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On the Sets Inhabited by Numbers[1]

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Summary. The information that all members of a set enjoy a property expressed by an adjective can be processed in a systematic way. The purpose of the work is to find out how to do that. If it works, 'membered' will become a reserved word and the work with it will be automated. I have chosen *membered* rather than *inhabited* because of the compatibility with the Automath terminology. The phrase τ *inhabits* θ could be translated to τ **is** θ in Mizar.

MML Identifier: `MEMBERED`.

The articles [6], [8], [4], [5], [3], [7], [1], and [2] provide the notation and terminology for this paper.

In this paper x, X, F denote sets.

Let X be a set. We say that X is complex-membered if and only if:

(Def. 1) If $x \in X$, then x is complex.

We say that X is real-membered if and only if:

(Def. 2) If $x \in X$, then x is real.

We say that X is rational-membered if and only if:

(Def. 3) If $x \in X$, then x is rational.

We say that X is integer-membered if and only if:

(Def. 4) If $x \in X$, then x is integer.

We say that X is natural-membered if and only if:

(Def. 5) If $x \in X$, then x is natural.

One can check the following observations:

* every set which is natural-membered is also integer-membered,
* every set which is integer-membered is also rational-membered,

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* every set which is rational-membered is also real-membered, and
* every set which is real-membered is also complex-membered.

Let us observe that there exists a set which is non empty and natural-membered.

One can verify the following observations:

* every subset of $\mathbb{C}$ is complex-membered,
* every subset of $\mathbb{R}$ is real-membered,
* every subset of $\mathbb{Q}$ is rational-membered,
* every subset of $\mathbb{Z}$ is integer-membered, and
* every subset of $\mathbb{N}$ is natural-membered.

One can verify the following observations:

* $\mathbb{C}$ is complex-membered,
* $\mathbb{R}$ is real-membered,
* $\mathbb{Q}$ is rational-membered,
* $\mathbb{Z}$ is integer-membered, and
* $\mathbb{N}$ is natural-membered.

Next we state several propositions:

(1) If X is complex-membered, then $X \subseteq \mathbb{C}$.

(2) If X is real-membered, then $X \subseteq \mathbb{R}$.

(3) If X is rational-membered, then $X \subseteq \mathbb{Q}$.

(4) If X is integer-membered, then $X \subseteq \mathbb{Z}$.

(5) If X is natural-membered, then $X \subseteq \mathbb{N}$.

Let X be a complex-membered set. One can check that every element of X is complex.

Let X be a real-membered set. One can verify that every element of X is real.

Let X be a rational-membered set. Note that every element of X is rational.

Let X be an integer-membered set. One can verify that every element of X is integer.

Let X be a natural-membered set. Observe that every element of X is natural.

For simplicity, we follow the rules: c, c_1, c_2, c_3 are complex numbers, r, r_1, r_2, r_3 are real numbers, w, w_1, w_2, w_3 are rational numbers, i, i_1, i_2, i_3 are integer numbers, and n, n_1, n_2, n_3 are natural numbers.

We now state a number of propositions:

(6) For every non empty complex-membered set X there exists c such that $c \in X$.

(7) For every non empty real-membered set X there exists r such that $r \in X$.

(8) For every non empty rational-membered set X there exists w such that $w \in X$.

(9) For every non empty integer-membered set X there exists i such that $i \in X$.

(10) For every non empty natural-membered set X there exists n such that $n \in X$.

(11) For every complex-membered set X such that for every c holds $c \in X$ holds $X = \mathbb{C}$.

(12) For every real-membered set X such that for every r holds $r \in X$ holds $X = \mathbb{R}$.

(13) For every rational-membered set X such that for every w holds $w \in X$ holds $X = \mathbb{Q}$.

(14) For every integer-membered set X such that for every i holds $i \in X$ holds $X = \mathbb{Z}$.

(15) For every natural-membered set X such that for every n holds $n \in X$ holds $X = \mathbb{N}$.

(16) For every complex-membered set Y such that $X \subseteq Y$ holds X is complex-membered.

(17) For every real-membered set Y such that $X \subseteq Y$ holds X is real-membered.

(18) For every rational-membered set Y such that $X \subseteq Y$ holds X is rational-membered.

(19) For every integer-membered set Y such that $X \subseteq Y$ holds X is integer-membered.

(20) For every natural-membered set Y such that $X \subseteq Y$ holds X is natural-membered.

One can verify that $\emptyset$ is natural-membered.

One can verify that every set which is empty is also natural-membered.

Let us consider c. One can verify that $\{c\}$ is complex-membered.

Let us consider r. One can verify that $\{r\}$ is real-membered.

Let us consider w. One can check that $\{w\}$ is rational-membered.

Let us consider i. One can verify that $\{i\}$ is integer-membered.

Let us consider n. Observe that $\{n\}$ is natural-membered.

Let us consider c_1, c_2. Note that $\{c_1, c_2\}$ is complex-membered.

Let us consider r_1, r_2. One can check that $\{r_1, r_2\}$ is real-membered.

Let us consider w_1, w_2. Observe that $\{w_1, w_2\}$ is rational-membered.

Let us consider i_1, i_2. One can verify that $\{i_1, i_2\}$ is integer-membered.

Let us consider n_1, n_2. Observe that $\{n_1, n_2\}$ is natural-membered.

Let us consider c_1, c_2, c_3. One can verify that $\{c_1, c_2, c_3\}$ is complex-membered.

Let us consider r_1, r_2, r_3. One can verify that $\{r_1, r_2, r_3\}$ is real-membered.

Let us consider w_1, w_2, w_3. Observe that $\{w_1, w_2, w_3\}$ is rational-membered.

Let us consider i_1, i_2, i_3. One can verify that $\{i_1, i_2, i_3\}$ is integer-membered.

Let us consider n_1, n_2, n_3. One can check that $\{n_1, n_2, n_3\}$ is natural-membered.

Let X be a complex-membered set. Note that every subset of X is complex-membered.

Let X be a real-membered set. One can verify that every subset of X is real-membered.

Let X be a rational-membered set. One can check that every subset of X is rational-membered.

Let X be an integer-membered set. Observe that every subset of X is integer-membered.

Let X be a natural-membered set. One can verify that every subset of X is natural-membered.

Let X, Y be complex-membered sets. Note that $X \cup Y$ is complex-membered.

Let X, Y be real-membered sets. Observe that $X \cup Y$ is real-membered.

Let X, Y be rational-membered sets. Note that $X \cup Y$ is rational-membered.

Let X, Y be integer-membered sets. Note that $X \cup Y$ is integer-membered.

Let X, Y be natural-membered sets. Observe that $X \cup Y$ is natural-membered.

Let X be a complex-membered set and let Y be a set. Note that $X \cap Y$ is complex-membered and $Y \cap X$ is complex-membered.

Let X be a real-membered set and let Y be a set. Note that $X \cap Y$ is real-membered and $Y \cap X$ is real-membered.

Let X be a rational-membered set and let Y be a set. Observe that $X \cap Y$ is rational-membered and $Y \cap X$ is rational-membered.

Let X be an integer-membered set and let Y be a set. Note that $X \cap Y$ is integer-membered and $Y \cap X$ is integer-membered.

Let X be a natural-membered set and let Y be a set. Observe that $X \cap Y$ is natural-membered and $Y \cap X$ is natural-membered.

Let X be a complex-membered set and let Y be a set. Note that $X \setminus Y$ is complex-membered.

Let X be a real-membered set and let Y be a set. Note that $X \setminus Y$ is real-membered.

Let X be a rational-membered set and let Y be a set. Observe that $X \setminus Y$ is rational-membered.

Let X be an integer-membered set and let Y be a set. Observe that $X \setminus Y$ is integer-membered.

Let X be a natural-membered set and let Y be a set. Observe that $X \setminus Y$ is natural-membered.

Let X, Y be complex-membered sets. Note that $X \dot{-} Y$ is complex-membered.

Let X, Y be real-membered sets. One can check that $X \dot{-} Y$ is real-membered.

Let X, Y be rational-membered sets. Note that $X \dot{-} Y$ is rational-membered.

Let X, Y be integer-membered sets. One can check that $X \dot{-} Y$ is integer-membered.

Let X, Y be natural-membered sets. One can verify that $X \dot{-} Y$ is natural-membered.

Let X, Y be complex-membered sets. Let us observe that $X \subseteq Y$ if and only if:

(Def. 6) If $c \in X$, then $c \in Y$.

Let X, Y be real-membered sets. Let us observe that $X \subseteq Y$ if and only if:

(Def. 7) If $r \in X$, then $r \in Y$.

Let X, Y be rational-membered sets. Let us observe that $X \subseteq Y$ if and only if:

(Def. 8) If $w \in X$, then $w \in Y$.

Let X, Y be integer-membered sets. Let us observe that $X \subseteq Y$ if and only if:

(Def. 9) If $i \in X$, then $i \in Y$.

Let X, Y be natural-membered sets. Let us observe that $X \subseteq Y$ if and only if:

(Def. 10) If $n \in X$, then $n \in Y$.

Let X, Y be complex-membered sets. Let us observe that $X = Y$ if and only if:

(Def. 11) $c \in X$ iff $c \in Y$.

Let X, Y be real-membered sets. Let us observe that $X = Y$ if and only if:

(Def. 12) $r \in X$ iff $r \in Y$.

Let X, Y be rational-membered sets. Let us observe that $X = Y$ if and only if:

(Def. 13) $w \in X$ iff $w \in Y$.

Let X, Y be integer-membered sets. Let us observe that $X = Y$ if and only if:

(Def. 14) $i \in X$ iff $i \in Y$.

Let X, Y be natural-membered sets. Let us observe that $X = Y$ if and only if:

(Def. 15) $n \in X$ iff $n \in Y$.

Let X, Y be complex-membered sets. Let us observe that X meets Y if and only if:

(Def. 16) There exists c such that $c \in X$ and $c \in Y$.

Let X, Y be real-membered sets. Let us observe that X meets Y if and only if:

(Def. 17) There exists r such that $r \in X$ and $r \in Y$.

Let X, Y be rational-membered sets. Let us observe that X meets Y if and only if:

(Def. 18) There exists w such that $w \in X$ and $w \in Y$.

Let X, Y be integer-membered sets. Let us observe that X meets Y if and only if:

(Def. 19) There exists i such that $i \in X$ and $i \in Y$.

Let X, Y be natural-membered sets. Let us observe that X meets Y if and only if:

(Def. 20) There exists n such that $n \in X$ and $n \in Y$.

One can prove the following propositions:

(21) If for every X such that $X \in F$ holds X is complex-membered, then $\bigcup F$ is complex-membered.

(22) If for every X such that $X \in F$ holds X is real-membered, then $\bigcup F$ is real-membered.

(23) If for every X such that $X \in F$ holds X is rational-membered, then $\bigcup F$ is rational-membered.

(24) If for every X such that $X \in F$ holds X is integer-membered, then $\bigcup F$ is integer-membered.

(25) If for every X such that $X \in F$ holds X is natural-membered, then $\bigcup F$ is natural-membered.

(26) For every X such that $X \in F$ and X is complex-membered holds $\bigcap F$ is complex-membered.

(27) For every X such that $X \in F$ and X is real-membered holds $\bigcap F$ is real-membered.

(28) For every X such that $X \in F$ and X is rational-membered holds $\bigcap F$ is rational-membered.

(29) For every X such that $X \in F$ and X is integer-membered holds $\bigcap F$ is integer-membered.

(30) For every X such that $X \in F$ and X is natural-membered holds $\bigcap F$ is natural-membered.

In this article we present several logical schemes. The scheme *CM Separation* concerns a unary predicate $\mathcal{P}$, and states that:

There exists a complex-membered set X such that for every c holds $c \in X$ iff $\mathcal{P}[c]$

for all values of the parameters.

The scheme *RM Separation* concerns a unary predicate $\mathcal{P}$, and states that:

There exists a real-membered set X such that for every r holds $r \in X$ iff $\mathcal{P}[r]$

for all values of the parameters.

The scheme *WM Separation* concerns a unary predicate $\mathcal{P}$, and states that:

There exists a rational-membered set X such that for every w holds $w \in X$ iff $\mathcal{P}[w]$

for all values of the parameters.

The scheme *IM Separation* concerns a unary predicate $\mathcal{P}$, and states that:

There exists an integer-membered set X such that for every i holds $i \in X$ iff $\mathcal{P}[i]$

for all values of the parameters.

The scheme *NM Separation* concerns a unary predicate $\mathcal{P}$, and states that:

There exists a natural-membered set X such that for every n holds $n \in X$ iff $\mathcal{P}[n]$

for all values of the parameters.

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Definition of Convex Function and Jensen's Inequality

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Summary. Convexity of a function in a real linear space is defined as convexity of its epigraph according to "Convex analysis" [24]. The epigraph of a function is a subset of the product of the function's domain space and the space of real numbers. Therefore, the product of two real linear spaces should be defined. The values of the functions under consideration are extended real numbers. We define the sum of a finite sequence of extended real numbers and get some properties of the sum. The relation between notions "function is convex" and "function is convex on set" (see definition 13 in [21]) is established. We obtain another version of the criterion for a set to be convex (see theorem 6 in [15] to compare) that may be more suitable in some cases. Finally, we prove Jensen's inequality (both strict and not strict) as criteria for functions to be convex.

MML Identifier: CONVFUN1.

The terminology and notation used here are introduced in the following articles: [27], [30], [25], [8], [18], [9], [3], [29], [14], [4], [31], [11], [6], [7], [19], [26], [22], [16], [5], [10], [21], [17], [2], [12], [28], [13], [1], [20], and [23].

1. Product of Two Real Linear Spaces

Let X, Y be non empty RLS structures. The functor AddInProdRLS(X,Y) yielding a binary operation on $[\!:$ the carrier of X, the carrier of $Y :\!]$ is defined by the condition (Def. 1).

(Def. 1) Let z_1, z_2 be elements of $[\!:$ the carrier of X, the carrier of $Y :\!]$, x_1, x_2 be vectors of X, and y_1, y_2 be vectors of Y. Suppose $z_1 = \langle x_1, y_1 \rangle$ and $z_2 = \langle x_2, y_2 \rangle$. Then $(\text{AddInProdRLS}(X,Y))(z_1, z_2) = \langle(\text{the addition of } X)(\langle x_1, x_2\rangle), (\text{the addition of } Y)(\langle y_1, y_2 \rangle)\rangle$.

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Let X, Y be non empty RLS structures. The functor MultInProdRLS(X,Y) yields a function from $[\!: \mathbb{R}, [\!:$ the carrier of X, the carrier of $Y :\!] :\!]$ into $[\!:$ the carrier of X, the carrier of $Y :\!]$ and is defined by the condition (Def. 2).

(Def. 2) Let a be a real number, z be an element of $[\!:$ the carrier of X, the carrier of $Y :\!]$, x be a vector of X, and y be a vector of Y. Suppose $z = \langle x, y \rangle$. Then (MultInProdRLS(X,Y))$(\langle a, z \rangle) = \langle$(the external multiplication of X)$(\langle a, x \rangle)$, (the external multiplication of Y)$(\langle a, y \rangle)\rangle$.

Let X, Y be non empty RLS structures. The functor ProdRLS(X,Y) yields an RLS structure and is defined by:

(Def. 3) ProdRLS(X,Y) $=$ $\langle [\!:$ the carrier of X, the carrier of $Y :\!], \langle 0_X, 0_Y \rangle$, AddInProdRLS$(X,Y)$, MultInProdRLS$(X,Y)\rangle$.

Let X, Y be non empty RLS structures. Note that ProdRLS(X,Y) is non empty.

Next we state two propositions:

(1) Let X, Y be non empty RLS structures, x be a vector of X, y be a vector of Y, u be a vector of ProdRLS(X,Y), and p be a real number. If $u = \langle x, y \rangle$, then $p \cdot u = \langle p \cdot x, p \cdot y \rangle$.

(2) Let X, Y be non empty RLS structures, x_1, x_2 be vectors of X, y_1, y_2 be vectors of Y, and u_1, u_2 be vectors of ProdRLS(X,Y). If $u_1 = \langle x_1, y_1 \rangle$ and $u_2 = \langle x_2, y_2 \rangle$, then $u_1 + u_2 = \langle x_1 + x_2, y_1 + y_2 \rangle$.

Let X, Y be Abelian non empty RLS structures. One can verify that ProdRLS(X,Y) is Abelian.

Let X, Y be add-associative non empty RLS structures. Observe that ProdRLS(X,Y) is add-associative.

Let X, Y be right zeroed non empty RLS structures. Observe that ProdRLS(X,Y) is right zeroed.

Let X, Y be right complementable non empty RLS structures. One can check that ProdRLS(X,Y) is right complementable.

Let X, Y be real linear space-like non empty RLS structures. Observe that ProdRLS(X,Y) is real linear space-like.

Next we state the proposition

(3) Let X, Y be real linear spaces, n be a natural number, x be a finite sequence of elements of the carrier of X, y be a finite sequence of elements of the carrier of Y, and z be a finite sequence of elements of the carrier of ProdRLS(X,Y). Suppose $\operatorname{len} x = n$ and $\operatorname{len} y = n$ and $\operatorname{len} z = n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $z(i) = \langle x(i), y(i) \rangle$. Then $\sum z = \langle \sum x, \sum y \rangle$.

2. Real Linear Space of Real Numbers

The non empty RLS structure $\mathbb{R}_{\mathrm{RLS}}$ is defined as follows:

(Def. 4) $\mathbb{R}_{\mathrm{RLS}} = \langle \mathbb{R}, 0, +_{\mathbb{R}}, \cdot_{\mathbb{R}} \rangle$.

Let us note that $\mathbb{R}_{\mathrm{RLS}}$ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

3. Sum of Finite Sequence of Extended Real Numbers

Let F be a finite sequence of elements of $\overline{\mathbb{R}}$. The functor $\sum F$ yields an extended real number and is defined by the condition (Def. 5).

(Def. 5) There exists a function f from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $\sum F = f(\operatorname{len} F)$ and $f(0) = 0_{\overline{\mathbb{R}}}$ and for every natural number i such that $i < \operatorname{len} F$ holds $f(i+1) = f(i) + F(i+1)$.

We now state several propositions:

(4) $\sum(\varepsilon_{\overline{\mathbb{R}}}) = 0_{\overline{\mathbb{R}}}$.

(5) For every extended real number a holds $\sum\langle a \rangle = a$.

(6) For all extended real numbers a, b holds $\sum\langle a, b \rangle = a + b$.

(7) For all finite sequences F, G of elements of $\overline{\mathbb{R}}$ such that $-\infty \notin \operatorname{rng} F$ and $-\infty \notin \operatorname{rng} G$ holds $\sum(F ^\frown G) = \sum F + \sum G$.

(8) Let F, G be finite sequences of elements of $\overline{\mathbb{R}}$ and s be a permutation of $\operatorname{dom} F$. If $G = F \cdot s$ and $-\infty \notin \operatorname{rng} F$, then $\sum F = \sum G$.

4. Definition of Convex Function

Let X be a non empty RLS structure and let f be a function from the carrier of X into $\overline{\mathbb{R}}$. The functor epigraph f yielding a subset of $\mathrm{ProdRLS}(X, \mathbb{R}_{\mathrm{RLS}})$ is defined as follows:

(Def. 6) epigraph $f = \{\langle x, y \rangle; x$ ranges over elements of X, y ranges over elements of $\mathbb{R}$: $f(x) \leqslant \overline{\mathbb{R}}(y)\}$.

Let X be a non empty RLS structure and let f be a function from the carrier of X into $\overline{\mathbb{R}}$. We say that f is convex if and only if:

(Def. 7) epigraph f is convex.

The following two propositions are true:

(9) Let X be a non empty RLS structure and f be a function from the carrier of X into $\overline{\mathbb{R}}$. Suppose that for every vector x of X holds $f(x) \neq -\infty$. Then f is convex if and only if for all vectors x_1, x_2 of X and for every real number p such that $0 < p$ and $p < 1$ holds $f(p \cdot x_1 + (1-p) \cdot x_2) \leqslant \overline{\mathbb{R}}(p) \cdot f(x_1) + \overline{\mathbb{R}}(1-p) \cdot f(x_2)$.

(10) Let X be a real linear space and f be a function from the carrier of X into $\overline{\mathbb{R}}$. Suppose that for every vector x of X holds $f(x) \neq -\infty$. Then f is convex if and only if for all vectors x_1, x_2 of X and for every real number p such that $0 \leqslant p$ and $p \leqslant 1$ holds $f(p \cdot x_1 + (1-p) \cdot x_2) \leqslant \overline{\mathbb{R}}(p) \cdot f(x_1) + \overline{\mathbb{R}}(1-p) \cdot f(x_2)$.

5. Relation between Notions "function is convex" and "function is convex on set"

We now state the proposition

(11) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, g be a function from the carrier of $\mathbb{R}_{\rm RLS}$ into $\overline{\mathbb{R}}$, and X be a subset of $\mathbb{R}_{\rm RLS}$. Suppose $X \subseteq \operatorname{dom} f$ and for every real number x holds if $x \in X$, then $g(x) = f(x)$ and if $x \notin X$, then $g(x) = +\infty$. Then g is convex if and only if the following conditions are satisfied:

(i) f is convex on X, and

(ii) X is convex.

6. Theorem 6 from [15] in Other Words

One can prove the following proposition

(12) Let X be a real linear space and M be a subset of X. Then M is convex if and only if for every non empty natural number n and for every finite sequence p of elements of $\mathbb{R}$ and for all finite sequences y, z of elements of the carrier of X such that $\operatorname{len} p = n$ and $\operatorname{len} y = n$ and $\operatorname{len} z = n$ and $\sum p = 1$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $p(i) > 0$ and $z(i) = p(i) \cdot y_i$ and $y_i \in M$ holds $\sum z \in M$.

