2 \cdot M_2 + r_3 \cdot M_3$ is convex.

(15) Let V be a non empty RLS structure and F be a family of subsets of V. Suppose that for every subset M of V such that $M \in F$ holds M is convex. Then $\bigcap F$ is convex.

(16) For every non empty RLS structure V and for every subset M of V such

that M is Affine holds M is convex.

Let V be a non empty RLS structure. Observe that there exists a subset of V which is convex.

Let V be a non empty RLS structure. Note that there exists a subset of V which is empty and convex.

Let V be a non empty RLS structure. One can check that there exists a subset of V which is non empty and convex.

The following four propositions are true:

(17) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $(u|v) \geqslant r\}$, then M is convex.

(18) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $(u|v) > r\}$, then M is convex.

(19) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $(u|v) \leqslant r\}$, then M is convex.

(20) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $(u|v) < r\}$, then M is convex.

2. Convex Combinations

Let V be a real linear space and let L be a linear combination of V. We say that L is convex if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exists a finite sequence F of elements of the carrier of V such that

(i) F is one-to-one,

(ii) $\operatorname{rng} F =$ the support of L, and

(iii) there exists a finite sequence f of elements of $\mathbb{R}$ such that $\operatorname{len} f = \operatorname{len} F$ and $\sum f = 1$ and for every natural number n such that $n \in \operatorname{dom} f$ holds $f(n) = L(F(n))$ and $f(n) \geqslant 0$.

One can prove the following propositions:

(21) Let V be a real linear space and L be a linear combination of V. If L is convex, then the support of $L \neq \emptyset$.

(22) Let V be a real linear space, L be a linear combination of V, and v be a vector of V. If L is convex and $L(v) \leqslant 0$, then $v \notin$ the support of L.

(23) For every real linear space V and for every linear combination L of V such that L is convex holds $L \neq \mathbf{0}_{\mathrm{LC}_V}$.

(24) Let V be a real linear space, v be a vector of V, and L be a linear combination of $\{v\}$. If L is convex, then $L(v) = 1$ and $\sum L = L(v) \cdot v$.

(25) Let V be a real linear space, v_1, v_2 be vectors of V, and L be a linear combination of $\{v_1, v_2\}$. Suppose $v_1 \neq v_2$ and L is convex. Then $L(v_1) + L(v_2) = 1$ and $L(v_1) \geqslant 0$ and $L(v_2) \geqslant 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

(26) Let V be a real linear space, v_1, v_2, v_3 be vectors of V, and L be a linear combination of $\{v_1, v_2, v_3\}$. Suppose $v_1 \neq v_2$ and $v_2 \neq v_3$ and $v_3 \neq v_1$ and L is convex. Then $L(v_1) + L(v_2) + L(v_3) = 1$ and $L(v_1) \geqslant 0$ and $L(v_2) \geqslant 0$ and $L(v_3) \geqslant 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2 + L(v_3) \cdot v_3$.

(27) Let V be a real linear space, v be a vector of V, and L be a linear combination of V. If L is convex and the support of $L = \{v\}$, then $L(v) = 1$.

(28) Let V be a real linear space, v_1, v_2 be vectors of V, and L be a linear combination of V. Suppose L is convex and the support of $L = \{v_1, v_2\}$ and $v_1 \neq v_2$. Then $L(v_1) + L(v_2) = 1$ and $L(v_1) \geqslant 0$ and $L(v_2) \geqslant 0$.

(29) Let V be a real linear space, v_1, v_2, v_3 be vectors of V, and L be a linear combination of V. Suppose L is convex and the support of $L = \{v_1, v_2, v_3\}$ and $v_1 \neq v_2$ and $v_2 \neq v_3$ and $v_3 \neq v_1$. Then $L(v_1) + L(v_2) + L(v_3) = 1$ and $L(v_1) \geqslant 0$ and $L(v_2) \geqslant 0$ and $L(v_3) \geqslant 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2 + L(v_3) \cdot v_3$.