7. Jensen's Inequality

One can prove the following two propositions:

(13) Let X be a real linear space and f be a function from the carrier of X into $\overline{\mathbb{R}}$. Suppose that for every vector x of X holds $f(x) \neq -\infty$. Then f is convex if and only if for every non empty natural number n and for every finite sequence p of elements of $\mathbb{R}$ and for every finite sequence F of elements of $\overline{\mathbb{R}}$ and for all finite sequences y, z of elements of the carrier of X such that $\operatorname{len} p = n$ and $\operatorname{len} F = n$ and $\operatorname{len} y = n$ and $\operatorname{len} z = n$ and $\sum p = 1$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $p(i) > 0$ and $z(i) = p(i) \cdot y_i$ and $F(i) = \overline{\mathbb{R}}(p(i)) \cdot f(y_i)$ holds $f(\sum z) \leqslant \sum F$.

(14) Let X be a real linear space and f be a function from the carrier of X into $\overline{\mathbb{R}}$. Suppose that for every vector x of X holds $f(x) \neq -\infty$. Then f is convex if and only if for every non empty natural number n and for every finite sequence p of elements of $\mathbb{R}$ and for every finite sequence F of elements of $\overline{\mathbb{R}}$ and for all finite sequences y, z of elements of the carrier of X such that $\operatorname{len} p = n$ and $\operatorname{len} F = n$ and $\operatorname{len} y = n$ and $\operatorname{len} z = n$ and $\sum p = 1$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $p(i) \geqslant 0$ and $z(i) = p(i) \cdot y_i$ and $F(i) = \overline{\mathbb{R}}(p(i)) \cdot f(y_i)$ holds $f(\sum z) \leqslant \sum F$.

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On Semilattice Structure of Mizar Types

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Summary. The aim of this paper is to develop a formal theory of Mizar types. The presented theory is an approach to the structure of Mizar types as a sup-semilattice with widening (subtyping) relation as the order. It is an abstraction from the existing implementation of the Mizar verifier and formalization of the ideas from [9].

MML Identifier: ABCMIZ_0.

The articles [20], [14], [24], [26], [23], [25], [3], [21], [1], [11], [12], [16], [10], [13], [18], [15], [4], [2], [19], [22], [5], [6], [7], [8], and [17] provide the terminology and notation for this paper.

1. Semilattice of Widening

Let us mention that every non empty relational structure which is trivial and reflexive is also complete.

Let T be a relational structure. A type of T is an element of T.

Let T be a relational structure. We say that T is Noetherian if and only if:

(Def. 1) The internal relation of T is reversely well founded.

Let us observe that every non empty relational structure which is trivial is also Noetherian.

Let T be a non empty relational structure. Let us observe that T is Noetherian if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let A be a non empty subset of T. Then there exists an element a of T such that $a \in A$ and for every element b of T such that $b \in A$ holds $a \not< b$.

Let T be a poset. We say that T is Mizar-widening-like if and only if:

(Def. 3) T is a sup-semilattice and Noetherian.

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Let us mention that every poset which is Mizar-widening-like is also Noetherian and upper-bounded and has l.u.b.'s.

Let us note that every sup-semilattice which is Noetherian is also Mizar-widening-like.

Let us observe that there exists a complete sup-semilattice which is Mizar-widening-like.

Let T be a Noetherian relational structure. One can check that the internal relation of T is reversely well founded.

Next we state the proposition

(1) For every Noetherian sup-semilattice T and for every ideal I of T holds $\sup I$ exists in T and $\sup I \in I$.

2. Adjectives

We consider adjective structures as systems

$\langle$ a set of adjectives, an operation non $\rangle$,

where the set of adjectives is a set and the operation non is a unary operation on the set of adjectives.

Let A be an adjective structure. We say that A is void if and only if:

(Def. 4) The set of adjectives of A is empty.

An adjective of A is an element of the set of adjectives of A.

The following proposition is true

(2) Let A_1, A_2 be adjective structures. Suppose the set of adjectives of A_1 = the set of adjectives of A_2. If A_1 is void, then A_2 is void.

Let A be an adjective structure and let a be an element of the set of adjectives of A. The functor $\mathrm{non}\, a$ yields an adjective of A and is defined as follows:

(Def. 5) $\mathrm{non}\, a =$ (the operation non of A)(a).

One can prove the following proposition

(3) Let A_1, A_2 be adjective structures. Suppose the adjective structure of A_1 = the adjective structure of A_2. Let a_1 be an adjective of A_1 and a_2 be an adjective of A_2. If $a_1 = a_2$, then $\mathrm{non}\, a_1 = \mathrm{non}\, a_2$.

Let A be an adjective structure. We say that A is involutive if and only if:

(Def. 6) For every adjective a of A holds $\mathrm{non}\,\mathrm{non}\, a = a$.

We say that A is without fixpoints if and only if:

(Def. 7) It is not true that there exists an adjective a of A such that $\mathrm{non}\, a = a$.

We now state three propositions:

(4) Let a_1, a_2 be sets. Suppose $a_1 \neq a_2$. Let A be an adjective structure. Suppose the set of adjectives of $A = \{a_1, a_2\}$ and (the operation non of A)(a_1) $= a_2$ and (the operation non of A)(a_2) $= a_1$. Then A is non void, involutive, and without fixpoints.

(5) Let A_1, A_2 be adjective structures. Suppose the adjective structure of $A_1 =$ the adjective structure of A_2. If A_1 is involutive, then A_2 is involutive.

(6) Let A_1, A_2 be adjective structures. Suppose the adjective structure of $A_1 =$ the adjective structure of A_2. If A_1 is without fixpoints, then A_2 is without fixpoints.

Let us observe that there exists a strict adjective structure which is non void, involutive, and without fixpoints.

Let A be a non void adjective structure. Observe that the set of adjectives of A is non empty.

We consider *TA*-structures as extensions of relational structure and adjective structure as systems

⟨ a carrier, a set of adjectives, an internal relation, an operation non, an adjective map ⟩,

where the carrier and the set of adjectives are sets, the internal relation is a binary relation on the carrier, the operation non is a unary operation on the set of adjectives, and the adjective map is a function from the carrier into Fin the set of adjectives.

Let X be a non empty set, let A be a set, let r be a binary relation on X, let n be a unary operation on A, and let a be a function from X into $\mathrm{Fin}\,A$. Observe that $\langle X, A, r, n, a\rangle$ is non empty.

Let X be a set, let A be a non empty set, let r be a binary relation on X, let n be a unary operation on A, and let a be a function from X into $\mathrm{Fin}\,A$. One can check that $\langle X, A, r, n, a\rangle$ is non void.

One can check that there exists a *TA*-structure which is trivial, reflexive, non empty, non void, involutive, without fixpoints, and strict.

Let T be a *TA*-structure and let t be an element of T. The functor $\mathrm{adjs}\,t$ yields a subset of the set of adjectives of T and is defined as follows:

(Def. 8) $\mathrm{adjs}\,t =$ (the adjective map of T)(t).

One can prove the following proposition

(7) Let T_1, T_2 be *TA*-structures. Suppose the *TA*-structure of $T_1 =$ the *TA*-structure of T_2. Let t_1 be a type of T_1 and t_2 be a type of T_2. If $t_1 = t_2$, then $\mathrm{adjs}\,t_1 = \mathrm{adjs}\,t_2$.

Let T be a *TA*-structure. We say that T is consistent if and only if:

(Def. 9) For every type t of T and for every adjective a of T such that $a \in \mathrm{adjs}\,t$ holds $\mathrm{non}\,a \notin \mathrm{adjs}\,t$.

Next we state the proposition

(8) Let T_1, T_2 be *TA*-structures. Suppose the *TA*-structure of $T_1 =$ the *TA*-structure of T_2. If T_1 is consistent, then T_2 is consistent.

Let T be a non empty *TA*-structure. We say that T has structured adjectives if and only if:

(Def. 10) The adjective map of T is a join-preserving map from T into $(2_{\subseteq}^{\text{the set of adjectives of } T})^{\text{op}}$.

We now state the proposition

(9) Let T_1, T_2 be non empty *TA*-structures. Suppose the *TA*-structure of $T_1 =$ the *TA*-structure of T_2. If T_1 has structured adjectives, then T_2 has structured adjectives.

Let T be a reflexive transitive antisymmetric *TA*-structure with l.u.b.'s. Let us observe that T has structured adjectives if and only if:

(Def. 11) For all types t_1, t_2 of T holds $\mathrm{adjs}(t_1 \sqcup t_2) = \mathrm{adjs}\, t_1 \cap \mathrm{adjs}\, t_2$.

One can prove the following proposition

(10) Let T be a reflexive transitive antisymmetric *TA*-structure with l.u.b.'s. Suppose T has structured adjectives. Let t_1, t_2 be types of T. If $t_1 \leqslant t_2$, then $\mathrm{adjs}\, t_2 \subseteq \mathrm{adjs}\, t_1$.

Let T be a *TA*-structure and let a be an element of the set of adjectives of T. The functor $\mathrm{types}\, a$ yields a subset of T and is defined as follows:

(Def. 12) For every set x holds $x \in \mathrm{types}\, a$ iff there exists a type t of T such that $x = t$ and $a \in \mathrm{adjs}\, t$.

Let T be a non empty *TA*-structure and let A be a subset of the set of adjectives of T. The functor $\mathrm{types}\, A$ yielding a subset of T is defined as follows:

(Def. 13) For every type t of T holds $t \in \mathrm{types}\, A$ iff for every adjective a of T such that $a \in A$ holds $t \in \mathrm{types}\, a$.

One can prove the following propositions:

(11) Let T_1, T_2 be *TA*-structures. Suppose the *TA*-structure of $T_1 =$ the *TA*-structure of T_2. Let a_1 be an adjective of T_1 and a_2 be an adjective of T_2. If $a_1 = a_2$, then $\mathrm{types}\, a_1 = \mathrm{types}\, a_2$.

(12) For every non empty *TA*-structure T and for every adjective a of T holds $\mathrm{types}\, a = \{t; t \text{ ranges over types of } T\colon a \in \mathrm{adjs}\, t\}$.

(13) Let T be a *TA*-structure, t be a type of T, and a be an adjective of T. Then $a \in \mathrm{adjs}\, t$ if and only if $t \in \mathrm{types}\, a$.

(14) Let T be a non empty *TA*-structure, t be a type of T, and A be a subset of the set of adjectives of T. Then $A \subseteq \mathrm{adjs}\, t$ if and only if $t \in \mathrm{types}\, A$.

(15) For every non void *TA*-structure T and for every type t of T holds $\mathrm{adjs}\, t = \{a; a \text{ ranges over adjectives of } T\colon t \in \mathrm{types}\, a\}$.

(16) Let T be a non empty *TA*-structure and t be a type of T. Then $\mathrm{types}(\emptyset_{\text{the set of adjectives of } T}) =$ the carrier of T.

Let T be a *TA*-structure. We say that T has typed adjectives if and only if:

(Def. 14) For every adjective a of T holds $\mathrm{types}\, a \cup \mathrm{types}\, \mathrm{non}\, a$ is non empty.

We now state the proposition

(17) Let T_1, T_2 be *TA*-structures. Suppose the *TA*-structure of T_1 = the *TA*-structure of T_2. If T_1 has typed adjectives, then T_2 has typed adjectives.

Let us mention that there exists a complete upper-bounded non empty trivial reflexive transitive antisymmetric strict *TA*-structure which is non void, Mizar-widening-like, involutive, without fixpoints, and consistent and has structured adjectives and typed adjectives.

Next we state the proposition

(18) For every consistent *TA*-structure T and for every adjective a of T holds $\mathrm{types}\, a$ misses $\mathrm{types}\, \mathrm{non}\, a$.

Let T be a reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives and let a be an adjective of T. Note that $\mathrm{types}\, a$ is lower and directed.

Let T be a reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives and let A be a subset of the set of adjectives of T. One can verify that $\mathrm{types}\, A$ is lower and directed.

We now state the proposition

(19) Let T be reflexive antisymmetric transitive *TA*-structure with l.u.b.'s with structured adjectives and a be an adjective of T. Then $\mathrm{types}\, a$ is empty or $\mathrm{types}\, a$ is an ideal of T.

3. Applicability of Adjectives

Let T be a *TA*-structure, let t be an element of T, and let a be an adjective of T. We say that a is applicable to t if and only if:

(Def. 15) There exists a type t' of T such that $t' \in \mathrm{types}\, a$ and $t' \leqslant t$.

Let T be a *TA*-structure, let t be a type of T, and let A be a subset of the set of adjectives of T. We say that A is applicable to t if and only if:

(Def. 16) There exists a type t' of T such that $A \subseteq \mathrm{adjs}\, t'$ and $t' \leqslant t$.

We now state the proposition

(20) Let T be a reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, a be an adjective of T, and t be a type of T. If a is applicable to t, then $\mathrm{types}\, a \cap {\downarrow} t$ is an ideal of T.

Let T be a non empty reflexive transitive *TA*-structure, let t be an element of T, and let a be an adjective of T. The functor $a * t$ yielding a type of T is defined by:

(Def. 17) $a * t = \sup(\mathrm{types}\, a \cap {\downarrow} t)$.

The following propositions are true:

(21) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and a be an adjective of T. If a is applicable to t, then $a * t \leqslant t$.

(22) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and a be an adjective of T. If a is applicable to t, then $a \in \mathrm{adjs}(a * t)$.

(23) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and a be an adjective of T. If a is applicable to t, then $a * t \in \mathrm{types}\, a$.

(24) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, a be an adjective of T, and t' be a type of T. If $t' \leqslant t$ and $a \in \mathrm{adjs}\, t'$, then a is applicable to t and $t' \leqslant a * t$.

(25) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and a be an adjective of T. If $a \in \mathrm{adjs}\, t$, then a is applicable to t and $a * t = t$.

(26) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and a, b be adjectives of T. Suppose a is applicable to t and b is applicable to $a * t$. Then b is applicable to t and a is applicable to $b*t$ and $a*(b*t) = b*(a*t)$.

(27) Let T be a reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, A be a subset of the set of adjectives of T, and t be a type of T. If A is applicable to t, then $\mathrm{types}\, A \cap {\downarrow}t$ is an ideal of T.

Let T be a non empty reflexive transitive *TA*-structure, let t be a type of T, and let A be a subset of the set of adjectives of T. The functor $A * t$ yielding a type of T is defined as follows:

(Def. 18) $A * t = \sup(\mathrm{types}\, A \cap {\downarrow}t)$.

Next we state the proposition

(28) Let T be a non empty reflexive transitive antisymmetric *TA*-structure and t be a type of T. Then $\emptyset_{\text{the set of adjectives of } T} * t = t$.

Let T be a non empty non void reflexive transitive *TA*-structure, let t be a type of T, and let p be a finite sequence of elements of the set of adjectives of T. The functor $\mathrm{apply}(p, t)$ yielding a finite sequence of elements of the carrier of T is defined by the conditions (Def. 19).

(Def. 19)(i) $\mathrm{len}\, \mathrm{apply}(p, t) = \mathrm{len}\, p + 1$,

(ii) $(\mathrm{apply}(p, t))(1) = t$, and

(iii) for every natural number i and for every adjective a of T and for every type t of T such that $i \in \mathrm{dom}\, p$ and $a = p(i)$ and $t = (\mathrm{apply}(p, t))(i)$ holds $(\mathrm{apply}(p, t))(i + 1) = a * t$.

Let T be a non empty non void reflexive transitive *TA*-structure, let t be a type of T, and let p be a finite sequence of elements of the set of adjectives of T. Note that $\mathrm{apply}(p, t)$ is non empty.

One can prove the following two propositions:

(29) Let T be a non empty non void reflexive transitive *TA*-structure and t be a type of T. Then $\mathrm{apply}(\varepsilon_{(\text{the set of adjectives of } T)}, t) = \langle t \rangle$.

(30) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and a be an adjective of T. Then $\mathrm{apply}(\langle a \rangle, t) = \langle t, a * t \rangle$.

Let T be a non empty non void reflexive transitive *TA*-structure, let t be a type of T, and let v be a finite sequence of elements of the set of adjectives of T. The functor $v * t$ yielding a type of T is defined by:

(Def. 20) $v * t = (\mathrm{apply}(v, t))(\mathrm{len}\, v + 1)$.

The following propositions are true:

(31) Let T be a non empty non void reflexive transitive *TA*-structure and t be a type of T. Then $\varepsilon_{(\text{the set of adjectives of } T)} * t = t$.

(32) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and a be an adjective of T. Then $\langle a \rangle * t = a * t$.

(33) For all finite sequences p, q and for every natural number i such that $i \geqslant 1$ and $i < \mathrm{len}\, p$ holds $(p \mathbin{{}^{\$\frown}} q)(i) = p(i)$.

(34) Let p be a non empty finite sequence, q be a finite sequence, and i be a natural number. If $i < \mathrm{len}\, q$, then $(p \mathbin{{}^{\$\frown}} q)(\mathrm{len}\, p + i) = q(i + 1)$.

(35) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Then $\mathrm{apply}(v_1 \frown v_2, t) = (\mathrm{apply}(v_1, t)) \mathbin{{}^{\$\frown}} \mathrm{apply}(v_2, v_1 * t)$.

(36) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, v_1, v_2 be finite sequences of elements of the set of adjectives of T, and i be a natural number. If $i \in \mathrm{dom}\, v_1$, then $(\mathrm{apply}(v_1 \frown v_2, t))(i) = (\mathrm{apply}(v_1, t))(i)$.

(37) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Then $(\mathrm{apply}(v_1 \frown v_2, t))(\mathrm{len}\, v_1 + 1) = v_1 * t$.

(38) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Then $v_2 * (v_1 * t) = (v_1 \frown v_2) * t$.

Let T be a non empty non void reflexive transitive *TA*-structure, let t be a type of T, and let v be a finite sequence of elements of the set of adjectives of T. We say that v is applicable to t if and only if the condition (Def. 21) is satisfied.

(Def. 21) Let i be a natural number, a be an adjective of T, and s be a type of T. If $i \in \mathrm{dom}\, v$ and $a = v(i)$ and $s = (\mathrm{apply}(v, t))(i)$, then a is applicable to s.

Next we state a number of propositions:

(39) Let T be a non empty non void reflexive transitive *TA*-structure and t be a type of T. Then $\varepsilon_{(\text{the set of adjectives of } T)}$ is applicable to t.

(40) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and a be an adjective of T. Then a is applicable to t if and only if $\langle a \rangle$ is applicable to t.

(41) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Suppose $v_1 \frown v_2$ is applicable to t. Then v_1 is applicable to t and v_2 is applicable to $v_1 * t$.

(42) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. Suppose v is applicable to t. Let i_1, i_2 be natural numbers. Suppose $1 \leqslant i_1$ and $i_1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} v + 1$. Let t_1, t_2 be types of T. If $t_1 = (\operatorname{apply}(v, t))(i_1)$ and $t_2 = (\operatorname{apply}(v, t))(i_2)$, then $t_2 \leqslant t_1$.

(43) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. Suppose v is applicable to t. Let s be a type of T. If $s \in \operatorname{rng} \operatorname{apply}(v, t)$, then $v * t \leqslant s$ and $s \leqslant t$.

(44) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. If v is applicable to t, then $v * t \leqslant t$.

(45) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. If v is applicable to t, then $\operatorname{rng} v \subseteq \operatorname{adjs}(v * t)$.

(46) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. Suppose v is applicable to t. Let A be a subset of the set of adjectives of T. If $A = \operatorname{rng} v$, then A is applicable to t.

(47) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Suppose v_1 is applicable to t and $\operatorname{rng} v_2 \subseteq \operatorname{rng} v_1$. Let s be a type of T. If $s \in \operatorname{rng} \operatorname{apply}(v_2, t)$, then $v_1 * t \leqslant s$.

(48) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-

structure with l.u.b.'s with structured adjectives, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. If $v_1 \frown v_2$ is applicable to t, then $v_2 \frown v_1$ is applicable to t.

(49) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. If $v_1 \frown v_2$ is applicable to t, then $(v_1 \frown v_2) * t = (v_2 \frown v_1) * t$.

(50) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. If A is applicable to t, then $A * t \leqslant t$.

(51) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. If A is applicable to t, then $A \subseteq \mathrm{adjs}(A * t)$.

(52) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. If A is applicable to t, then $A * t \in \mathrm{types}\, A$.

(53) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, A be a subset of the set of adjectives of T, and t' be a type of T. If $t' \leqslant t$ and $A \subseteq \mathrm{adjs}\, t'$, then A is applicable to t and $t' \leqslant A * t$.

(54) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. If $A \subseteq \mathrm{adjs}\, t$, then A is applicable to t and $A * t = t$.

(55) Let T be a *TA*-structure, t be a type of T, and A, B be subsets of the set of adjectives of T. If A is applicable to t and $B \subseteq A$, then B is applicable to t.

(56) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, a be an adjective of T, and A, B be subsets of the set of adjectives of T. If $B = A \cup \{a\}$ and B is applicable to t, then $a * (A * t) = B * t$.

(57) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. Suppose v is applicable to t. Let A be a subset of the set of adjectives of T. If $A = \mathrm{rng}\, v$, then $v * t = A * t$.

4. Subject Function

Let T be a non empty non void *TA*-structure. The functor $\operatorname{sub} T$ yields a function from the set of adjectives of T into the carrier of T and is defined as follows:

(Def. 22) For every adjective a of T holds $(\operatorname{sub} T)(a) = \sup(\operatorname{types} a \cup \operatorname{types} \operatorname{non} a)$.

We introduce *TAS*-structures which are extensions of *TA*-structure and are systems

⟨ a carrier, a set of adjectives, an internal relation, an operation non, an adjective map, a subject map ⟩,

where the carrier and the set of adjectives are sets, the internal relation is a binary relation on the carrier, the operation non is a unary operation on the set of adjectives, the adjective map is a function from the carrier into Fin the set of adjectives, and the subject map is a function from the set of adjectives into the carrier.

Let us observe that there exists a *TAS*-structure which is non void, reflexive, trivial, non empty, and strict.

Let T be a non empty non void *TAS*-structure and let a be an adjective of T. The functor $\operatorname{sub} a$ yields a type of T and is defined as follows:

(Def. 23) $\operatorname{sub} a =$ (the subject map of T)(a).

Let T be a non empty non void *TAS*-structure. We say that T is absorbing non if and only if:

(Def. 24) (The subject map of T) $\cdot$ (the operation non of T) = the subject map of T.

We say that T is subjected if and only if:

(Def. 25) For every adjective a of T holds $\operatorname{types} a \cup \operatorname{types} \operatorname{non} a \leqslant \operatorname{sub} a$ and if $\operatorname{types} a \neq \emptyset$ and $\operatorname{types} \operatorname{non} a \neq \emptyset$, then $\operatorname{sub} a = \sup(\operatorname{types} a \cup \operatorname{types} \operatorname{non} a)$.

Let T be a non empty non void *TAS*-structure. Let us observe that T is absorbing non if and only if:

(Def. 26) For every adjective a of T holds $\operatorname{sub} \operatorname{non} a = \operatorname{sub} a$.

Let T be a non empty non void *TAS*-structure, let t be an element of T, and let a be an adjective of T. We say that a is properly applicable to t if and only if:

(Def. 27) $t \leqslant \operatorname{sub} a$ and a is applicable to t.

Let T be a non empty non void reflexive transitive *TAS*-structure, let t be a type of T, and let v be a finite sequence of elements of the set of adjectives of T. We say that v is properly applicable to t if and only if the condition (Def. 28) is satisfied.

(Def. 28) Let i be a natural number, a be an adjective of T, and s be a type of T. If $i \in \operatorname{dom} v$ and $a = v(i)$ and $s = (\operatorname{apply}(v, t))(i)$, then a is properly applicable to s.

One can prove the following propositions:

(58) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. If v is properly applicable to t, then v is applicable to t.

(59) Let T be a non empty non void reflexive transitive *TAS*-structure and t be a type of T. Then $\varepsilon_{(\text{the set of adjectives of } T)}$ is properly applicable to t.

(60) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and a be an adjective of T. Then a is properly applicable to t if and only if $\langle a \rangle$ is properly applicable to t.

(61) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Suppose $v_1 \frown v_2$ is properly applicable to t. Then v_1 is properly applicable to t and v_2 is properly applicable to $v_1 * t$.

(62) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Suppose v_1 is properly applicable to t and v_2 is properly applicable to $v_1 * t$. Then $v_1 \frown v_2$ is properly applicable to t.

Let T be a non empty non void reflexive transitive *TAS*-structure, let t be a type of T, and let A be a subset of the set of adjectives of T. We say that A is properly applicable to t if and only if the condition (Def. 29) is satisfied.

(Def. 29) There exists a finite sequence s of elements of the set of adjectives of T such that $\operatorname{rng} s = A$ and s is properly applicable to t.

Next we state two propositions:

(63) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and A be a subset of the set of adjectives of T. If A is properly applicable to t, then A is finite.

(64) Let T be a non empty non void reflexive transitive *TAS*-structure and t be a type of T. Then $\emptyset_{\text{the set of adjectives of } T}$ is properly applicable to t.

The scheme *MinimalFiniteSet* concerns a unary predicate $\mathcal{P}$, and states that:

There exists a finite set A such that $\mathcal{P}[A]$ and for every set B such that $B \subseteq A$ and $\mathcal{P}[B]$ holds $B = A$

provided the following requirement is met:

- There exists a finite set A such that $\mathcal{P}[A]$.

One can prove the following proposition

(65) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and A be a subset of the set of adjectives of T. Suppose A is properly applicable to t. Then there exists a subset B of the set of adjectives of T such that

(i) $B \subseteq A$,

(ii) B is properly applicable to t,

(iii) $A * t = B * t$, and

(iv) for every subset C of the set of adjectives of T such that $C \subseteq B$ and C is properly applicable to t and $A * t = C * t$ holds $C = B$.

Let T be a non empty non void reflexive transitive *TAS*-structure. We say that T is commutative if and only if the condition (Def. 30) is satisfied.

(Def. 30) Let t_1, t_2 be types of T and a be an adjective of T. Suppose a is properly applicable to t_1 and $a * t_1 \leqslant t_2$. Then there exists a finite subset A of the set of adjectives of T such that A is properly applicable to $t_1 \sqcup t_2$ and $A * (t_1 \sqcup t_2) = t_2$.

Let us observe that there exists a complete upper-bounded non empty non void trivial reflexive transitive antisymmetric strict *TAS*-structure which is Mizar-widening-like, involutive, without fixpoints, consistent, absorbing non, subjected, and commutative and has structured adjectives and typed adjectives.

Next we state the proposition

(66) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. Suppose A is properly applicable to t. Then there exists an one-to-one finite sequence s of elements of the set of adjectives of T such that $\operatorname{rng} s = A$ and s is properly applicable to t.

5. Reduction of Adjectives

Let T be a non empty non void reflexive transitive *TAS*-structure. The functor $\circ\!\!\rightarrow_T$ yields a binary relation on T and is defined by the condition (Def. 31).

(Def. 31) Let t_1, t_2 be types of T. Then $\langle t_1, t_2 \rangle \in \circ\!\!\rightarrow_T$ if and only if there exists an adjective a of T such that $a \notin \operatorname{adjs} t_2$ and a is properly applicable to t_2 and $a * t_2 = t_1$.

Next we state the proposition

(67) Let T be an antisymmetric non void reflexive transitive Noetherian *TAS*-structure with l.u.b.'s with structured adjectives. Then $\circ\!\!\rightarrow_T \subseteq$ the internal relation of T.

The scheme *RedInd* deals with a non empty set $\mathcal{A}$, a binary relation $\mathcal{B}$ on $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

For all elements x, y of $\mathcal{A}$ such that $\mathcal{B}$ reduces x to y holds $\mathcal{P}[x, y]$

provided the parameters have the following properties:

- For all elements x, y of $\mathcal{A}$ such that $\langle x, y \rangle \in \mathcal{B}$ holds $\mathcal{P}[x, y]$,
- For every element x of $\mathcal{A}$ holds $\mathcal{P}[x, x]$, and
- For all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

We now state a number of propositions:

(68) Let T be an antisymmetric non void reflexive transitive Noetherian *TAS*-structure with l.u.b.'s with structured adjectives and t_1, t_2 be types of T. If $\circ\!\!\to_T$ reduces t_1 to t_2, then $t_1 \leqslant t_2$.

(69) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives. Then $\circ\!\!\to_T$ is irreflexive.

(70) Let T be an antisymmetric non void reflexive transitive Noetherian *TAS*-structure with l.u.b.'s with structured adjectives. Then $\circ\!\!\to_T$ is strongly-normalizing.

(71) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a finite subset of the set of adjectives of T. Suppose that for every subset C of the set of adjectives of T such that $C \subseteq A$ and C is properly applicable to t and $A * t = C * t$ holds $C = A$. Let s be an one-to-one finite sequence of elements of the set of adjectives of T. Suppose $\operatorname{rng} s = A$ and s is properly applicable to t. Let i be a natural number. If $1 \leqslant i$ and $i \leqslant \operatorname{len} s$, then $\langle (\operatorname{apply}(s,t))(i+1),\ (\operatorname{apply}(s,t))(i) \rangle \in \circ\!\!\to_T$.

(72) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a finite subset of the set of adjectives of T. Suppose that for every subset C of the set of adjectives of T such that $C \subseteq A$ and C is properly applicable to t and $A * t = C * t$ holds $C = A$. Let s be an one-to-one finite sequence of elements of the set of adjectives of T. Suppose $\operatorname{rng} s = A$ and s is properly applicable to t. Then $\operatorname{Rev}(\operatorname{apply}(s,t))$ is a reduction sequence w.r.t. $\circ\!\!\to_T$.

(73) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a finite subset of the set of adjectives of T. If A is properly applicable to t, then $\circ\!\!\to_T$ reduces $A * t$ to t.

(74) Let X be a non empty set, R be a binary relation on X, and r be a reduction sequence w.r.t. R. If $r(1) \in X$, then r is a finite sequence of elements of X.

(75) Let X be a non empty set, R be a binary relation on X, x be an element of X, and y be a set. If R reduces x to y, then $y \in X$.

(76) Let X be a non empty set and R be a binary relation on X. Suppose R is weakly-normalizing and has unique normal form property. Let x be an element of X. Then $\operatorname{nf}_R(x) \in X$.

(77) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives and t_1, t_2 be types of T. Suppose $\circ\!\!\to_T$ reduces t_1 to t_2. Then there exists a finite subset A of the set

of adjectives of T such that A is properly applicable to t_2 and $t_1 = A * t_2$.

(78) Let T be an antisymmetric commutative non void reflexive transitive Noetherian *TAS*-structure with l.u.b.'s with structured adjectives. Then $\circ\!\!\to_T$ has Church-Rosser property and unique normal form property.

6. Radix Types

Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s and let t be a type of T. The functor $\operatorname{radix} t$ yielding a type of T is defined by:

(Def. 32) $\operatorname{radix} t = \mathrm{nf}_{\circ\!\to_T}(t)$.

We now state several propositions:

(79) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s and t be a type of T. Then $\circ\!\!\to_T$ reduces t to $\operatorname{radix} t$.

(80) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s and t be a type of T. Then $t \leqslant \operatorname{radix} t$.

(81) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s, t be a type of T, and X be a set. Suppose $X = \{t'; t'$ ranges over types of T: $\bigvee_{A:\ \text{finite subset of the set of adjectives of } T} (A$ is properly applicable to $t' \ \wedge\ A * t' = t)\}$. Then $\sup X$ exists in T and $\operatorname{radix} t = \bigsqcup_T X$.

(82) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s, t_1, t_2 be types of T, and a be an adjective of T. If a is properly applicable to t_1 and $a * t_1 \leqslant \operatorname{radix} t_2$, then $t_1 \leqslant \operatorname{radix} t_2$.

(83) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s and t_1, t_2 be types of T. If $t_1 \leqslant t_2$, then $\operatorname{radix} t_1 \leqslant \operatorname{radix} t_2$.

(84) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s, t be a type of T, and a be an adjective of T. If a is properly applicable to t, then $\operatorname{radix}(a * t) = \operatorname{radix} t$.

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Lines in n-Dimensional Euclidean Spaces

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Summary. In this paper, we define the line of n-dimensional Euclidean space and we introduce basic properties of affine space on this space. Next, we define the inner product of elements of this space. At the end, we introduce orthogonality of lines of this space.

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The papers [13], [4], [15], [2], [12], [8], [5], [11], [10], [3], [6], [1], [14], [7], and [9] provide the terminology and notation for this paper.

We adopt the following rules: a, b, l_1 are real numbers, n is a natural number, and x, x_1, x_2, y_1, y_2 are elements of $\mathcal{R}^n$.

Next we state several propositions:

(1) $0 \cdot x + x = x$ and $x + \langle \underbrace{0, \ldots, 0}_{n} \rangle = x$.

(2) $a \cdot \langle \underbrace{0, \ldots, 0}_{n} \rangle = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(3) $1 \cdot x = x$ and $0 \cdot x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(4) $(a \cdot b) \cdot x = a \cdot (b \cdot x)$.

(5) If $a \cdot x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$, then $a = 0$ or $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(6) $a \cdot (x_1 + x_2) = a \cdot x_1 + a \cdot x_2$.

(7) $(a + b) \cdot x = a \cdot x + b \cdot x$.

(8) If $a \cdot x_1 = a \cdot x_2$, then $a = 0$ or $x_1 = x_2$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. The functor $\mathrm{Line}(x_1, x_2)$ yields a subset of $\mathcal{R}^n$ and is defined by:

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(Def. 1) $\mathrm{Line}(x_1, x_2) = \{(1 - l_1) \cdot x_1 + l_1 \cdot x_2\}$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. Observe that $\mathrm{Line}(x_1, x_2)$ is non empty.

The following proposition is true

(9) $\mathrm{Line}(x_1, x_2) = \mathrm{Line}(x_2, x_1)$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. Let us observe that the functor $\mathrm{Line}(x_1, x_2)$ is commutative.

One can prove the following propositions:

(10) $x_1 \in \mathrm{Line}(x_1, x_2)$ and $x_2 \in \mathrm{Line}(x_1, x_2)$.

(11) If $y_1 \in \mathrm{Line}(x_1, x_2)$ and $y_2 \in \mathrm{Line}(x_1, x_2)$, then $\mathrm{Line}(y_1, y_2) \subseteq \mathrm{Line}(x_1, x_2)$.

(12) If $y_1 \in \mathrm{Line}(x_1, x_2)$ and $y_2 \in \mathrm{Line}(x_1, x_2)$ and $y_1 \neq y_2$, then $\mathrm{Line}(x_1, x_2) \subseteq \mathrm{Line}(y_1, y_2)$.

Let us consider n and let A be a subset of $\mathcal{R}^n$. We say that A is line if and only if:

(Def. 2) There exist x_1, x_2 such that $x_1 \neq x_2$ and $A = \mathrm{Line}(x_1, x_2)$.

We introduce A is a line as a synonym of A is line.

Next we state three propositions:

(13) Let A, C be subsets of $\mathcal{R}^n$ and given x_1, x_2. Suppose A is a line and C is a line and $x_1 \in A$ and $x_2 \in A$ and $x_1 \in C$ and $x_2 \in C$. Then $x_1 = x_2$ or $A = C$.

(14) For every subset A of $\mathcal{R}^n$ such that A is a line there exist x_1, x_2 such that $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$.

(15) For every subset A of $\mathcal{R}^n$ such that A is a line there exists x_2 such that $x_1 \neq x_2$ and $x_2 \in A$.

Let us consider n and let x be an element of $\mathcal{R}^n$. The functor $\mathrm{Rn2Fin}(x)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by:

(Def. 3) $\mathrm{Rn2Fin}(x) = x$.

Let us consider n and let x be an element of $\mathcal{R}^n$. The functor $|x|$ yields a real number and is defined as follows:

(Def. 4) $|x| = |\mathrm{Rn2Fin}(x)|$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. The functor $|(x_1, x_2)|$ yielding a real number is defined by:

(Def. 5) $|(x_1, x_2)| = |(\mathrm{Rn2Fin}(x_1), \mathrm{Rn2Fin}(x_2))|$.

Let us observe that the functor $|(x_1, x_2)|$ is commutative.

We now state a number of propositions:

(16) For all elements x_1, x_2 of $\mathcal{R}^n$ holds $|(x_1, x_2)| = \frac{1}{4} \cdot (|x_1 + x_2|^{\mathbf{2}} - |x_1 - x_2|^{\mathbf{2}})$.

(17) For every element x of $\mathcal{R}^n$ holds $|(x, x)| \geqslant 0$.

(18) For every element x of $\mathcal{R}^n$ holds $|x|^{\mathbf{2}} = |(x, x)|$.

(19) For every element x of $\mathcal{R}^n$ holds $0 \leqslant |x|$.

(20) For every element x of $\mathcal{R}^n$ holds $|x| = \sqrt{|(x, x)|}$.

(21) For every element x of $\mathcal{R}^n$ holds $|(x, x)| = 0$ iff $|x| = 0$.

(22) For every element x of $\mathcal{R}^n$ holds $|(x, x)| = 0$ iff $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(23) For every element x of $\mathcal{R}^n$ holds $|(x, \langle \underbrace{0, \ldots, 0}_{n} \rangle)| = 0$.

(24) For every element x of $\mathcal{R}^n$ holds $|(\langle \underbrace{0, \ldots, 0}_{n} \rangle, x)| = 0$.

(25) For all elements x_1, x_2, x_3 of $\mathcal{R}^n$ holds $|(x_1 + x_2, x_3)| = |(x_1, x_3)| + |(x_2, x_3)|$.

(26) For all elements x_1, x_2 of $\mathcal{R}^n$ and for every real number a holds $|(a \cdot x_1, x_2)| = a \cdot |(x_1, x_2)|$.

(27) For all elements x_1, x_2 of $\mathcal{R}^n$ and for every real number a holds $|(x_1, a \cdot x_2)| = a \cdot |(x_1, x_2)|$.

(28) For all elements x_1, x_2 of $\mathcal{R}^n$ holds $|(-x_1, x_2)| = -|(x_1, x_2)|$.

(29) For all elements x_1, x_2 of $\mathcal{R}^n$ holds $|(x_1, -x_2)| = -|(x_1, x_2)|$.

(30) For all elements x_1, x_2 of $\mathcal{R}^n$ holds $|(-x_1, -x_2)| = |(x_1, x_2)|$.

(31) For all elements x_1, x_2, x_3 of $\mathcal{R}^n$ holds $|(x_1 - x_2, x_3)| = |(x_1, x_3)| - |(x_2, x_3)|$.

(32) For all real numbers a, b and for all elements x_1, x_2, x_3 of $\mathcal{R}^n$ holds $|(a \cdot x_1 + b \cdot x_2, x_3)| = a \cdot |(x_1, x_3)| + b \cdot |(x_2, x_3)|$.

(33) For all elements x_1, y_1, y_2 of $\mathcal{R}^n$ holds $|(x_1, y_1 + y_2)| = |(x_1, y_1)| + |(x_1, y_2)|$.

(34) For all elements x_1, y_1, y_2 of $\mathcal{R}^n$ holds $|(x_1, y_1 - y_2)| = |(x_1, y_1)| - |(x_1, y_2)|$.

(35) For all elements x_1, x_2, y_1, y_2 of $\mathcal{R}^n$ holds $|(x_1 + x_2, y_1 + y_2)| = |(x_1, y_1)| + |(x_1, y_2)| + |(x_2, y_1)| + |(x_2, y_2)|$.

(36) For all elements x_1, x_2, y_1, y_2 of $\mathcal{R}^n$ holds $|(x_1 - x_2, y_1 - y_2)| = (|(x_1, y_1)| - |(x_1, y_2)| - |(x_2, y_1)|) + |(x_2, y_2)|$.

(37) For all elements x, y of $\mathcal{R}^n$ holds $|(x + y, x + y)| = |(x, x)| + 2 \cdot |(x, y)| + |(y, y)|$.

(38) For all elements x, y of $\mathcal{R}^n$ holds $|(x - y, x - y)| = (|(x, x)| - 2 \cdot |(x, y)|) + |(y, y)|$.

(39) For all elements x, y of $\mathcal{R}^n$ holds $|x + y|^{\mathbf{2}} = |x|^{\mathbf{2}} + 2 \cdot |(x, y)| + |y|^{\mathbf{2}}$.

(40) For all elements x, y of $\mathcal{R}^n$ holds $|x - y|^{\mathbf{2}} = (|x|^{\mathbf{2}} - 2 \cdot |(x, y)|) + |y|^{\mathbf{2}}$.

(41) For all elements x, y of $\mathcal{R}^n$ holds $|x + y|^{\mathbf{2}} + |x - y|^{\mathbf{2}} = 2 \cdot (|x|^{\mathbf{2}} + |y|^{\mathbf{2}})$.

(42) For all elements x, y of $\mathcal{R}^n$ holds $|x + y|^{\mathbf{2}} - |x - y|^{\mathbf{2}} = 4 \cdot |(x, y)|$.

(43) For all elements x, y of $\mathcal{R}^n$ holds $||(x, y)|| \leqslant |x| \cdot |y|$.

(44) For all elements x, y of $\mathcal{R}^n$ holds $|x+y| \leqslant |x| + |y|$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. We say that x_1, x_2 are orthogonal if and only if:

(Def. 6) $|(x_1, x_2)| = 0$.

Let us note that the predicate x_1, x_2 are orthogonal is symmetric.

We now state the proposition

(45) Let R be a subset of $\mathbb{R}$ and x_1, x_2, y_1 be elements of $\mathcal{R}^n$. Suppose $R = \{|y_1 - x|; x$ ranges over elements of $\mathcal{R}^n$: $x \in \mathrm{Line}(x_1, x_2)\}$. Then there exists an element y_2 of $\mathcal{R}^n$ such that $y_2 \in \mathrm{Line}(x_1, x_2)$ and $|y_1 - y_2| = \inf R$ and $x_1 - x_2$, $y_1 - y_2$ are orthogonal.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{Line}(p_1, p_2)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def. 7) $\mathrm{Line}(p_1, p_2) = \{(1 - l_1) \cdot p_1 + l_1 \cdot p_2\}$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Observe that $\mathrm{Line}(p_1, p_2)$ is non empty.

In the sequel p_1, p_2, q_1, q_2 are points of $\mathcal{E}^n_{\mathrm{T}}$.

The following proposition is true

(46) $\mathrm{Line}(p_1, p_2) = \mathrm{Line}(p_2, p_1)$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Let us observe that the functor $\mathrm{Line}(p_1, p_2)$ is commutative.

One can prove the following three propositions:

(47) $p_1 \in \mathrm{Line}(p_1, p_2)$ and $p_2 \in \mathrm{Line}(p_1, p_2)$.

(48) If $q_1 \in \mathrm{Line}(p_1, p_2)$ and $q_2 \in \mathrm{Line}(p_1, p_2)$, then $\mathrm{Line}(q_1, q_2) \subseteq \mathrm{Line}(p_1, p_2)$.

(49) If $q_1 \in \mathrm{Line}(p_1, p_2)$ and $q_2 \in \mathrm{Line}(p_1, p_2)$ and $q_1 \neq q_2$, then $\mathrm{Line}(p_1, p_2) \subseteq \mathrm{Line}(q_1, q_2)$.

Let us consider n and let A be a subset of $\mathcal{E}^n_{\mathrm{T}}$. We say that A is line if and only if:

(Def. 8) There exist p_1, p_2 such that $p_1 \neq p_2$ and $A = \mathrm{Line}(p_1, p_2)$.

We introduce A is a line as a synonym of A is line.