3. Convex Hull

In this article we present several logical schemes. The scheme *SubFamExRLS* deals with an RLS structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a family F of subsets of $\mathcal{A}$ such that for every subset B of the carrier of $\mathcal{A}$ holds $B \in F$ iff $\mathcal{P}[B]$

for all values of the parameters.

The scheme *SubFamExRLS2* deals with an RLS structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a family F of subsets of $\mathcal{A}$ such that for every subset B of $\mathcal{A}$ holds $B \in F$ iff $\mathcal{P}[B]$

for all values of the parameters.

Let V be a non empty RLS structure and let M be a subset of V. The functor Convex-Family M yields a family of subsets of V and is defined as follows:

(Def. 4) For every subset N of V holds $N \in$ Convex-Family M iff N is convex and $M \subseteq N$.

Let V be a non empty RLS structure and let M be a subset of V. The functor conv M yields a convex subset of V and is defined by:

(Def. 5) conv $M = \bigcap$ Convex-Family M.

The following proposition is true

(30) Let V be a non empty RLS structure, M be a subset of V, and N be a convex subset of V. If $M \subseteq N$, then $\operatorname{conv} M \subseteq N$.

4. Miscellaneous

One can prove the following propositions:

(31) Let p be a finite sequence and x, y, z be sets. Suppose p is one-to-one and $\operatorname{rng} p = \{x, y, z\}$ and $x \neq y$ and $y \neq z$ and $z \neq x$. Then $p = \langle x, y, z \rangle$ or $p = \langle x, z, y \rangle$ or $p = \langle y, x, z \rangle$ or $p = \langle y, z, x \rangle$ or $p = \langle z, x, y \rangle$ or $p = \langle z, y, x \rangle$.

(32) For every real linear space-like non empty RLS structure V and for every subset M of V holds $1 \cdot M = M$.

(33) For every non empty RLS structure V and for every empty subset M of V and for every real number r holds $r \cdot M = \emptyset$.

(34) For every real linear space V and for every non empty subset M of V holds $0 \cdot M = \{0_V\}$.

(35) For every right zeroed non empty loop structure V and for every subset M of V holds $M + \{0_V\} = M$.

(36) For every add-associative non empty loop structure V and for all subsets M_1, M_2, M_3 of V holds $(M_1 + M_2) + M_3 = M_1 + (M_2 + M_3)$.

(37) Let V be a real linear space-like non empty RLS structure, M be a subset of V, and r_1, r_2 be real numbers. Then $r_1 \cdot (r_2 \cdot M) = (r_1 \cdot r_2) \cdot M$.

(38) Let V be a real linear space-like non empty RLS structure, M_1, M_2 be subsets of V, and r be a real number. Then $r \cdot (M_1 + M_2) = r \cdot M_1 + r \cdot M_2$.

(39) Let V be a non empty RLS structure, M, N be subsets of V, and r be a real number. If $M \subseteq N$, then $r \cdot M \subseteq r \cdot N$.

(40) For every non empty loop structure V and for every empty subset M of V and for every subset N of V holds $M + N = \emptyset$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[5] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[6] Noboru Endou, Takashi Mitsuishi, and Yasunari Shidama. Dimension of real unitary space. *Formalized Mathematics*, 11(**1**):23–28, 2003.
[7] Noboru Endou, Takashi Mitsuishi, and Yasunari Shidama. Topology of real unitary space. *Formalized Mathematics*, 11(**1**):33–38, 2003.

[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[9] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[10] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[11] Jan Popiołek. Introduction to Banach and Hilbert spaces - part I. *Formalized Mathematics*, 2(**4**):511–516, 1991.
[12] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[14] Wojciech A. Trybulec. Linear combinations in real linear space. *Formalized Mathematics*, 1(**3**):581–588, 1990.
[15] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(**2**):297–301, 1990.
[16] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received November 5, 2002