We now state three propositions:

(50) For all subsets A, C of $\mathcal{E}^n_{\mathrm{T}}$ such that A is a line and C is a line and $p_1 \in A$ and $p_2 \in A$ and $p_1 \in C$ and $p_2 \in C$ holds $p_1 = p_2$ or $A = C$.

(51) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ such that A is a line there exist p_1, p_2 such that $p_1 \in A$ and $p_2 \in A$ and $p_1 \neq p_2$.

(52) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ such that A is a line there exists p_2 such that $p_1 \neq p_2$ and $p_2 \in A$.

Let us consider n and let p be a point of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{TPn2Rn}(p)$ yields an element of $\mathcal{R}^n$ and is defined as follows:

(Def. 9) $\mathrm{TPn2Rn}(p) = p$.

Let us consider n and let p be a point of $\mathcal{E}^n_{\mathrm{T}}$. The functor $|p|$ yields a real number and is defined as follows:

(Def. 10) $|p| = |\,\mathrm{TPn2Rn}(p)|$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $|(p_1, p_2)|$ yields a real number and is defined as follows:

(Def. 11) $|(p_1, p_2)| = |(\mathrm{TPn2Rn}(p_1), \mathrm{TPn2Rn}(p_2))|$.

Let us observe that the functor $|(p_1, p_2)|$ is commutative.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. We say that p_1, p_2 are orthogonal if and only if:

(Def. 12) $|(p_1, p_2)| = 0$.

Let us note that the predicate p_1, p_2 are orthogonal is symmetric.

Next we state the proposition

(53) Let R be a subset of $\mathbb{R}$ and p_1, p_2, q_1 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $R = \{|q_1 - p|; p$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $p \in \mathrm{Line}(p_1, p_2)\}$. Then there exists a point q_2 of $\mathcal{E}^n_{\mathrm{T}}$ such that $q_2 \in \mathrm{Line}(p_1, p_2)$ and $|q_1 - q_2| = \inf R$ and $p_1 - p_2$, $q_1 - q_2$ are orthogonal.

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Banach Space of Absolute Summable Real Sequences

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Summary. A continuation of [5]. As the example of real norm spaces, we introduce the arithmetic addition and multiplication in the set of absolute summable real sequences and also introduce the norm. This set has the structure of the Banach space.

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The notation and terminology used here are introduced in the following papers: [14], [17], [4], [1], [13], [7], [2], [3], [18], [16], [10], [15], [11], [9], [8], [12], and [6].

1. The Space of Absolute Summable Real Sequences

The subset the set of l1-real sequences of the linear space of real sequences is defined by the condition (Def. 1).

(Def. 1) Let x be a set. Then $x \in$ the set of l1-real sequences if and only if $x \in$ the set of real sequences and $\mathrm{id}_{\mathrm{seq}}(x)$ is absolutely summable.

Let us observe that the set of l1-real sequences is non empty.

One can prove the following two propositions:

(1) The set of l1-real sequences is linearly closed.

(2) $\langle$the set of l1-real sequences, Zero_(the set of l1-real sequences, the linear space of real sequences), Add_(the set of l1-real sequences, the linear space

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of real sequences), Mult_(the set of l1-real sequences, the linear space of real sequences)⟩ is a subspace of the linear space of real sequences.

One can check that ⟨the set of l1-real sequences, Zero_(the set of l1-real sequences, the linear space of real sequences), Add_(the set of l1-real sequences, the linear space of real sequences), Mult_(the set of l1-real sequences, the linear space of real sequences)⟩ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

One can prove the following proposition

(3) ⟨the set of l1-real sequences, Zero_(the set of l1-real sequences, the linear space of real sequences), Add_(the set of l1-real sequences, the linear space of real sequences), Mult_(the set of l1-real sequences, the linear space of real sequences)⟩ is a real linear space.

The function $\mathrm{norm}_{\mathrm{seq}}$ from the set of l1-real sequences into $\mathbb{R}$ is defined by:

(Def. 2) For every set x such that $x \in$ the set of l1-real sequences holds $\mathrm{norm}_{\mathrm{seq}}(x) = \sum|\mathrm{id}_{\mathrm{seq}}(x)|$.

Let X be a non empty set, let Z be an element of X, let A be a binary operation on X, let M be a function from $[:\mathbb{R}, X:]$ into X, and let N be a function from X into $\mathbb{R}$. One can check that $\langle X, Z, A, M, N\rangle$ is non empty.

Next we state four propositions:

(4) Let l be a normed structure. Suppose ⟨the carrier of l, the zero of l, the addition of l, the external multiplication of l⟩ is a real linear space. Then l is a real linear space.

(5) Let r_1 be a sequence of real numbers. Suppose that for every natural number n holds $r_1(n) = 0$. Then r_1 is absolutely summable and $\sum|r_1| = 0$.

(6) Let r_1 be a sequence of real numbers. Suppose r_1 is absolutely summable and $\sum|r_1| = 0$. Let n be a natural number. Then $r_1(n) = 0$.

(7) ⟨the set of l1-real sequences, Zero_(the set of l1-real sequences, the linear space of real sequences), Add_(the set of l1-real sequences, the linear space of real sequences), Mult_(the set of l1-real sequences, the linear space of real sequences), $\mathrm{norm}_{\mathrm{seq}}$⟩ is a real linear space.

The non empty normed structure l1-Space is defined by the condition (Def. 3).

(Def. 3) l1-Space = ⟨the set of l1-real sequences, Zero_(the set of l1-real sequences, the linear space of real sequences), Add_(the set of l1-real sequences, the linear space of real sequences), Mult_(the set of l1-real sequences, the linear space of real sequences), $\mathrm{norm}_{\mathrm{seq}}$⟩.

2. The Space is Banach Space

One can prove the following two propositions:

(8) The carrier of l1-Space = the set of l1-real sequences and for every set x holds x is an element of l1-Space iff x is a sequence of real numbers and $\mathrm{id_{seq}}(x)$ is absolutely summable and for every set x holds x is a vector of l1-Space iff x is a sequence of real numbers and $\mathrm{id_{seq}}(x)$ is absolutely summable and $0_{\text{l1-Space}} = \text{Zeroseq}$ and for every vector u of l1-Space holds $u = \mathrm{id_{seq}}(u)$ and for all vectors u, v of l1-Space holds $u + v = \mathrm{id_{seq}}(u) + \mathrm{id_{seq}}(v)$ and for every real number r and for every vector u of l1-Space holds $r \cdot u = r\ \mathrm{id_{seq}}(u)$ and for every vector u of l1-Space holds $-u = -\mathrm{id_{seq}}(u)$ and $\mathrm{id_{seq}}(-u) = -\mathrm{id_{seq}}(u)$ and for all vectors u, v of l1-Space holds $u - v = \mathrm{id_{seq}}(u) - \mathrm{id_{seq}}(v)$ and for every vector v of l1-Space holds $\mathrm{id_{seq}}(v)$ is absolutely summable and for every vector v of l1-Space holds $\|v\| = \sum|\mathrm{id_{seq}}(v)|$.

(9) Let x, y be points of l1-Space and a be a real number. Then $\|x\| = 0$ iff $x = 0_{\text{l1-Space}}$ and $0 \leqslant \|x\|$ and $\|x + y\| \leqslant \|x\| + \|y\|$ and $\|a \cdot x\| = |a| \cdot \|x\|$.

Let us observe that l1-Space is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

Let X be a non empty normed structure and let x, y be points of X. The functor $\rho(x, y)$ yields a real number and is defined by:

(Def. 4) $\rho(x, y) = \|x - y\|$.

Let N_1 be a non empty normed structure and let s_1 be a sequence of N_1. We say that s_1 is CCauchy if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let r_2 be a real number. Suppose $r_2 > 0$. Then there exists a natural number k_1 such that for all natural numbers n_1, m_1 if $n_1 \geqslant k_1$ and $m_1 \geqslant k_1$, then $\rho(s_1(n_1), s_1(m_1)) < r_2$.

We introduce s_1 is Cauchy sequence by norm as a synonym of s_1 is CCauchy.

In the sequel N_1 denotes a non empty real normed space and s_2 denotes a sequence of N_1.

We now state two propositions:

(10) s_2 is Cauchy sequence by norm if and only if for every real number r such that $r > 0$ there exists a natural number k such that for all natural numbers n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\|s_2(n) - s_2(m)\| < r$.

(11) For every sequence v_1 of l1-Space such that v_1 is Cauchy sequence by norm holds v_1 is convergent.

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Cross Products and Tripple Vector Products in 3-dimensional Euclidean Space

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Summary. First, we extend the basic theorems of 3-dimensional Euclidean space, and then define the cross product in the same space and relative vector relations using the above definition.

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The articles [14], [2], [12], [9], [6], [4], [3], [5], [13], [10], [11], [7], [8], and [1] provide the terminology and notation for this paper.

We adopt the following convention: x, y, z denote real numbers, x_3, y_3 denote elements of $\mathbb{R}$, and p denotes a point of $\mathcal{E}^3_{\mathrm{T}}$.

We now state the proposition

(1) There exist x, y, z such that $p = \langle x, y, z \rangle$.

Let us consider p. The functor $p_{\mathbf{1}}$ yielding a real number is defined as follows:

(Def. 1) For every finite sequence f such that $p = f$ holds $p_{\mathbf{1}} = f(1)$.

The functor $p_{\mathbf{2}}$ yields a real number and is defined by:

(Def. 2) For every finite sequence f such that $p = f$ holds $p_{\mathbf{2}} = f(2)$.

The functor $p_{\mathbf{3}}$ yields a real number and is defined by:

(Def. 3) For every finite sequence f such that $p = f$ holds $p_{\mathbf{3}} = f(3)$.

Let us consider x, y, z. The functor $[x, y, z]$ yields a point of $\mathcal{E}^3_{\mathrm{T}}$ and is defined as follows:

(Def. 4) $[x, y, z] = \langle x, y, z \rangle$.

One can prove the following three propositions:

(2) $[x, y, z]_{\mathbf{1}} = x$ and $[x, y, z]_{\mathbf{2}} = y$ and $[x, y, z]_{\mathbf{3}} = z$.

(3) $p = [p_{\mathbf{1}}, p_{\mathbf{2}}, p_{\mathbf{3}}]$.

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(4) $0_{\mathcal{E}^3_{\rm T}} = [0,0,0]$.

We adopt the following rules: p_1, p_2, p_3, p_4 are points of $\mathcal{E}^3_{\rm T}$ and x_1, x_2, y_1, y_2, z_1, z_2 are real numbers.

Next we state several propositions:

(5) $p_1 + p_2 = [(p_1)_{\bf 1} + (p_2)_{\bf 1}, (p_1)_{\bf 2} + (p_2)_{\bf 2}, (p_1)_{\bf 3} + (p_2)_{\bf 3}]$.

(6) $[x_1, y_1, z_1] + [x_2, y_2, z_2] = [x_1 + x_2, y_1 + y_2, z_1 + z_2]$.

(7) $x \cdot p = [x \cdot p_{\bf 1}, x \cdot p_{\bf 2}, x \cdot p_{\bf 3}]$.

(8) $x \cdot [x_1, y_1, z_1] = [x \cdot x_1, x \cdot y_1, x \cdot z_1]$.

(9) $(x \cdot p)_{\bf 1} = x \cdot p_{\bf 1}$ and $(x \cdot p)_{\bf 2} = x \cdot p_{\bf 2}$ and $(x \cdot p)_{\bf 3} = x \cdot p_{\bf 3}$.

(10) $-p = [-p_{\bf 1}, -p_{\bf 2}, -p_{\bf 3}]$.

(11) $-[x_1, y_1, z_1] = [-x_1, -y_1, -z_1]$.

(12) $p_1 - p_2 = [(p_1)_{\bf 1} - (p_2)_{\bf 1}, (p_1)_{\bf 2} - (p_2)_{\bf 2}, (p_1)_{\bf 3} - (p_2)_{\bf 3}]$.

(13) $[x_1, y_1, z_1] - [x_2, y_2, z_2] = [x_1 - x_2, y_1 - y_2, z_1 - z_2]$.

Let us consider p_1, p_2. The functor $p_1 \times p_2$ yielding a point of $\mathcal{E}^3_{\rm T}$ is defined by:

(Def. 5) $p_1 \times p_2 = [(p_1)_{\bf 2} \cdot (p_2)_{\bf 3} - (p_1)_{\bf 3} \cdot (p_2)_{\bf 2}, (p_1)_{\bf 3} \cdot (p_2)_{\bf 1} - (p_1)_{\bf 1} \cdot (p_2)_{\bf 3}, (p_1)_{\bf 1} \cdot (p_2)_{\bf 2} - (p_1)_{\bf 2} \cdot (p_2)_{\bf 1}]$.

The following propositions are true:

(14) If $p = [x, y, z]$, then $p_{\bf 1} = x$ and $p_{\bf 2} = y$ and $p_{\bf 3} = z$.

(15) $[x_1, y_1, z_1] \times [x_2, y_2, z_2] = [y_1 \cdot z_2 - z_1 \cdot y_2, z_1 \cdot x_2 - x_1 \cdot z_2, x_1 \cdot y_2 - y_1 \cdot x_2]$.

(16) $(x \cdot p_1) \times p_2 = x \cdot (p_1 \times p_2)$ and $(x \cdot p_1) \times p_2 = p_1 \times (x \cdot p_2)$.

(17) $p_1 \times p_2 = -p_2 \times p_1$.

(18) $(-p_1) \times p_2 = p_1 \times -p_2$.

(19) $[0,0,0] \times [x, y, z] = 0_{\mathcal{E}^3_{\rm T}}$.

(20) $[x_1, 0, 0] \times [x_2, 0, 0] = 0_{\mathcal{E}^3_{\rm T}}$.

(21) $[0, y_1, 0] \times [0, y_2, 0] = 0_{\mathcal{E}^3_{\rm T}}$.

(22) $[0, 0, z_1] \times [0, 0, z_2] = 0_{\mathcal{E}^3_{\rm T}}$.

(23) $p_1 \times (p_2 + p_3) = p_1 \times p_2 + p_1 \times p_3$.

(24) $(p_1 + p_2) \times p_3 = p_1 \times p_3 + p_2 \times p_3$.

(25) $p_1 \times p_1 = 0_{\mathcal{E}^3_{\rm T}}$.

(26) $(p_1 + p_2) \times (p_3 + p_4) = p_1 \times p_3 + p_1 \times p_4 + p_2 \times p_3 + p_2 \times p_4$.

(27) $p = \langle p_{\bf 1}, p_{\bf 2}, p_{\bf 3} \rangle$.

(28) For all finite sequences f_1, f_2 of elements of $\mathbb{R}$ such that $\operatorname{len} f_1 = 3$ and $\operatorname{len} f_2 = 3$ holds $f_1 \bullet f_2 = \langle f_1(1) \cdot f_2(1), f_1(2) \cdot f_2(2), f_1(3) \cdot f_2(3) \rangle$.

(29) $|(p_1, p_2)| = (p_1)_{\bf 1} \cdot (p_2)_{\bf 1} + (p_1)_{\bf 2} \cdot (p_2)_{\bf 2} + (p_1)_{\bf 3} \cdot (p_2)_{\bf 3}$.

(30) $|([x_1, x_2, x_3], [y_1, y_2, y_3])| = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$.

Let us consider p_1, p_2, p_3. The functor $\langle|p_1, p_2, p_3|\rangle$ yielding a real number is defined as follows:

(Def. 6) $\langle|p_1, p_2, p_3|\rangle = |(p_1, p_2 \times p_3)|$.

The following propositions are true:

(31) $\langle|p_1, p_1, p_2|\rangle = 0$ and $\langle|p_2, p_1, p_2|\rangle = 0$.

(32) $p_1 \times (p_2 \times p_3) = |(p_1, p_3)| \cdot p_2 - |(p_1, p_2)| \cdot p_3$.

(33) $\langle|p_1, p_2, p_3|\rangle = \langle|p_2, p_3, p_1|\rangle$.

(34) $\langle|p_1, p_2, p_3|\rangle = \langle|p_3, p_1, p_2|\rangle$.

(35) $\langle|p_1, p_2, p_3|\rangle = |(p_1 \times p_2, p_3)|$.

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Calculation of Matrices of Field Elements. Part I

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Summary. This article gives property of calculation of matrices.

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The articles [8], [3], [10], [11], [4], [1], [5], [2], [13], [6], [7], [12], and [9] provide the notation and terminology for this paper.

In this paper i denotes a natural number.

Let K be a field and let M_1, M_2 be matrices over K. The functor $M_1 - M_2$ yielding a matrix over K is defined by:

(Def. 1) $M_1 - M_2 = M_1 + -M_2$.

One can prove the following propositions:

(1) For every field K and for every matrix M over K such that $\operatorname{len} M > 0$ holds $--M = M$.

(2) For every field K and for every matrix M over K such that $\operatorname{len} M > 0$ holds $M + -M = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} M)\times(\operatorname{width} M)}$.

(3) For every field K and for every matrix M over K such that $\operatorname{len} M > 0$ holds $M - M = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} M)\times(\operatorname{width} M)}$.

(4) Let K be a field and M_1, M_2, M_3 be matrices over K. Suppose $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$ and $M_1 + M_3 = M_2 + M_3$. Then $M_1 = M_2$.

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(5) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{len} M_2 > 0$ holds $M_1 - -M_2 = M_1 + M_2$.

(6) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ and $M_1 = M_1 + M_2$ holds $M_2 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)}$.

(7) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ and $M_1 - M_2 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)}$ holds $M_1 = M_2$.

(8) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ and $M_1 + M_2 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)}$ holds $M_2 = -M_1$.

(9) For all natural numbers n, m and for every field K such that $n > 0$ holds $-\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n\times m} = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n\times m}$.

(10) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ and $M_2 - M_1 = M_2$ holds $M_1 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)}$.

(11) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ holds $M_1 = M_1 - (M_2 - M_2)$.

(12) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ holds $-(M_1 + M_2) = -M_1 + -M_2$.

(13) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ holds $M_1 - (M_1 - M_2) = M_2$.

(14) Let K be a field and M_1, M_2, M_3 be matrices over K. Suppose $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$ and $M_1 - M_3 = M_2 - M_3$. Then $M_1 = M_2$.

(15) Let K be a field and M_1, M_2, M_3 be matrices over K. Suppose $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$ and $M_3 - M_1 = M_3 - M_2$. Then $M_1 = M_2$.

(16) Let K be a field and M_1, M_2, M_3 be matrices over K. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$, then $M_1 - M_2 - M_3 = M_1 - M_3 - M_2$.

(17) Let K be a field and M_1, M_2, M_3 be matrices over K. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$, then $M_1 - M_3 = M_1 - M_2 - (M_3 - M_2)$.

(18) Let K be a field and M_1, M_2, M_3 be matrices over K. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$, then $M_3 - M_1 - (M_3 - M_2) = M_2 - M_1$.

(19) Let K be a field and M_1, M_2, M_3, M_4 be matrices over K. Suppose $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{len} M_3 = \operatorname{len} M_4$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{width} M_3 = \operatorname{width} M_4$ and $\operatorname{len} M_1 > 0$ and $M_1 - M_2 = M_3 - M_4$. Then $M_1 - M_3 = M_2 - M_4$.

(20) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ holds $M_1 = M_1 + (M_2 - M_2)$.

(21) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ holds $M_1 = (M_1 + M_2) - M_2$.

(22) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ holds $M_1 = (M_1 - M_2) + M_2$.

(23) Let K be a field and M_1, M_2, M_3 be matrices over K. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$, then $M_1 + M_3 = M_1 + M_2 + (M_3 - M_2)$.

(24) Let K be a field and M_1, M_2, M_3 be matrices over K. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$, then $(M_1 + M_2) - M_3 = (M_1 - M_3) + M_2$.

(25) Let K be a field and M_1, M_2, M_3 be matrices over K. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$, then $(M_1 - M_2) + M_3 = (M_3 - M_2) + M_1$.

(26) Let K be a field and M_1, M_2, M_3 be matrices over K. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$, then $M_1 + M_3 = (M_1 + M_2) - (M_2 - M_3)$.

(27) Let K be a field and M_1, M_2, M_3 be matrices over K. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$

and $\operatorname{len} M_1 > 0$, then $M_1 - M_3 = (M_1 + M_2) - (M_3 + M_2)$.

(28) Let K be a field and M_1, M_2, M_3, M_4 be matrices over K. Suppose $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{len} M_3 = \operatorname{len} M_4$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{width} M_3 = \operatorname{width} M_4$ and $\operatorname{len} M_1 > 0$ and $M_1 + M_2 = M_3 + M_4$. Then $M_1 - M_3 = M_4 - M_2$.

(29) Let K be a field and M_1, M_2, M_3, M_4 be matrices over K. Suppose $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{len} M_3 = \operatorname{len} M_4$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{width} M_3 = \operatorname{width} M_4$ and $\operatorname{len} M_1 > 0$ and $M_1 - M_3 = M_4 - M_2$. Then $M_1 + M_2 = M_3 + M_4$.

(30) Let K be a field and M_1, M_2, M_3, M_4 be matrices over K. Suppose $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{len} M_3 = \operatorname{len} M_4$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{width} M_3 = \operatorname{width} M_4$ and $\operatorname{len} M_1 > 0$ and $M_1 + M_2 = M_3 - M_4$. Then $M_1 + M_4 = M_3 - M_2$.

(31) Let K be a field and M_1, M_2, M_3 be matrices over K. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$, then $M_1 - (M_2 + M_3) = M_1 - M_2 - M_3$.

(32) Let K be a field and M_1, M_2, M_3 be matrices over K. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$, then $M_1 - (M_2 - M_3) = (M_1 - M_2) + M_3$.

(33) Let K be a field and M_1, M_2, M_3 be matrices over K. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$, then $M_1 - (M_2 - M_3) = M_1 + (M_3 - M_2)$.

(34) Let K be a field and M_1, M_2, M_3 be matrices over K. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$, then $M_1 - M_3 = (M_1 - M_2) + (M_2 - M_3)$.

(35) Let K be a field and M_1, M_2, M_3 be matrices over K. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$ and $-M_1 = -M_2$, then $M_1 = M_2$.

(36) For every field K and for every matrix M over K such that $\operatorname{len} M > 0$ and $-M = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{(\operatorname{len} M)\times(\operatorname{width} M)}$ holds $M = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{(\operatorname{len} M)\times(\operatorname{width} M)}$.

(37) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{len} M_1 =$

len M_2 and width M_1 = width M_2 and len $M_1 > 0$ and $M_1 + -M_2 = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} M_1)\times(\operatorname{width} M_1)}$ holds $M_1 = M_2$.

(38) For every field K and for all matrices M_1, M_2 over K such that len M_1 = len M_2 and width M_1 = width M_2 and len $M_1 > 0$ holds $M_1 = M_1 + M_2 + -M_2$.

(39) For every field K and for all matrices M_1, M_2 over K such that len M_1 = len M_2 and width M_1 = width M_2 and len $M_1 > 0$ holds $M_1 = M_1 + (M_2 + -M_2)$.

(40) For every field K and for all matrices M_1, M_2 over K such that len M_1 = len M_2 and width M_1 = width M_2 and len $M_1 > 0$ holds $M_1 = -M_2 + M_1 + M_2$.

(41) For every field K and for all matrices M_1, M_2 over K such that len M_1 = len M_2 and width M_1 = width M_2 and len $M_1 > 0$ holds $-(-M_1 + M_2) = M_1 + -M_2$.

(42) For every field K and for all matrices M_1, M_2 over K such that len M_1 = len M_2 and width M_1 = width M_2 and len $M_1 > 0$ holds $M_1 + M_2 = -(-M_1 + -M_2)$.

(43) For every field K and for all matrices M_1, M_2 over K such that len M_1 = len M_2 and width M_1 = width M_2 and len $M_1 > 0$ holds $-(M_1 - M_2) = M_2 - M_1$.

(44) For every field K and for all matrices M_1, M_2 over K such that len M_1 = len M_2 and width M_1 = width M_2 and len $M_1 > 0$ holds $-M_1 - M_2 = -M_2 - M_1$.

(45) For every field K and for all matrices M_1, M_2 over K such that len M_1 = len M_2 and width M_1 = width M_2 and len $M_1 > 0$ holds $M_1 = -M_2 - (-M_1 - M_2)$.

(46) Let K be a field and M_1, M_2, M_3 be matrices over K. If len M_1 = len M_2 and len M_2 = len M_3 and width M_1 = width M_2 and width M_2 = width M_3 and len $M_1 > 0$, then $-M_1 - M_2 - M_3 = -M_1 - M_3 - M_2$.

(47) Let K be a field and M_1, M_2, M_3 be matrices over K. If len M_1 = len M_2 and len M_2 = len M_3 and width M_1 = width M_2 and width M_2 = width M_3 and len $M_1 > 0$, then $-M_1 - M_2 - M_3 = -M_2 - M_3 - M_1$.

(48) Let K be a field and M_1, M_2, M_3 be matrices over K. If len M_1 = len M_2 and len M_2 = len M_3 and width M_1 = width M_2 and width M_2 = width M_3 and len $M_1 > 0$, then $-M_1 - M_2 - M_3 = -M_3 - M_2 - M_1$.

(49) Let K be a field and M_1, M_2, M_3 be matrices over K. If len M_1 = len M_2 and len M_2 = len M_3 and width M_1 = width M_2 and width M_2 = width M_3 and len $M_1 > 0$, then $M_3 - M_1 - (M_3 - M_2) = -(M_1 - M_2)$.

(50) For every field K and for every matrix M over K such that $\operatorname{len} M > 0$ holds $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{(\operatorname{len} M)\times(\operatorname{width} M)} - M = -M$.

(51) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ holds $M_1 + M_2 = M_1 - -M_2$.

(52) For every field K and for all matrices M_1, M_2 over K such that $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{len} M_1 > 0$ holds $M_1 = M_1 - (M_2 + -M_2)$.

(53) Let K be a field and M_1, M_2, M_3 be matrices over K. Suppose $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$ and $M_1 - M_3 = M_2 + -M_3$. Then $M_1 = M_2$.

(54) Let K be a field and M_1, M_2, M_3 be matrices over K. Suppose $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{len} M_2 = \operatorname{len} M_3$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $\operatorname{width} M_2 = \operatorname{width} M_3$ and $\operatorname{len} M_1 > 0$ and $M_3 - M_1 = M_3 + -M_2$. Then $M_1 = M_2$.

(55) Let K be a field and A, B be matrices over K. If $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$, then the indices of $A =$ the indices of B.

(56) Let K be a field and x, y, z be finite sequences of elements of the carrier of K. If $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$, then $(x + y) \bullet z = x \bullet z + y \bullet z$.

(57) Let K be a field and x, y, z be finite sequences of elements of the carrier of K. If $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$, then $z \bullet (x + y) = z \bullet x + z \bullet y$.

(58) Let D be a non empty set and M be a matrix over D. Suppose $\operatorname{len} M > 0$. Let n be a natural number. Then M is a matrix over D of dimension $n \times \operatorname{width} M$ if and only if $n = \operatorname{len} M$.

(59) Let K be a field, j be a natural number, and A, B be matrices over K. Suppose $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ and there exists a natural number j such that $\langle i, j\rangle \in$ the indices of A. Then $\operatorname{Line}(A + B, i) = \operatorname{Line}(A, i) + \operatorname{Line}(B, i)$.

(60) Let K be a field, j be a natural number, and A, B be matrices over K. Suppose $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ and there exists a natural number i such that $\langle i, j\rangle \in$ the indices of A. Then $(A + B)_{\square,j} = A_{\square,j} + B_{\square,j}$.

(61) Let V_1 be a field and P_1, P_2 be finite sequences of elements of the carrier of V_1. If $\operatorname{len} P_1 = \operatorname{len} P_2$, then $\sum(P_1 + P_2) = \sum P_1 + \sum P_2$.

(62) Let K be a field and A, B, C be matrices over K. If $\operatorname{len} B = \operatorname{len} C$ and $\operatorname{width} B = \operatorname{width} C$ and $\operatorname{width} A = \operatorname{len} B$ and $\operatorname{len} A > 0$ and $\operatorname{len} B > 0$, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

(63) Let K be a field and A, B, C be matrices over K. If $\operatorname{len} B = \operatorname{len} C$ and

width B = width C and len A = width B and len $B > 0$ and len $A > 0$, then $(B + C) \cdot A = B \cdot A + C \cdot A$.

(64) Let K be a field, n, m, k be natural numbers, M_1 be a matrix over K of dimension $n \times m$, and M_2 be a matrix over K of dimension $m \times k$. Suppose width M_1 = len M_2 and $0 <$ len M_1 and $0 <$ len M_2. Then $M_1 \cdot M_2$ is a matrix over K of dimension $n \times k$.

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Lattice of Fuzzy Sets[1]

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Summary. This article concerns a connection of fuzzy logic and lattice theory. Namely, the fuzzy sets form a Heyting lattice with union and intersection of fuzzy sets as meet and join operations. The lattice of fuzzy sets is defined as the product of interval posets. As the final result, we have characterized the composition of fuzzy relations in terms of lattice theory and proved its associativity.

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The notation and terminology used in this paper are introduced in the following articles: [18], [9], [23], [6], [7], [17], [1], [8], [22], [16], [20], [15], [24], [21], [14], [19], [2], [3], [4], [12], [10], [5], [13], and [11].

1. Posets of Real Numbers

Let R be a relational structure. We say that R is real if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) The carrier of $R \subseteq \mathbb{R}$, and

(ii) for all real numbers x, y such that $x \in$ the carrier of R and $y \in$ the carrier of R holds $\langle x, y\rangle \in$ the internal relation of R iff $x \leqslant y$.

Let R be a relational structure. We say that R is interval if and only if:

(Def. 2) R is real and there exist real numbers a, b such that $a \leqslant b$ and the carrier of $R = [a, b]$.

Let us mention that every relational structure which is interval is also real and non empty.

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Let us observe that every relational structure which is empty is also real.

One can prove the following proposition

(1) For every subset X of $\mathbb{R}$ there exists a strict relational structure R such that the carrier of $R = X$ and R is real.

Let us note that there exists a relational structure which is interval and strict.

The following proposition is true

(2) Let R_1, R_2 be real relational structures. Suppose the carrier of $R_1 =$ the carrier of R_2. Then the relational structure of $R_1 =$ the relational structure of R_2.

Let R be a non empty real relational structure. Observe that every element of R is real.

Let X be a subset of $\mathbb{R}$. The functor RealPoset X yields a real strict relational structure and is defined as follows:

(Def. 3) The carrier of RealPoset $X = X$.

Let X be a non empty subset of $\mathbb{R}$. Note that RealPoset X is non empty.

Let R be a relational structure and let x, y be elements of R. We introduce $x \preceq y$ and $y \succeq x$ as synonyms of $x \leqslant y$.

Let x, y be real numbers. We introduce $x \leqslant_{\mathbb{R}} y$ and $y \geqslant_{\mathbb{R}} x$ as synonyms of $x \leqslant y$. We introduce $y <_{\mathbb{R}} x$ and $x >_{\mathbb{R}} y$ as antonyms of $x \leqslant y$.

We now state the proposition

(3) For every non empty real relational structure R and for all elements x, y of R holds $x \leqslant_{\mathbb{R}} y$ iff $x \preceq y$.

Let us observe that every relational structure which is real is also reflexive, antisymmetric, and transitive.

Let us observe that every real non empty relational structure is connected.

Let R be a non empty real relational structure and let x, y be elements of R. Then $\max(x, y)$ is an element of R.

Let R be a non empty real relational structure and let x, y be elements of R. Then $\min(x, y)$ is an element of R.

Let us note that every real non empty relational structure has l.u.b.'s and g.l.b.'s.

We follow the rules: x, y denote real numbers, R denotes a real non empty relational structure, and a, b denote elements of R.

One can prove the following four propositions:

(4) $a \sqcup b = \max(a, b)$.

(5) $a \sqcap b = \min(a, b)$.

(6) There exists x such that $x \in$ the carrier of R and for every y such that $y \in$ the carrier of R holds $x \leqslant y$ if and only if R is lower-bounded.

(7) There exists x such that $x \in$ the carrier of R and for every y such that $y \in$ the carrier of R holds $x \geqslant y$ if and only if R is upper-bounded.

Let us observe that every non empty relational structure which is interval is also bounded.

The following proposition is true

(8) For every interval non empty relational structure R and for every set X holds sup X exists in R.

Let us observe that every interval non empty relational structure is complete.

Let us note that every chain is distributive.

One can check that every interval non empty relational structure is Heyting.

One can verify that $[0, 1]$ is non empty.

Let us observe that RealPoset$[0, 1]$ is interval.

2. Product of Heyting Lattices

We now state several propositions:

(9) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a sup-semilattice. Then $\prod J$ has l.u.b.'s.

(10) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a semilattice. Then $\prod J$ has g.l.b.'s.

(11) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a semilattice. Let f, g be elements of $\prod J$ and i be an element of I. Then $(f \sqcap g)(i) = f(i) \sqcap g(i)$.

(12) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a sup-semilattice. Let f, g be elements of $\prod J$ and i be an element of I. Then $(f \sqcup g)(i) = f(i) \sqcup g(i)$.

(13) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a Heyting complete lattice. Then $\prod J$ is complete and Heyting.

Let A be a non empty set and let R be a complete Heyting lattice. Observe that R^A is Heyting.

3. Lattice of Fuzzy Sets

Let A be a non empty set. The functor FuzzyLattice A yielding a Heyting complete lattice is defined by:

(Def. 4) FuzzyLattice $A = (\mathrm{RealPoset}[0,1])^A$.

We now state the proposition

(14) For every non empty set A holds the carrier of FuzzyLattice $A = [0,1]^A$.

Let A be a non empty set. Note that FuzzyLattice A is constituted functions.

Next we state the proposition

(15) Let R be a complete Heyting lattice, X be a subset of R, and y be an element of R. Then $\bigsqcup_R X \sqcap y = \bigsqcup_R \{x \sqcap y; x$ ranges over elements of R: $x \in X\}$.

Let X be a non empty set and let a be an element of FuzzyLattice X. The functor ${}^@a$ yields a membership function of X and is defined by:

(Def. 5) ${}^@a = a$.

Let X be a non empty set and let f be a membership function of X. The functor $f^@$ yielding an element of FuzzyLattice X is defined by:

(Def. 6) $f^@ = f$.

Let X be a non empty set, let f be a membership function of X, and let x be an element of X. Then $f(x)$ is an element of RealPoset$[0,1]$.

Let X be a non empty set, let f be an element of FuzzyLattice X, and let x be an element of X. Then $f(x)$ is an element of RealPoset$[0,1]$.

For simplicity, we follow the rules: C is a non empty set, c is an element of C, f, g are membership functions of C, and s, t are elements of FuzzyLattice C.

Next we state several propositions:

(16) For every c holds $f(c) \leqslant_{\mathbb{R}} g(c)$ iff $f^@ \preceq g^@$.

(17) $s \preceq t$ iff for every c holds $({}^@s)(c) \leqslant_{\mathbb{R}} ({}^@t)(c)$.

(18) $\max(f, g) = f^@ \sqcup g^@$.

(19) $s \sqcup t = \max({}^@s, {}^@t)$.

(20) $\min(f, g) = f^@ \sqcap g^@$.

(21) $s \sqcap t = \min({}^@s, {}^@t)$.

4. Associativity of Composition of Fuzzy Relations

In this article we present several logical schemes. The scheme *SupDistributivity* deals with a complete lattice $\mathcal{A}$, non empty sets $\mathcal{B}$, $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{F}(x, y); y$ ranges over elements of $\mathcal{C} : \mathcal{Q}[y]\}; x$ ranges over elements of $\mathcal{B} : \mathcal{P}[x]\} = \bigsqcup_{\mathcal{A}}\{\mathcal{F}(x, y); x$ ranges over elements of $\mathcal{B}, y$ ranges over elements of $\mathcal{C} : \mathcal{P}[x] \wedge \mathcal{Q}[y]\}$

for all values of the parameters.

The scheme *SupDistributivity'* deals with a complete lattice $\mathcal{A}$, non empty sets $\mathcal{B}$, $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{F}(x, y); x$ ranges over elements of $\mathcal{B} : \mathcal{P}[x]\}; y$ ranges over elements of $\mathcal{C} : \mathcal{Q}[y]\} = \bigsqcup_{\mathcal{A}}\{\mathcal{F}(x, y); x$ ranges over elements of $\mathcal{B}, y$ ranges over elements of $\mathcal{C} : \mathcal{P}[x] \wedge \mathcal{Q}[y]\}$

for all values of the parameters.

The scheme *FraenkelF'R'* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, two binary functors $\mathcal{F}$ and $\mathcal{G}$ yielding sets, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(u_1, v_1); u_1$ ranges over elements of $\mathcal{A}, v_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_1, v_1]\} = \{\mathcal{G}(u_2, v_2); u_2$ ranges over elements of $\mathcal{A}, v_2$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_2, v_2]\}$

provided the parameters meet the following condition:

- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ such that $\mathcal{P}[u, v]$ holds $\mathcal{F}(u, v) = \mathcal{G}(u, v)$.

The scheme *FraenkelF6"R* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, two binary functors $\mathcal{F}$ and $\mathcal{G}$ yielding sets, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(u_1, v_1); u_1$ ranges over elements of $\mathcal{A}, v_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_1, v_1]\} = \{\mathcal{G}(u_2, v_2); u_2$ ranges over elements of $\mathcal{A}, v_2$ ranges over elements of $\mathcal{B} : \mathcal{Q}[u_2, v_2]\}$

provided the following requirements are met:

- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{P}[u, v]$ iff $\mathcal{Q}[u, v]$, and
- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ such that $\mathcal{P}[u, v]$ holds $\mathcal{F}(u, v) = \mathcal{G}(u, v)$.

The scheme *SupCommutativity* deals with a complete lattice $\mathcal{A}$, non empty sets $\mathcal{B}$, $\mathcal{C}$, two binary functors $\mathcal{F}$ and $\mathcal{G}$ yielding elements of $\mathcal{A}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{F}(x, y); y$ ranges over elements of $\mathcal{C} : \mathcal{Q}[y]\}; x$ ranges over elements of $\mathcal{B} : \mathcal{P}[x]\} = \bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{G}(x', y'); x'$ ranges over elements of $\mathcal{B} : \mathcal{P}[x']\}; y'$ ranges over elements of $\mathcal{C} : \mathcal{Q}[y']\}$

provided the parameters meet the following condition:

- For every element x of $\mathcal{B}$ and for every element y of $\mathcal{C}$ such that $\mathcal{P}[x]$ and $\mathcal{Q}[y]$ holds $\mathcal{F}(x, y) = \mathcal{G}(x, y)$.

One can prove the following propositions:

(22) Let X, Y, Z be non empty sets, R be a membership function of X, Y, S be a membership function of Y, Z, x be an element of X, and z be an element of Z. Then $(R\,S)(\langle x, z\rangle) = \bigsqcup_{\mathrm{RealPoset}[0,1]}\{R(\langle x, y\rangle) \sqcap S(\langle y, z\rangle) : y$ ranges over elements of $Y\}$.

(23) Let X, Y, Z, W be non empty sets, R be a membership function of X, Y, S be a membership function of Y, Z, and T be a membership function of Z, W. Then $(R\,S)\,T = R\,(S\,T)$.

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On the Kuratowski Limit Operators[1]

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Summary. In the paper we give formal descriptions of the two Kuratowski limit oprators: Li S and Ls S, where S is an arbitrary sequence of subsets of a fixed topological space. In the two last sections we prove basic properties of these lower and upper topological limits, which may be found e.g. in [19]. In the sections 2–4, we present three operators which are associated in some sense with the above mentioned, that is lim inf F, lim sup F, and limes F, where F is a sequence of subsets of a fixed 1-sorted structure.

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The articles [30], [33], [2], [29], [9], [1], [22], [24], [35], [12], [34], [6], [4], [18], [8], [7], [16], [5], [13], [25], [31], [21], [10], [23], [14], [15], [20], [17], [27], [28], [26], [11], [3], and [32] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following four propositions:

(1) For all sets X, x and for every subset A of X such that $x \notin A$ and $x \in X$ holds $x \in A^{\rm c}$.

(2) For every function F and for every set i such that $i \in \operatorname{dom} F$ holds $\bigcap F \subseteq F(i)$.

(3) Let T be a non empty 1-sorted structure and S_1, S_2 be sequences of subsets of the carrier of T. Then $S_1 = S_2$ if and only if for every natural number n holds $S_1(n) = S_2(n)$.

(4) For all sets A, B, C, D such that A meets B and C meets D holds $[\!: A, C :\!]$ meets $[\!: B, D :\!]$.

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Let X be a 1-sorted structure. Note that every sequence of subsets of the carrier of X is non empty.

Let T be a non empty 1-sorted structure. One can check that there exists a sequence of subsets of the carrier of T which is non-empty.

Let T be a non empty 1-sorted structure.

(Def. 1) A sequence of subsets of the carrier of T is said to be a sequence of subsets of T.

In this article we present several logical schemes. The scheme *LambdaSSeq* deals with a non empty 1-sorted structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a subset of $\mathcal{A}$, and states that:

There exists a sequence f of subsets of $\mathcal{A}$ such that for every natural number n holds $f(n) = \mathcal{F}(n)$

for all values of the parameters.

The scheme *ExTopStrSeq* deals with a non empty topological space $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a subset of $\mathcal{A}$, and states that:

There exists a sequence S of subsets of the carrier of $\mathcal{A}$ such that for every natural number n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

We now state the proposition

(5) Let X be a non empty 1-sorted structure and F be a sequence of subsets of the carrier of X. Then $\operatorname{rng} F$ is a family of subsets of X.

Let X be a non empty 1-sorted structure and let F be a sequence of subsets of the carrier of X. Then $\bigcup F$ is a subset of X. Then $\bigcap F$ is a subset of X.

2. Lower and Upper Limit of Sequences of Subsets

Let X be a non empty set, let S be a function from $\mathbb{N}$ into X, and let k be a natural number. The functor $S \uparrow k$ yields a function from $\mathbb{N}$ into X and is defined as follows:

(Def. 2) For every natural number n holds $(S \uparrow k)(n) = S(n + k)$.

Let X be a non empty 1-sorted structure and let F be a sequence of subsets of the carrier of X. The functor $\liminf F$ yields a subset of X and is defined as follows:

(Def. 3) There exists a sequence f of subsets of X such that $\liminf F = \bigcup f$ and for every natural number n holds $f(n) = \bigcap(F \uparrow n)$.

The functor $\limsup F$ yields a subset of X and is defined by:

(Def. 4) There exists a sequence f of subsets of X such that $\limsup F = \bigcap f$ and for every natural number n holds $f(n) = \bigcup(F \uparrow n)$.

Next we state a number of propositions:

(6) Let X be a non empty 1-sorted structure, F be a sequence of subsets of the carrier of X, and x be a set. Then $x \in \bigcap F$ if and only if for every natural number z holds $x \in F(z)$.

(7) Let X be a non empty 1-sorted structure, F be a sequence of subsets of the carrier of X, and x be a set. Then $x \in \liminf F$ if and only if there exists a natural number n such that for every natural number k holds $x \in F(n + k)$.

(8) Let X be a non empty 1-sorted structure, F be a sequence of subsets of the carrier of X, and x be a set. Then $x \in \limsup F$ if and only if for every natural number n there exists a natural number k such that $x \in F(n+k)$.

(9) For every non empty 1-sorted structure X and for every sequence F of subsets of the carrier of X holds $\liminf F \subseteq \limsup F$.

(10) For every non empty 1-sorted structure X and for every sequence F of subsets of the carrier of X holds $\bigcap F \subseteq \liminf F$.

(11) For every non empty 1-sorted structure X and for every sequence F of subsets of the carrier of X holds $\limsup F \subseteq \bigcup F$.

(12) For every non empty 1-sorted structure X and for every sequence F of subsets of the carrier of X holds $\liminf F = (\limsup \mathrm{Complement}\, F)^{\mathrm{c}}$.

(13) Let X be a non empty 1-sorted structure and A, B, C be sequences of subsets of the carrier of X. If for every natural number n holds $C(n) = A(n) \cap B(n)$, then $\liminf C = \liminf A \cap \liminf B$.

(14) Let X be a non empty 1-sorted structure and A, B, C be sequences of subsets of the carrier of X. If for every natural number n holds $C(n) = A(n) \cup B(n)$, then $\limsup C = \limsup A \cup \limsup B$.

(15) Let X be a non empty 1-sorted structure and A, B, C be sequences of subsets of the carrier of X. If for every natural number n holds $C(n) = A(n) \cup B(n)$, then $\liminf A \cup \liminf B \subseteq \liminf C$.

(16) Let X be a non empty 1-sorted structure and A, B, C be sequences of subsets of the carrier of X. If for every natural number n holds $C(n) = A(n) \cap B(n)$, then $\limsup C \subseteq \limsup A \cap \limsup B$.

(17) Let X be a non empty 1-sorted structure, A be a sequence of subsets of the carrier of X, and B be a subset of X. If for every natural number n holds $A(n) = B$, then $\limsup A = B$.

(18) Let X be a non empty 1-sorted structure, A be a sequence of subsets of the carrier of X, and B be a subset of X. If for every natural number n holds $A(n) = B$, then $\liminf A = B$.

(19) Let X be a non empty 1-sorted structure, A, B be sequences of subsets of the carrier of X, and C be a subset of X. If for every natural number n holds $B(n) = C \dot{-} A(n)$, then $C \dot{-} \liminf A \subseteq \limsup B$.

(20) Let X be a non empty 1-sorted structure, A, B be sequences of subsets

of the carrier of X, and C be a subset of X. If for every natural number n holds $B(n) = C \dot{-} A(n)$, then $C \dot{-} \limsup A \subseteq \limsup B$.

3. Ascending and Descending Families of Subsets

Let T be a non empty 1-sorted structure and let S be a sequence of subsets of T. We say that S is descending if and only if:

(Def. 5) For every natural number i holds $S(i+1) \subseteq S(i)$.

We say that S is ascending if and only if:

(Def. 6) For every natural number i holds $S(i) \subseteq S(i+1)$.

Next we state several propositions:

(21) Let f be a function. Suppose that for every natural number i holds $f(i+1) \subseteq f(i)$. Let i, j be natural numbers. If $i \leqslant j$, then $f(j) \subseteq f(i)$.

(22) Let T be a non empty 1-sorted structure and C be a sequence of subsets of T. Suppose C is descending. Let i, m be natural numbers. If $i \geqslant m$, then $C(i) \subseteq C(m)$.

(23) Let T be a non empty 1-sorted structure and C be a sequence of subsets of T. Suppose C is ascending. Let i, m be natural numbers. If $i \geqslant m$, then $C(m) \subseteq C(i)$.

(24) Let T be a non empty 1-sorted structure, F be a sequence of subsets of T, and x be a set. Suppose F is descending and there exists a natural number k such that for every natural number n such that $n > k$ holds $x \in F(n)$. Then $x \in \bigcap F$.

(25) Let T be a non empty 1-sorted structure and F be a sequence of subsets of T. If F is descending, then $\liminf F = \bigcap F$.

(26) Let T be a non empty 1-sorted structure and F be a sequence of subsets of T. If F is ascending, then $\limsup F = \bigcup F$.

4. Constant and Convergent Sequences

Let T be a non empty 1-sorted structure and let S be a sequence of subsets of T. We say that S is convergent if and only if:

(Def. 7) $\limsup S = \liminf S$.

We now state the proposition

(27) Let T be a non empty 1-sorted structure and S be a sequence of subsets of T. If S is constant, then the value of S is a subset of T.

Let T be a non empty 1-sorted structure and let S be a sequence of subsets of T. Let us observe that S is constant if and only if:

(Def. 8) There exists a subset A of T such that for every natural number n holds $S(n) = A$.

Let T be a non empty 1-sorted structure. Observe that every sequence of subsets of T which is constant is also convergent, ascending, and descending.

Let T be a non empty 1-sorted structure. Note that there exists a sequence of subsets of T which is constant and non empty.

Let T be a non empty 1-sorted structure and let S be a convergent sequence of subsets of T. The functor limes S yields a subset of T and is defined as follows:

(Def. 9) limes $S = \limsup S$ and limes $S = \liminf S$.

One can prove the following proposition

(28) Let X be a non empty 1-sorted structure, F be a convergent sequence of subsets of X, and x be a set. Then $x \in$ limes F if and only if there exists a natural number n such that for every natural number k holds $x \in F(n+k)$.

5. Topological Lemmas

In the sequel n denotes a natural number.

Let f be a finite sequence of elements of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. One can check that $\widetilde{\mathcal{L}}(f)$ is closed.

We now state several propositions:

(29) Let r be a real number, M be a non empty Reflexive metric structure, and x be an element of M. If $0 < r$, then $x \in \mathrm{Ball}(x, r)$.

(30) For every point x of $\mathcal{E}^n$ and for every real number r holds $\mathrm{Ball}(x, r)$ is an open subset of $\mathcal{E}^n_{\mathrm{T}}$.

(31) For all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ and for all points p', q' of $\mathcal{E}^n$ such that $p = p'$ and $q = q'$ holds $\rho(p', q') = |p - q|$.

(32) Let p be a point of $\mathcal{E}^n$, x, p' be points of $\mathcal{E}^n_{\mathrm{T}}$, and r be a real number. If $p = p'$ and $x \in \mathrm{Ball}(p, r)$, then $|x - p'| < r$.

(33) Let p be a point of $\mathcal{E}^n$, x, p' be points of $\mathcal{E}^n_{\mathrm{T}}$, and r be a real number. If $p = p'$ and $|x - p'| < r$, then $x \in \mathrm{Ball}(p, r)$.

(34) Let n be a natural number, r be a point of $\mathcal{E}^n_{\mathrm{T}}$, and X be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $r \in \overline{X}$. Then there exists a sequence s_1 in $\mathcal{E}^n_{\mathrm{T}}$ such that $\mathrm{rng}\, s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 = r$.

Let M be a non empty metric space. Note that M_{top} is first-countable.

Let n be a natural number. Note that $\mathcal{E}^n_{\mathrm{T}}$ is first-countable.

Next we state several propositions:

(35) Let p be a point of $\mathcal{E}^n$, q be a point of $\mathcal{E}^n_{\mathrm{T}}$, and r be a real number. If $p = q$ and $r > 0$, then $\mathrm{Ball}(p, r)$ is a neighbourhood of q.

(36) Let A be a subset of $\mathcal{E}^n_{\rm T}$, p be a point of $\mathcal{E}^n_{\rm T}$, and p' be a point of $\mathcal{E}^n$. Suppose $p = p'$. Then $p \in \overline{A}$ if and only if for every real number r such that $r > 0$ holds $\mathrm{Ball}(p', r)$ meets A.

(37) Let x, y be points of $\mathcal{E}^n_{\rm T}$ and x' be a point of $\mathcal{E}^n$. If $x' = x$ and $x \neq y$, then there exists a real number r such that $y \notin \mathrm{Ball}(x', r)$.

(38) Let S be a subset of $\mathcal{E}^n_{\rm T}$. Then S is non Bounded if and only if for every real number r such that $r > 0$ there exist points x, y of $\mathcal{E}^n$ such that $x \in S$ and $y \in S$ and $\rho(x, y) > r$.

(39) For all real numbers a, b and for all points x, y of $\mathcal{E}^n$ such that $\mathrm{Ball}(x, a)$ meets $\mathrm{Ball}(y, b)$ holds $\rho(x, y) < a + b$.

(40) Let a, b, c be real numbers and x, y, z be points of $\mathcal{E}^n$. If $\mathrm{Ball}(x, a)$ meets $\mathrm{Ball}(z, c)$ and $\mathrm{Ball}(z, c)$ meets $\mathrm{Ball}(y, b)$, then $\rho(x, y) < a + b + 2 \cdot c$.

(41) Let X, Y be non empty topological spaces, x be a point of X, y be a point of Y, and V be a subset of $[\!: X, Y :\!]$. Then V is a neighbourhood of $[\!: \{x\}, \{y\} :\!]$ if and only if V is a neighbourhood of $\langle x, y \rangle$.

Now we present two schemes. The scheme *TSubsetEx* deals with a non empty topological structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a subset X of $\mathcal{A}$ such that for every point x of $\mathcal{A}$ holds $x \in X$ iff $\mathcal{P}[x]$

for all values of the parameters.

The scheme *TSubsetUniq* deals with a topological structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> Let A_1, A_2 be subsets of $\mathcal{A}$. Suppose for every point x of $\mathcal{A}$ holds $x \in A_1$ iff $\mathcal{P}[x]$ and for every point x of $\mathcal{A}$ holds $x \in A_2$ iff $\mathcal{P}[x]$. Then $A_1 = A_2$

for all values of the parameters.

Let T be a non empty topological structure, let S be a sequence of subsets of the carrier of T, and let i be a natural number. Then $S(i)$ is a subset of T.

One can prove the following two propositions:

(42) Let T be a non empty 1-sorted structure, S be a sequence of subsets of the carrier of T, and R be a sequence of naturals. Then $S \cdot R$ is a sequence of subsets of T.

(43) $\mathrm{id}_{\mathbb{N}}$ is an increasing sequence of naturals.

Let us observe that $\mathrm{id}_{\mathbb{N}}$ is real-yielding.

6. Subsequences

Let T be a non empty 1-sorted structure and let S be a sequence of subsets of the carrier of T. A sequence of subsets of T is said to be a subsequence of S if:

(Def. 10) There exists an increasing sequence N_1 of naturals such that it $= S \cdot N_1$.

We now state several propositions:

(44) For every non empty 1-sorted structure T holds every sequence S of subsets of the carrier of T is a subsequence of S.

(45) Let T be a non empty 1-sorted structure, S be a sequence of subsets of T, and S_1 be a subsequence of S. Then $\operatorname{rng} S_1 \subseteq \operatorname{rng} S$.

(46) Let T be a non empty 1-sorted structure, S_1 be a sequence of subsets of the carrier of T, and S_2 be a subsequence of S_1. Then every subsequence of S_2 is a subsequence of S_1.

(47) Let T be a non empty 1-sorted structure, F, G be sequences of subsets of the carrier of T, and A be a subset of T. Suppose G is a subsequence of F and for every natural number i holds $F(i) = A$. Then $G = F$.

(48) Let T be a non empty 1-sorted structure, A be a constant sequence of subsets of T, and B be a subsequence of A. Then $A = B$.

(49) Let T be a non empty 1-sorted structure, S be a sequence of subsets of the carrier of T, R be a subsequence of S, and n be a natural number. Then there exists a natural number m such that $m \geqslant n$ and $R(n) = S(m)$.

Let T be a non empty 1-sorted structure and let X be a constant sequence of subsets of T. Note that every subsequence of X is constant.

The scheme *SubSeqChoice* deals with a non empty topological space $\mathcal{A}$, a sequence $\mathcal{B}$ of subsets of the carrier of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a subsequence S_1 of $\mathcal{B}$ such that for every natural number n holds $\mathcal{P}[S_1(n)]$

provided the following condition is satisfied:

- For every natural number n there exists a natural number m such that $n \leqslant m$ and $\mathcal{P}[\mathcal{B}(m)]$.

7. The Lower Topological Limit

Let T be a non empty topological space and let S be a sequence of subsets of the carrier of T. The functor $\operatorname{Li} S$ yielding a subset of T is defined by the condition (Def. 11).

(Def. 11) Let p be a point of T. Then $p \in \operatorname{Li} S$ if and only if for every neighbourhood G of p there exists a natural number k such that for every natural number m such that $m > k$ holds $S(m)$ meets G.

The following propositions are true:

(50) Let S be a sequence of subsets of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, p be a point of $\mathcal{E}^n_{\mathrm{T}}$, and p' be a point of $\mathcal{E}^n$. Suppose $p = p'$. Then $p \in \operatorname{Li} S$ if and only if for every real number r such that $r > 0$ there exists a natural number k

such that for every natural number m such that $m > k$ holds $S(m)$ meets $\mathrm{Ball}(p', r)$.

(51) For every non empty topological space T and for every sequence S of subsets of the carrier of T holds $\overline{\mathrm{Li}\ S} = \mathrm{Li}\ S$.

(52) For every non empty topological space T and for every sequence S of subsets of the carrier of T holds Li S is closed.

(53) Let T be a non empty topological space and R, S be sequences of subsets of the carrier of T. If R is a subsequence of S, then $\mathrm{Li}\ S \subseteq \mathrm{Li}\ R$.

(54) Let T be a non empty topological space and A, B be sequences of subsets of the carrier of T. If for every natural number i holds $A(i) \subseteq B(i)$, then $\mathrm{Li}\ A \subseteq \mathrm{Li}\ B$.

(55) Let T be a non empty topological space and A, B, C be sequences of subsets of the carrier of T. If for every natural number i holds $C(i) = A(i) \cup B(i)$, then $\mathrm{Li}\ A \cup \mathrm{Li}\ B \subseteq \mathrm{Li}\ C$.

(56) Let T be a non empty topological space and A, B, C be sequences of subsets of the carrier of T. If for every natural number i holds $C(i) = A(i) \cap B(i)$, then $\mathrm{Li}\ C \subseteq \mathrm{Li}\ A \cap \mathrm{Li}\ B$.

(57) Let T be a non empty topological space and F, G be sequences of subsets of the carrier of T. If for every natural number i holds $G(i) = \overline{F(i)}$, then $\mathrm{Li}\ G = \mathrm{Li}\ F$.

(58) Let S be a sequence of subsets of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and p be a point of $\mathcal{E}^n_{\mathrm{T}}$. Given a sequence s in $\mathcal{E}^n_{\mathrm{T}}$ such that s is convergent and for every natural number x holds $s(x) \in S(x)$ and $p = \lim s$. Then $p \in \mathrm{Li}\ S$.

(59) Let T be a non empty topological space, P be a subset of T, and s be a sequence of subsets of the carrier of T. If for every natural number i holds $s(i) \subseteq P$, then $\mathrm{Li}\ s \subseteq \overline{P}$.

(60) Let T be a non empty topological space, F be a sequence of subsets of the carrier of T, and A be a subset of T. If for every natural number i holds $F(i) = A$, then $\mathrm{Li}\ F = \overline{A}$.

(61) Let T be a non empty topological space, F be a sequence of subsets of the carrier of T, and A be a closed subset of T. If for every natural number i holds $F(i) = A$, then $\mathrm{Li}\ F = A$.

(62) Let S be a sequence of subsets of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and P be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose P is Bounded and for every natural number i holds $S(i) \subseteq P$. Then Li S is Bounded.

(63) Let S be a sequence of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is Bounded and for every natural number i holds $S(i) \subseteq P$ and for every natural number i holds $S(i)$ is compact. Then Li S is compact.

(64) Let A, B be sequences of subsets of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and C be a sequence of subsets of the carrier of $[:\mathcal{E}^n_{\mathrm{T}}, \mathcal{E}^n_{\mathrm{T}}:]$. If for every natural number i holds

$C(i) = [\!: A(i), B(i) :\!]$, then $[\!: \mathrm{Li}\ A, \mathrm{Li}\ B :\!] = \mathrm{Li}\ C$.

(65) For every sequence S of subsets of $\mathcal{E}^2_{\mathrm{T}}$ holds $\liminf S \subseteq \mathrm{Li}\ S$.

(66) For every simple closed curve C and for every natural number i holds $\mathrm{Fr}((\mathrm{UBD}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i)))^{\mathrm{c}}) = \widetilde{\mathcal{L}}(\mathrm{Cage}(C,i))$.

8. The Upper Topological Limit

Let T be a non empty topological space and let S be a sequence of subsets of the carrier of T. The functor $\mathrm{Ls}\ S$ yields a subset of T and is defined as follows:

(Def. 12) For every set x holds $x \in \mathrm{Ls}\ S$ iff there exists a subsequence A of S such that $x \in \mathrm{Li}\ A$.

One can prove the following propositions:

(67) Let N be a natural number, F be a sequence of $\mathcal{E}^N_{\mathrm{T}}$, x be a point of $\mathcal{E}^N_{\mathrm{T}}$, and x' be a point of $\mathcal{E}^N$. Suppose $x = x'$. Then x is a cluster point of F if and only if for every real number r and for every natural number n such that $r > 0$ there exists a natural number m such that $n \leqslant m$ and $F(m) \in \mathrm{Ball}(x', r)$.

(68) For every non empty topological space T and for every sequence A of subsets of the carrier of T holds $\mathrm{Li}\ A \subseteq \mathrm{Ls}\ A$.

(69) Let A, B, C be sequences of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. Suppose for every natural number i holds $A(i) \subseteq B(i)$ and C is a subsequence of A. Then there exists a subsequence D of B such that for every natural number i holds $C(i) \subseteq D(i)$.

(70) Let A, B, C be sequences of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. Suppose for every natural number i holds $A(i) \subseteq B(i)$ and C is a subsequence of B. Then there exists a subsequence D of A such that for every natural number i holds $D(i) \subseteq C(i)$.

(71) Let A, B be sequences of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. If for every natural number i holds $A(i) \subseteq B(i)$, then $\mathrm{Ls}\ A \subseteq \mathrm{Ls}\ B$.

(72) Let A, B, C be sequences of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. If for every natural number i holds $C(i) = A(i) \cup B(i)$, then $\mathrm{Ls}\ A \cup \mathrm{Ls}\ B \subseteq \mathrm{Ls}\ C$.

(73) Let A, B, C be sequences of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. If for every natural number i holds $C(i) = A(i) \cap B(i)$, then $\mathrm{Ls}\ C \subseteq \mathrm{Ls}\ A \cap \mathrm{Ls}\ B$.

(74) Let A, B be sequences of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and C, C_1 be sequences of subsets of the carrier of $[\!: \mathcal{E}^2_{\mathrm{T}}, \mathcal{E}^2_{\mathrm{T}} :\!]$. Suppose for every natural number i holds $C(i) = [\!: A(i), B(i) :\!]$ and C_1 is a subsequence of C. Then there exist sequences A_1, B_1 of subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ such that A_1 is a subsequence of A and B_1 is a subsequence of B and for every natural number i holds $C_1(i) = [\!: A_1(i), B_1(i) :\!]$.

(75) Let A, B be sequences of subsets of the carrier of $\mathcal{E}^2_{\rm T}$ and C be a sequence of subsets of the carrier of $[\!:\mathcal{E}^2_{\rm T}, \mathcal{E}^2_{\rm T}:\!]$. If for every natural number i holds $C(i) = [\!: A(i), B(i) :\!]$, then $\operatorname{Ls} C \subseteq [\!: \operatorname{Ls} A, \operatorname{Ls} B :\!]$.

(76) Let T be a non empty topological space, F be a sequence of subsets of the carrier of T, and A be a subset of T. If for every natural number i holds $F(i) = A$, then $\operatorname{Li} F = \operatorname{Ls} F$.

(77) Let F be a sequence of subsets of the carrier of $\mathcal{E}^2_{\rm T}$ and A be a subset of $\mathcal{E}^2_{\rm T}$. If for every natural number i holds $F(i) = A$, then $\operatorname{Ls} F = \overline{A}$.

(78) Let F, G be sequences of subsets of the carrier of $\mathcal{E}^2_{\rm T}$. If for every natural number i holds $G(i) = \overline{F(i)}$, then $\operatorname{Ls} G = \operatorname{Ls} F$.

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On the Segmentation of a Simple Closed Curve[1]

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Summary. The main goal of the work was to introduce the concept of the segmentation of a simple closed curve into (arbitrary small) arcs. The existence of it has been proved by Yatsuka Nakamura [21]. The concept of the gap of a segmentation is also introduced. It is the smallest distance between disjoint segments in the segmentation. For this purpose, the relationship between segments of an arc [24] and segments on a simple closed curve [21] has been shown.

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The papers [30], [35], [10], [3], [2], [29], [1], [13], [8], [9], [7], [4], [34], [25], [33], [22], [20], [28], [15], [26], [27], [18], [6], [12], [31], [19], [14], [16], [17], [23], [5], [24], [21], [11], and [32] provide the notation and terminology for this paper.

1. Preliminaries

The scheme *AndScheme* deals with a non empty set $\mathcal{A}$ and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{a; a$ ranges over elements of $\mathcal{A} : \mathcal{P}[a] \ \wedge \ \mathcal{Q}[a]\} = \{a_1; a_1$ ranges over elements of $\mathcal{A} : \mathcal{P}[a_1]\} \cap \{a_2; a_2$ ranges over elements of $\mathcal{A} : \mathcal{Q}[a_2]\}$

for all values of the parameters.

For simplicity, we follow the rules: C is a simple closed curve, p, q are points of $\mathcal{E}^2_{\rm T}$, i, j, k, n are natural numbers, and e is a real number.

The following proposition is true

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(1) For all finite non empty subsets A, B of $\mathbb{R}$ holds $\min(A \cup B) = \min(\min A, \min B)$.

Let T be a non empty topological space. One can check that there exists a subset of T which is compact and non empty.

Next we state several propositions:

(2) Let T be a non empty topological space, f be a continuous real map of T, and A be a compact subset of T. Then $f^\circ A$ is compact.

(3) For every compact subset A of $\mathbb{R}$ and for every non empty subset B of $\mathbb{R}$ such that $B \subseteq A$ holds $\inf B \in A$.

(4) Let A, B be compact non empty subsets of $\mathcal{E}^n_{\rm T}$, f be a continuous real map of $[\!:\mathcal{E}^n_{\rm T}, \mathcal{E}^n_{\rm T}:\!]$, and g be a real map of $\mathcal{E}^n_{\rm T}$. Suppose that for every point p of $\mathcal{E}^n_{\rm T}$ there exists a subset G of $\mathbb{R}$ such that $G = \{f(p, q); q$ ranges over points of $\mathcal{E}^n_{\rm T}$: $q \in B\}$ and $g(p) = \inf G$. Then $\inf(f^\circ[\!: A, B :\!]) = \inf(g^\circ A)$.

(5) Let A, B be compact non empty subsets of $\mathcal{E}^n_{\rm T}$, f be a continuous real map of $[\!:\mathcal{E}^n_{\rm T}, \mathcal{E}^n_{\rm T}:\!]$, and g be a real map of $\mathcal{E}^n_{\rm T}$. Suppose that for every point q of $\mathcal{E}^n_{\rm T}$ there exists a subset G of $\mathbb{R}$ such that $G = \{f(p, q); p$ ranges over points of $\mathcal{E}^n_{\rm T}$: $p \in A\}$ and $g(q) = \inf G$. Then $\inf(f^\circ[\!: A, B :\!]) = \inf(g^\circ B)$.

(6) If $q \in \mathrm{LowerArc}(C)$ and $q \neq \mathrm{W_{min}}(C)$, then $\mathrm{E_{max}}(C) \leqslant_C q$.

(7) If $q \in \mathrm{UpperArc}(C)$, then $q \leqslant_C \mathrm{E_{max}}(C)$.

2. The Euclidean Distance

Let us consider n. The functor $\mathrm{EuclDist}(n)$ yielding a real map of $[\!:\mathcal{E}^n_{\rm T}, \mathcal{E}^n_{\rm T}:\!]$ is defined as follows:

(Def. 1) For all points p, q of $\mathcal{E}^n_{\rm T}$ holds $(\mathrm{EuclDist}(n))(p, q) = |p - q|$.

Let T be a non empty topological space and let f be a real map of T. Let us observe that f is continuous if and only if:

(Def. 2) For every point p of T and for every neighbourhood N of $f(p)$ there exists a neighbourhood V of p such that $f^\circ V \subseteq N$.

Let us consider n. Note that $\mathrm{EuclDist}(n)$ is continuous.

3. On the Distance between Subsets of a Euclidean Space

The following proposition is true

(8) For all non empty compact subsets A, B of $\mathcal{E}^n_{\rm T}$ such that A misses B holds $\mathrm{dist_{min}}(A, B) > 0$.

4. On the Segments

The following propositions are true:

(9) If $p \leqslant_C q$ and $q \leqslant_C \mathrm{E_{max}}(C)$ and $p \neq q$, then $\mathrm{Segment}(p, q, C) = \mathrm{Segment}(\mathrm{UpperArc}(C), \mathrm{W_{min}}(C), \mathrm{E_{max}}(C), p, q)$.

(10) If $\mathrm{E_{max}}(C) \leqslant_C q$, then $\mathrm{Segment}(\mathrm{E_{max}}(C), q, C) = \mathrm{Segment}(\mathrm{LowerArc}(C), \mathrm{E_{max}}(C), \mathrm{W_{min}}(C), \mathrm{E_{max}}(C), q)$.

(11) If $\mathrm{E_{max}}(C) \leqslant_C q$, then $\mathrm{Segment}(q, \mathrm{W_{min}}(C), C) = \mathrm{Segment}(\mathrm{LowerArc}(C), \mathrm{E_{max}}(C), \mathrm{W_{min}}(C), q, \mathrm{W_{min}}(C))$.

(12) If $p \leqslant_C q$ and $\mathrm{E_{max}}(C) \leqslant_C p$, then $\mathrm{Segment}(p, q, C) = \mathrm{Segment}(\mathrm{LowerArc}(C), \mathrm{E_{max}}(C), \mathrm{W_{min}}(C), p, q)$.

(13) If $p \leqslant_C \mathrm{E_{max}}(C)$ and $\mathrm{E_{max}}(C) \leqslant_C q$, then $\mathrm{Segment}(p, q, C) = \mathrm{RSegment}(\mathrm{UpperArc}(C), \mathrm{W_{min}}(C), \mathrm{E_{max}}(C), p) \cup \mathrm{LSegment}(\mathrm{LowerArc}(C), \mathrm{E_{max}}(C), \mathrm{W_{min}}(C), q)$.

(14) If $p \leqslant_C \mathrm{E_{max}}(C)$, then $\mathrm{Segment}(p, \mathrm{W_{min}}(C), C) = \mathrm{RSegment}(\mathrm{UpperArc}(C), \mathrm{W_{min}}(C), \mathrm{E_{max}}(C), p) \cup \mathrm{LSegment}(\mathrm{LowerArc}(C), \mathrm{E_{max}}(C), \mathrm{W_{min}}(C), \mathrm{W_{min}}(C))$.

(15) $\mathrm{RSegment}(\mathrm{UpperArc}(C), \mathrm{W_{min}}(C), \mathrm{E_{max}}(C), p) = \mathrm{Segment}(\mathrm{UpperArc}(C), \mathrm{W_{min}}(C), \mathrm{E_{max}}(C), p, \mathrm{E_{max}}(C))$.

(16) $\mathrm{LSegment}(\mathrm{LowerArc}(C), \mathrm{E_{max}}(C), \mathrm{W_{min}}(C), p) = \mathrm{Segment}(\mathrm{LowerArc}(C), \mathrm{E_{max}}(C), \mathrm{W_{min}}(C), \mathrm{E_{max}}(C), p)$.

(17) For every point p of $\mathcal{E}^2_\mathrm{T}$ such that $p \in C$ and $p \neq \mathrm{W_{min}}(C)$ holds $\mathrm{Segment}(p, \mathrm{W_{min}}(C), C)$ is an arc from p to $\mathrm{W_{min}}(C)$.

(18) For all points p, q of $\mathcal{E}^2_\mathrm{T}$ such that $p \neq q$ and $p \leqslant_C q$ holds $\mathrm{Segment}(p, q, C)$ is an arc from p to q.

(19) $C = \mathrm{Segment}(\mathrm{W_{min}}(C), \mathrm{W_{min}}(C), C)$.

(20) For every point q of $\mathcal{E}^2_\mathrm{T}$ such that $q \in C$ holds $\mathrm{Segment}(q, \mathrm{W_{min}}(C), C)$ is compact.

(21) For all points q_1, q_2 of $\mathcal{E}^2_\mathrm{T}$ such that $q_1 \leqslant_C q_2$ holds $\mathrm{Segment}(q_1, q_2, C)$ is compact.

5. The Concept of a Segmentation

Let us consider C. A finite sequence of elements of $\mathcal{E}^2_\mathrm{T}$ is said to be a segmentation of C if it satisfies the conditions (Def. 3).

(Def. 3) $\mathrm{It}_1 = \mathrm{W_{min}}(C)$ and it is one-to-one and $8 \leqslant \mathrm{len}\,\mathrm{it}$ and $\mathrm{rng}\,\mathrm{it} \subseteq C$ and for every natural number i such that $1 \leqslant i$ and $i < \mathrm{len}\,\mathrm{it}$ holds $\mathrm{it}_i \leqslant_C \mathrm{it}_{i+1}$ and for every natural number i such that $1 \leqslant i$ and $i + 1 < \mathrm{len}\,\mathrm{it}$ holds $\mathrm{Segment}(\mathrm{it}_i, \mathrm{it}_{i+1}, C) \cap \mathrm{Segment}(\mathrm{it}_{i+1}, \mathrm{it}_{i+2}, C) =$

$\{\text{it}_{i+1}\}$ and $\text{Segment}(\text{it}_{\text{len it}}, \text{it}_1, C) \cap \text{Segment}(\text{it}_1, \text{it}_2, C) = \{\text{it}_1\}$ and $\text{Segment}(\text{it}_{\text{len it} -' 1}, \text{it}_{\text{len it}}, C) \cap \text{Segment}(\text{it}_{\text{len it}}, \text{it}_1, C) = \{\text{it}_{\text{len it}}\}$ and $\text{Segment}(\text{it}_{\text{len it} -' 1}, \text{it}_{\text{len it}}, C)$ misses $\text{Segment}(\text{it}_1, \text{it}_2, C)$ and for all natural numbers i, j such that $1 \leqslant i$ and $i < j$ and $j < \text{len it}$ and i and j are not adjacent holds $\text{Segment}(\text{it}_i, \text{it}_{i+1}, C)$ misses $\text{Segment}(\text{it}_j, \text{it}_{j+1}, C)$ and for every natural number i such that $1 < i$ and $i + 1 < \text{len it}$ holds $\text{Segment}(\text{it}_{\text{len it}}, \text{it}_1, C)$ misses $\text{Segment}(\text{it}_i, \text{it}_{i+1}, C)$.

Let us consider C. One can verify that every segmentation of C is non trivial.

One can prove the following proposition

(22) For every segmentation S of C and for every i such that $1 \leqslant i$ and $i \leqslant \text{len } S$ holds $S_i \in C$.

6. The Segments of a Segmentation

Let us consider C, let i be a natural number, and let S be a segmentation of C. The functor $\text{Segm}(S, i)$ yields a subset of $\mathcal{E}^2_{\text{T}}$ and is defined by:

(Def. 4) $$\text{Segm}(S, i) = \begin{cases} \text{Segment}(S_i, S_{i+1}, C), & \text{if } 1 \leqslant i \text{ and } i < \text{len } S, \\ \text{Segment}(S_{\text{len } S}, S_1, C), & \text{otherwise.} \end{cases}$$

The following proposition is true

(23) For every segmentation S of C such that $i \in \text{dom } S$ holds $\text{Segm}(S, i) \subseteq C$.

Let us consider C, let S be a segmentation of C, and let us consider i. Note that $\text{Segm}(S, i)$ is non empty and compact.

We now state several propositions:

(24) For every segmentation S of C and for every p such that $p \in C$ there exists a natural number i such that $i \in \text{dom } S$ and $p \in \text{Segm}(S, i)$.

(25) Let S be a segmentation of C and given i, j. Suppose $1 \leqslant i$ and $i < j$ and $j < \text{len } S$ and i and j are not adjacent. Then $\text{Segm}(S, i)$ misses $\text{Segm}(S, j)$.

(26) For every segmentation S of C and for every j such that $1 < j$ and $j < \text{len } S -' 1$ holds $\text{Segm}(S, \text{len } S)$ misses $\text{Segm}(S, j)$.

(27) Let S be a segmentation of C and given i, j. Suppose $1 \leqslant i$ and $i < j$ and $j < \text{len } S$ and i and j are adjacent. Then $\text{Segm}(S, i) \cap \text{Segm}(S, j) = \{S_{i+1}\}$.

(28) Let S be a segmentation of C and given i, j. Suppose $1 \leqslant i$ and $i < j$ and $j < \text{len } S$ and i and j are adjacent. Then $\text{Segm}(S, i)$ meets $\text{Segm}(S, j)$.

(29) For every segmentation S of C holds $\text{Segm}(S, \text{len } S) \cap \text{Segm}(S, 1) = \{S_1\}$.

(30) For every segmentation S of C holds $\text{Segm}(S, \text{len } S)$ meets $\text{Segm}(S, 1)$.

(31) For every segmentation S of C holds $\text{Segm}(S, \text{len } S) \cap \text{Segm}(S, \text{len } S -' 1) = \{S_{\text{len } S}\}$.

(32) For every segmentation S of C holds $\text{Segm}(S, \text{len } S)$ meets $\text{Segm}(S, \text{len } S -' 1)$.

7. The Diameter of a Segmentation

Let us consider n and let C be a subset of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\emptyset C$ yielding a real number is defined by:

(Def. 5) There exists a subset W of $\mathcal{E}^n$ such that $W = C$ and $\emptyset C = \emptyset W$.

Let us consider C and let S be a segmentation of C. The functor $\emptyset S$ yielding a real number is defined as follows:

(Def. 6) There exists a non empty finite subset S_1 of $\mathbb{R}$ such that $S_1 = \{\emptyset\, \mathrm{Segm}(S, i) : i \in \mathrm{dom}\, S\}$ and $\emptyset S = \max S_1$.

We now state three propositions:

(33) For every segmentation S of C and for every i holds $\emptyset\, \mathrm{Segm}(S, i) \leqslant \emptyset S$.

(34) For every segmentation S of C and for every real number e such that for every i holds $\emptyset\, \mathrm{Segm}(S, i) < e$ holds $\emptyset S < e$.

(35) For every real number e such that $e > 0$ there exists a segmentation S of C such that $\emptyset S < e$.

8. The Concept of the Gap of a Segmentation

Let us consider C and let S be a segmentation of C. The functor $\mathrm{Gap}(S)$ yields a real number and is defined by the condition (Def. 7).

(Def. 7) There exist non empty finite subsets S_1, S_2 of $\mathbb{R}$ such that $S_1 = \{\mathrm{dist}_{\min}(\mathrm{Segm}(S, i), \mathrm{Segm}(S, j)) : 1 \leqslant i \ \wedge \ i < j \ \wedge \ j < \mathrm{len}\, S \ \wedge \ i$ and j are not adjacent$\}$ and $S_2 = \{\mathrm{dist}_{\min}(\mathrm{Segm}(S, \mathrm{len}\, S), \mathrm{Segm}(S, k)) : 1 < k \ \wedge \ k < \mathrm{len}\, S -' 1\}$ and $\mathrm{Gap}(S) = \min(\min S_1, \min S_2)$.

Next we state two propositions:

(36) Let S be a segmentation of C. Then there exists a finite non empty subset F of $\mathbb{R}$ such that $F = \{\mathrm{dist}_{\min}(\mathrm{Segm}(S, i), \mathrm{Segm}(S, j)) : 1 \leqslant i \wedge i < j \ \wedge \ j \leqslant \mathrm{len}\, S \ \wedge \ \mathrm{Segm}(S, i)$ misses $\mathrm{Segm}(S, j)\}$ and $\mathrm{Gap}(S) = \min F$.

(37) For every segmentation S of C holds $\mathrm{Gap}(S) > 0$.

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On the Calculus of Binary Arithmetics

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Summary. In this paper, we have binary arithmetic and its related operations. We include some theorems concerning logical operators.

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The notation and terminology used in this paper have been introduced in the following articles: [3], [4], [2], and [1].

Let x, y be boolean sets. The functor x 'nand' y is defined as follows:

(Def. 1) x 'nand' $y = \neg(x \wedge y)$.

Let us note that the functor x 'nand' y is commutative.

Let x, y be boolean sets. Note that x 'nand' y is boolean.

Let x, y be elements of *Boolean*. Then x 'nand' y is an element of *Boolean*.

Let x, y be boolean sets. The functor x 'nor' y is defined by:

(Def. 2) x 'nor' $y = \neg(x \vee y)$.

Let us note that the functor x 'nor' y is commutative.

Let x, y be boolean sets. Note that x 'nor' y is boolean.

Let x, y be elements of *Boolean*. Then x 'nor' y is an element of *Boolean*.

Let x, y be boolean sets. The functor x 'xnor' y is defined as follows:

(Def. 3) x 'xnor' $y = \neg(x \oplus y)$.

Let us observe that the functor x 'xnor' y is commutative.

Let x, y be boolean sets. Note that x 'xnor' y is boolean.

Let x, y be elements of *Boolean*. Then x 'xnor' y is an element of *Boolean*.

In the sequel x, y, z, w are boolean sets.

The following propositions are true:

(1) *true* 'nand' $x = \neg x$.

(2) *false* 'nand' $x =$ *true*.

(3) x 'nand' $x = \neg x$ and $\neg(x$ 'nand' $x) = x$.

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(4) $\neg(x \text{ 'nand' } y) = x \wedge y$.

(5) $x \text{ 'nand' } \neg x = \mathit{true}$ and $\neg(x \text{ 'nand' } \neg x) = \mathit{false}$.

(6) $x \text{ 'nand' } y \wedge z = \neg(x \wedge y \wedge z)$.

(7) $x \text{ 'nand' } y \wedge z = x \wedge y \text{ 'nand' } z$.

(8) $x \text{ 'nand' } (y \vee z) = \neg(x \wedge y) \wedge \neg(x \wedge z)$.

(9) $x \text{ 'nand' } (y \oplus z) = x \wedge y \Leftrightarrow x \wedge z$.

(10) $\mathit{true} \text{ 'nor' } x = \mathit{false}$.

(11) $\mathit{false} \text{ 'nor' } x = \neg x$.

(12) $x \text{ 'nor' } x = \neg x$ and $\neg(x \text{ 'nor' } x) = x$.

(13) $\neg(x \text{ 'nor' } y) = x \vee y$.

(14) $x \text{ 'nor' } \neg x = \mathit{false}$ and $\neg(x \text{ 'nor' } \neg x) = \mathit{true}$.

(15) $x \text{ 'nor' } y \wedge z = \neg(x \vee y) \vee \neg(x \vee z)$.

(16) $x \text{ 'nor' } (y \vee z) = \neg(x \vee y \vee z)$.

(17) $\mathit{true} \text{ 'xnor' } x = x$.

(18) $\mathit{false} \text{ 'xnor' } x = \neg x$.

(19) $x \text{ 'xnor' } x = \mathit{true}$ and $\neg(x \text{ 'xnor' } x) = \mathit{false}$.

(20) $\neg(x \text{ 'xnor' } y) = x \oplus y$.

(21) $x \text{ 'xnor' } \neg x = \mathit{false}$ and $\neg(x \text{ 'xnor' } \neg x) = \mathit{true}$.

(22) $x \Subset y \Rightarrow z$ iff $x \wedge y \Subset z$.

(23) $x \Leftrightarrow y = (x \Rightarrow y) \wedge (y \Rightarrow x)$.

(24) $x \Leftrightarrow y = \mathit{true}$ iff $x \Rightarrow y = \mathit{true}$ and $y \Rightarrow x = \mathit{true}$.

(25) If $x \Rightarrow y = \mathit{true}$ and $y \Rightarrow x = \mathit{true}$, then $x = y$.

(26) If $x \Rightarrow y = \mathit{true}$ and $y \Rightarrow z = \mathit{true}$, then $x \Rightarrow z = \mathit{true}$.

(27) If $x \Leftrightarrow y = \mathit{true}$ and $y \Leftrightarrow z = \mathit{true}$, then $x \Leftrightarrow z = \mathit{true}$.

(28) $x \Rightarrow y = \neg y \Rightarrow \neg x$.

(29) $x \Leftrightarrow y = \neg x \Leftrightarrow \neg y$.

(30) If $x \Leftrightarrow y = \mathit{true}$ and $z \Leftrightarrow w = \mathit{true}$, then $x \wedge z \Leftrightarrow y \wedge w = \mathit{true}$.

(31) If $x \Leftrightarrow y = \mathit{true}$ and $z \Leftrightarrow w = \mathit{true}$, then $x \Rightarrow z \Leftrightarrow y \Rightarrow w = \mathit{true}$.

(32) If $x \Leftrightarrow y = \mathit{true}$ and $z \Leftrightarrow w = \mathit{true}$, then $x \vee z \Leftrightarrow y \vee w = \mathit{true}$.

(33) If $x \Leftrightarrow y = \mathit{true}$ and $z \Leftrightarrow w = \mathit{true}$, then $x \Leftrightarrow z \Leftrightarrow y \Leftrightarrow w = \mathit{true}$.

(34) If $x = \mathit{true}$ and $x \Rightarrow y = \mathit{true}$, then $y = \mathit{true}$.

(35) If $y = \mathit{true}$, then $x \Rightarrow y = \mathit{true}$.

(36) If $\neg x = \mathit{true}$, then $x \Rightarrow y = \mathit{true}$.

(37) $x \Rightarrow x = \mathit{true}$.

(38) If $x \Rightarrow y = \mathit{true}$ and $x \Rightarrow \neg y = \mathit{true}$, then $\neg x = \mathit{true}$.

(39) $\neg x \Rightarrow x \Rightarrow x = \mathit{true}$.

(40) $x \Rightarrow y \Rightarrow \neg(y \wedge z) \Rightarrow \neg(x \wedge z) = \mathit{true}$.

(41) $x \Rightarrow y \Rightarrow y \Rightarrow z \Rightarrow x \Rightarrow z = true$.

(42) If $x \Rightarrow y = true$, then $y \Rightarrow z \Rightarrow x \Rightarrow z = true$.

(43) $y \Rightarrow x \Rightarrow y = true$.

(44) $x \Rightarrow y \Rightarrow z \Rightarrow y \Rightarrow z = true$.

(45) $y \Rightarrow y \Rightarrow x \Rightarrow x = true$.

(46) $z \Rightarrow y \Rightarrow x \Rightarrow y \Rightarrow z \Rightarrow x = true$.

(47) $y \Rightarrow z \Rightarrow x \Rightarrow y \Rightarrow x \Rightarrow z = true$.

(48) $y \Rightarrow y \Rightarrow z \Rightarrow y \Rightarrow z = true$.

(49) $x \Rightarrow y \Rightarrow z \Rightarrow x \Rightarrow y \Rightarrow x \Rightarrow z = true$.

(50) If $x = true$, then $x \Rightarrow y \Rightarrow y = true$.

(51) If $z \Rightarrow y \Rightarrow x = true$, then $y \Rightarrow z \Rightarrow x = true$.

(52) If $z \Rightarrow y \Rightarrow x = true$ and $y = true$, then $z \Rightarrow x = true$.

(53) If $z \Rightarrow y \Rightarrow x = true$ and $y = true$ and $z = true$, then $x = true$.

(54) If $y \Rightarrow y \Rightarrow z = true$, then $y \Rightarrow z = true$.

(55) If $x \Rightarrow y \Rightarrow z = true$, then $x \Rightarrow y \Rightarrow x \Rightarrow z = true$.

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SCMPDS Is Not Standard

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Summary. The aim of the paper is to show that SCMPDS ([8]) does not belong to the class of standard computers ([16]).

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The terminology and notation used in this paper are introduced in the following papers: [14], [19], [11], [3], [2], [13], [6], [12], [17], [1], [5], [9], [18], [20], [7], [4], [10], [15], [8], and [16].

1. Preliminaries

In this paper r, s are real numbers.

We now state several propositions:

(1) $0 \leqslant r + |r|$.

(2) $0 \leqslant -r + |r|$.

(3) If $|r| = |s|$, then $r = s$ or $r = -s$.

(4) For all natural numbers i, j such that $i < j$ and $i \neq 0$ holds $\frac{i}{j}$ is not integer.

(5) $\{2 \cdot k; k$ ranges over natural numbers: $k > 1\}$ is infinite.

(6) For every function f and for all sets a, b, c such that $a \neq c$ holds $(f + \cdot (a \dot{\longmapsto} b))(c) = f(c)$.

(7) For every function f and for all sets a, b, c, d such that $a \neq b$ holds $(f + \cdot [a \longmapsto c, b \longmapsto d])(a) = c$ and $(f + \cdot [a \longmapsto c, b \longmapsto d])(b) = d$.

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2. SCMPDS

For simplicity, we adopt the following rules: a, b are Int positions, i is an instruction of SCMPDS, l is an instruction-location of SCMPDS, and k, k_1, k_2 are integers.

Let l_1, l_2 be Int positions and let a, b be integers. Then $[l_1 \longmapsto a, l_2 \longmapsto b]$ is a finite partial state of SCMPDS.

One can verify that SCMPDS has non trivial instruction locations.

Let l be an instruction-location of SCMPDS. The functor locnum(l) yields a natural number and is defined by:

(Def. 1) $\mathbf{i}_{\mathrm{locnum}(l)} = l$.

Let l be an instruction-location of SCMPDS. Then locnum(l) is an element of $\mathbb{N}$.

We now state a number of propositions:

(8) $l = 2 \cdot \mathrm{locnum}(l) + 2$.

(9) For all instruction-locations l_3, l_4 of SCMPDS such that $l_3 \neq l_4$ holds $\mathrm{locnum}(l_3) \neq \mathrm{locnum}(l_4)$.

(10) For all instruction-locations l_3, l_4 of SCMPDS such that $l_3 \neq l_4$ holds $\mathrm{Next}(l_3) \neq \mathrm{Next}(l_4)$.

(11) Let N be a set with non empty elements, S be an IC-Ins-separated definite non empty non void AMI over N, i be an instruction of S, and l be an instruction-location of S. Then $\mathrm{JUMP}(i) \subseteq \mathrm{NIC}(i, l)$.

(12) If for every state s of SCMPDS such that $\mathbf{IC}_s = l$ and $s(l) = i$ holds $(\mathrm{Exec}(i, s))(\mathbf{IC}_{\mathrm{SCMPDS}}) = \mathrm{Next}(\mathbf{IC}_s)$, then $\mathrm{NIC}(i, l) = \{\mathrm{Next}(l)\}$.

(13) If for every instruction-location l of SCMPDS holds $\mathrm{NIC}(i, l) = \{\mathrm{Next}(l)\}$, then JUMP($i$) is empty.

(14) $\mathrm{NIC}(\mathrm{goto}\ k, l) = \{2 \cdot |k + \mathrm{locnum}(l)| + 2\}$.

(15) $\mathrm{NIC}(\mathrm{return}\, a, l) = \{2 \cdot k; k$ ranges over natural numbers: $k > 1\}$.

(16) $\mathrm{NIC}(\mathrm{saveIC}(a, k_1), l) = \{\mathrm{Next}(l)\}$.

(17) $\mathrm{NIC}(a{:=}k_1, l) = \{\mathrm{Next}(l)\}$.

(18) $\mathrm{NIC}(a_{k_1}{:=}k_2, l) = \{\mathrm{Next}(l)\}$.

(19) $\mathrm{NIC}((a, k_1) := (b, k_2), l) = \{\mathrm{Next}(l)\}$.

(20) $\mathrm{NIC}(\mathrm{AddTo}(a, k_1, k_2), l) = \{\mathrm{Next}(l)\}$.

(21) $\mathrm{NIC}(\mathrm{AddTo}(a, k_1, b, k_2), l) = \{\mathrm{Next}(l)\}$.

(22) $\mathrm{NIC}(\mathrm{SubFrom}(a, k_1, b, k_2), l) = \{\mathrm{Next}(l)\}$.

(23) $\mathrm{NIC}(\mathrm{MultBy}(a, k_1, b, k_2), l) = \{\mathrm{Next}(l)\}$.

(24) $\mathrm{NIC}(\mathrm{Divide}(a, k_1, b, k_2), l) = \{\mathrm{Next}(l)\}$.

(25) $\mathrm{NIC}((a, k_1) <> 0\_\mathrm{goto}\, k_2, l) = \{\mathrm{Next}(l), |2 \cdot (k_2 + \mathrm{locnum}(l))| + 2\}$.

(26) $\mathrm{NIC}((a, k_1) <= 0\_\mathrm{goto}\, k_2, l) = \{\mathrm{Next}(l), |2 \cdot (k_2 + \mathrm{locnum}(l))| + 2\}$.

(27) $\mathrm{NIC}((a, k_1) >= 0\_\mathrm{goto}\, k_2, l) = \{\mathrm{Next}(l), |2 \cdot (k_2 + \mathrm{locnum}(l))| + 2\}$.

Let us consider k. Observe that JUMP(goto k) is empty.

Next we state the proposition

(28) $\mathrm{JUMP}(\mathrm{return}\, a) = \{2 \cdot k; k$ ranges over natural numbers: $k > 1\}$.

Let us consider a. Note that JUMP(return a) is infinite.

Let us consider a, k_1. One can verify that $\mathrm{JUMP}(\mathrm{saveIC}(a, k_1))$ is empty.

Let us consider a, k_1. Observe that $\mathrm{JUMP}(a{:=}k_1)$ is empty.

Let us consider a, k_1, k_2. Note that $\mathrm{JUMP}(a_{k_1}{:=}k_2)$ is empty.

Let us consider a, b, k_1, k_2. One can check that $\mathrm{JUMP}((a, k_1) := (b, k_2))$ is empty.

Let us consider a, k_1, k_2. One can verify that $\mathrm{JUMP}(\mathrm{AddTo}(a, k_1, k_2))$ is empty.

Let us consider a, b, k_1, k_2. One can verify the following observations:

* $\mathrm{JUMP}(\mathrm{AddTo}(a, k_1, b, k_2))$ is empty,
* $\mathrm{JUMP}(\mathrm{SubFrom}(a, k_1, b, k_2))$ is empty,
* $\mathrm{JUMP}(\mathrm{MultBy}(a, k_1, b, k_2))$ is empty, and
* $\mathrm{JUMP}(\mathrm{Divide}(a, k_1, b, k_2))$ is empty.

Let us consider a, k_1, k_2. One can verify the following observations:

* $\mathrm{JUMP}((a, k_1) <> 0\_\mathrm{goto}\, k_2)$ is empty,
* $\mathrm{JUMP}((a, k_1) <= 0\_\mathrm{goto}\, k_2)$ is empty, and
* $\mathrm{JUMP}((a, k_1) >= 0\_\mathrm{goto}\, k_2)$ is empty.

Next we state two propositions:

(29) $\mathrm{SUCC}(l) =$ the instruction locations of SCMPDS.

(30) Let N be a set with non empty elements, S be an IC-Ins-separated definite non empty non void AMI over N, and l_3, l_4 be instruction-locations of S. If $\mathrm{SUCC}(l_3) =$ the instruction locations of S, then $l_3 \leqslant l_4$.

Let us mention that SCMPDS is non InsLoc-antisymmetric.

One can verify that SCMPDS is non standard.

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On the Upper and Lower Approximations of the Curve[1]

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The papers [28], [32], [2], [15], [1], [5], [6], [4], [31], [16], [29], [17], [27], [13], [3], [25], [26], [10], [11], [8], [30], [14], [20], [18], [12], [23], [22], [24], [7], [9], [19], and [21] provide the terminology and notation for this paper.

In this paper n denotes a natural number.

Let C be a simple closed curve. The functor UpperAppr(C) yields a sequence of subsets of the carrier of $\mathcal{E}^2_{\rm T}$ and is defined as follows:

(Def. 1) For every natural number i holds $(\mathrm{UpperAppr}(C))(i) = \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i)))$.

The functor LowerAppr(C) yielding a sequence of subsets of the carrier of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 2) For every natural number i holds $(\mathrm{LowerAppr}(C))(i) = \mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i)))$.

Let C be a simple closed curve. The functor NorthArc(C) yields a subset of $\mathcal{E}^2_{\rm T}$ and is defined by:

(Def. 3) $\mathrm{NorthArc}(C) = \mathrm{Li}\ \mathrm{UpperAppr}(C)$.

The functor SouthArc(C) yielding a subset of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 4) $\mathrm{SouthArc}(C) = \mathrm{Li}\ \mathrm{LowerAppr}(C)$.

We now state a number of propositions:

(1) For all natural numbers n, m such that $n \leqslant m$ and $n \neq 0$ holds $\frac{n+1}{n} \geqslant \frac{m+1}{m}$.

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(2) Let E be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and m, j be natural numbers. Suppose $1 \leqslant m$ and $m \leqslant n$ and $1 \leqslant j$ and $j \leqslant$ width Gauge(E, n). Then $\mathcal{L}($Gauge$(E, n) \circ ($Center Gauge$(E, n),$ width Gauge$(E, n)),$ Gauge$(E, n) \circ ($Center Gauge$(E, n), j)) \subseteq \mathcal{L}($Gauge$(E, m) \circ ($Center Gauge$(E, m),$ width Gauge$(E, m)),$ Gauge$(E, n) \circ ($Center Gauge$(E, n), j))$.

(3) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and i, j be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant$ len Gauge(C, n) and $1 \leqslant j$ and $j \leqslant$ width Gauge(C, n) and Gauge$(C, n) \circ (i, j) \in \widetilde{\mathcal{L}}($Cage$(C, n))$. Then $\mathcal{L}($Gauge$(C, n) \circ (i,$ width Gauge$(C, n)),$ Gauge$(C, n) \circ (i, j))$ meets $\widetilde{\mathcal{L}}($UpperSeq$(C, n))$.

(4) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and n be a natural number. Suppose $n > 0$. Let i, j be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant$ len Gauge(C, n) and $1 \leqslant j$ and $j \leqslant$ width Gauge(C, n) and Gauge$(C, n) \circ (i, j) \in \widetilde{\mathcal{L}}($Cage$(C, n))$. Then $\mathcal{L}($Gauge$(C, n) \circ (i,$ width Gauge$(C, n)),$ Gauge$(C, n) \circ (i, j))$ meets UpperArc$(\widetilde{\mathcal{L}}($Cage$(C, n)))$.

(5) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and j be a natural number. Suppose Gauge$(C, n + 1) \circ ($Center Gauge$(C, n + 1), j) \in$ LowerArc$(\widetilde{\mathcal{L}}($Cage$(C, n + 1)))$ and $1 \leqslant j$ and $j \leqslant$ width Gauge$(C, n + 1)$. Then $\mathcal{L}($Gauge$(C, 1) \circ ($Center Gauge$(C, 1),$ width Gauge$(C, 1)),$ Gauge$(C, n+1) \circ ($Center Gauge $(C, n+1), j))$ meets UpperArc$(\widetilde{\mathcal{L}}($Cage$(C, n+1)))$.

(6) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$, f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, and k be a natural number. Suppose $1 \leqslant k$ and $k+1 \leqslant$ len f and f is a sequence which elements belong to Gauge(C, n). Then $\rho(f_k, f_{k+1}) = \frac{\text{N-bound}(C) - \text{S-bound}(C)}{2^n}$ or $\rho(f_k, f_{k+1}) = \frac{\text{E-bound}(C) - \text{W-bound}(C)}{2^n}$.

(7) Let M be a symmetric triangle metric structure, r be a real number, and p, q, x be elements of M. If $p \in$ Ball(x, r) and $q \in$ Ball(x, r), then $\rho(p, q) < 2 \cdot r$.

(8) Let A be a subset of $\mathcal{E}^n_{\rm T}$, p be a point of $\mathcal{E}^n_{\rm T}$, and p' be a point of $\mathcal{E}^n$. Suppose $p = p'$. Let s be a real number. Suppose $s > 0$. Then $p \in \overline{A}$ if and only if for every real number r such that $0 < r$ and $r < s$ holds Ball(p', r) meets A.

(9) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds N-bound$(C) <$ N-bound$(\widetilde{\mathcal{L}}($Cage$(C, n)))$.

(10) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds E-bound$(C) <$ E-bound$(\widetilde{\mathcal{L}}($Cage$(C, n)))$.

(11) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$

holds S-bound($\widetilde{\mathcal{L}}$(Cage(C, n))) $<$ S-bound(C).

(12) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\rm T}$ holds W-bound($\widetilde{\mathcal{L}}$(Cage(C, n))) $<$ W-bound(C).

(13) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i <$ len Gauge(C, n) and $1 \leqslant k$ and $k \leqslant j$ and $j \leqslant$ width Gauge(C, n) and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, k), Gauge(C, n) $\circ$ (i, j)) $\cap$ $\widetilde{\mathcal{L}}$(UpperSeq(C, n)) $=$ {Gauge(C, n) $\circ$ (i, k)} and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, k), Gauge(C, n) $\circ$ (i, j)) $\cap$ $\widetilde{\mathcal{L}}$(LowerSeq(C, n)) $=$ {Gauge(C, n) $\circ$ (i, j)}. Then $\mathcal{L}$(Gauge(C, n) $\circ$ (i, k), Gauge(C, n) $\circ$ (i, j)) meets UpperArc(C).

(14) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i <$ len Gauge(C, n) and $1 \leqslant k$ and $k \leqslant j$ and $j \leqslant$ width Gauge(C, n) and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, k), Gauge(C, n) $\circ$ (i, j)) $\cap$ $\widetilde{\mathcal{L}}$(UpperSeq(C, n)) $=$ {Gauge(C, n) $\circ$ (i, k)} and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, k), Gauge(C, n) $\circ$ (i, j)) $\cap$ $\widetilde{\mathcal{L}}$(LowerSeq(C, n)) $=$ {Gauge(C, n) $\circ$ (i, j)}. Then $\mathcal{L}$(Gauge(C, n) $\circ$ (i, k), Gauge(C, n) $\circ$ (i, j)) meets LowerArc(C).

(15) Let C be a simple closed curve and i, j, k be natural numbers. Suppose that $1 < i$ and $i <$ len Gauge(C, n) and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant$ width Gauge(C, n) and $n > 0$ and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) $\cap$ LowerArc($\widetilde{\mathcal{L}}$(Cage(C, n))) $=$ {Gauge(C, n) $\circ$ (i, k)} and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) $\cap$ UpperArc($\widetilde{\mathcal{L}}$(Cage(C, n))) $=$ {Gauge(C, n) $\circ$ (i, j)}. Then $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) meets UpperArc(C).

(16) Let C be a simple closed curve and i, j, k be natural numbers. Suppose that $1 < i$ and $i <$ len Gauge(C, n) and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant$ width Gauge(C, n) and $n > 0$ and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) $\cap$ LowerArc($\widetilde{\mathcal{L}}$(Cage(C, n))) $=$ {Gauge(C, n) $\circ$ (i, k)} and $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) $\cap$ UpperArc($\widetilde{\mathcal{L}}$(Cage(C, n))) $=$ {Gauge(C, n) $\circ$ (i, j)}. Then $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) meets LowerArc(C).

(17) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i <$ len Gauge(C, n) and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant$ width Gauge(C, n) and Gauge(C, n) $\circ$ (i, k) $\in$ $\widetilde{\mathcal{L}}$(LowerSeq(C, n)) and Gauge(C, n) $\circ$ (i, j) $\in$ $\widetilde{\mathcal{L}}$(UpperSeq(C, n)). Then $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) meets UpperArc(C).

(18) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i <$ len Gauge(C, n) and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant$ width Gauge(C, n) and Gauge(C, n) $\circ$ (i, k) $\in$ $\widetilde{\mathcal{L}}$(LowerSeq(C, n)) and Gauge(C, n) $\circ$ (i, j) $\in$ $\widetilde{\mathcal{L}}$(UpperSeq(C, n)). Then $\mathcal{L}$(Gauge(C, n) $\circ$ (i, j), Gauge(C, n) $\circ$ (i, k)) meets LowerArc(C).

(19) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i <$ len Gauge(C, n) and $1 \leqslant j$

and $j \leqslant k$ and $k \leqslant \operatorname{width}\operatorname{Gauge}(C,n)$ and $n > 0$ and $\operatorname{Gauge}(C,n) \circ (i,k) \in \operatorname{LowerArc}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))$ and $\operatorname{Gauge}(C,n) \circ (i,j) \in \operatorname{UpperArc}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))$. Then $\mathcal{L}(\operatorname{Gauge}(C,n)\circ(i,j), \operatorname{Gauge}(C,n)\circ(i,k))$ meets $\operatorname{UpperArc}(C)$.

(20) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \operatorname{len}\operatorname{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width}\operatorname{Gauge}(C,n)$ and $n > 0$ and $\operatorname{Gauge}(C,n) \circ (i,k) \in \operatorname{LowerArc}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))$ and $\operatorname{Gauge}(C,n) \circ (i,j) \in \operatorname{UpperArc}(\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n)))$. Then $\mathcal{L}(\operatorname{Gauge}(C,n)\circ(i,j), \operatorname{Gauge}(C,n)\circ(i,k))$ meets $\operatorname{LowerArc}(C)$.

(21) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 \leqslant i_2$ and $i_2 < \operatorname{len}\operatorname{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width}\operatorname{Gauge}(C,n)$ and $(\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,j), \operatorname{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,k), \operatorname{Gauge}(C,n)\circ(i_2,k))) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n)) = \{\operatorname{Gauge}(C,n)\circ(i_1,j)\}$ and $(\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,j), \operatorname{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,k), \operatorname{Gauge}(C,n)\circ(i_2,k))) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n)) = \{\operatorname{Gauge}(C,n)\circ(i_2,k)\}$. Then $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,j), \operatorname{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,k), \operatorname{Gauge}(C,n) \circ (i_2,k))$ meets $\operatorname{UpperArc}(C)$.

(22) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 \leqslant i_2$ and $i_2 < \operatorname{len}\operatorname{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width}\operatorname{Gauge}(C,n)$ and $(\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,j), \operatorname{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,k), \operatorname{Gauge}(C,n)\circ(i_2,k))) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n)) = \{\operatorname{Gauge}(C,n)\circ(i_1,j)\}$ and $(\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,j), \operatorname{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,k), \operatorname{Gauge}(C,n)\circ(i_2,k))) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n)) = \{\operatorname{Gauge}(C,n)\circ(i_2,k)\}$. Then $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,j), \operatorname{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,k), \operatorname{Gauge}(C,n) \circ (i_2,k))$ meets $\operatorname{LowerArc}(C)$.

(23) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_2$ and $i_2 \leqslant i_1$ and $i_1 < \operatorname{len}\operatorname{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width}\operatorname{Gauge}(C,n)$ and $(\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,j), \operatorname{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,k), \operatorname{Gauge}(C,n)\circ(i_2,k))) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n)) = \{\operatorname{Gauge}(C,n)\circ(i_1,j)\}$ and $(\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,j), \operatorname{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,k), \operatorname{Gauge}(C,n)\circ(i_2,k))) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n)) = \{\operatorname{Gauge}(C,n)\circ(i_2,k)\}$. Then $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,j), \operatorname{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,k), \operatorname{Gauge}(C,n) \circ (i_2,k))$ meets $\operatorname{UpperArc}(C)$.

(24) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_2$ and $i_2 \leqslant i_1$ and $i_1 < \operatorname{len}\operatorname{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width}\operatorname{Gauge}(C,n)$ and $(\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i_1,j), \operatorname{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\operatorname{Gauge}(C,n) \circ$

$(i_1, k), \mathrm{Gauge}(C, n) \circ (i_2, k))) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C, n)) = \{\mathrm{Gauge}(C, n) \circ (i_1, j)\}$ and $(\mathcal{L}(\mathrm{Gauge}(C, n) \circ (i_1, j), \mathrm{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\mathrm{Gauge}(C, n) \circ (i_1, k), \mathrm{Gauge}(C, n) \circ (i_2, k))) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C, n)) = \{\mathrm{Gauge}(C, n) \circ (i_2, k)\}$. Then $\mathcal{L}(\mathrm{Gauge}(C, n) \circ (i_1, j), \mathrm{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\mathrm{Gauge}(C, n) \circ (i_1, k), \mathrm{Gauge}(C, n) \circ (i_2, k))$ meets $\mathrm{LowerArc}(C)$.

(25) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 < \mathrm{len}\,\mathrm{Gauge}(C, n+1)$ and $1 < i_2$ and $i_2 < \mathrm{len}\,\mathrm{Gauge}(C, n+1)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C, n+1)$ and $\mathrm{Gauge}(C, n+1) \circ (i_1, k) \in \mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n+1)))$ and $\mathrm{Gauge}(C, n+1) \circ (i_2, j) \in \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n+1)))$. Then $\mathcal{L}(\mathrm{Gauge}(C, n+1) \circ (i_2, j), \mathrm{Gauge}(C, n+1) \circ (i_2, k)) \cup \mathcal{L}(\mathrm{Gauge}(C, n+1) \circ (i_2, k), \mathrm{Gauge}(C, n+1) \circ (i_1, k))$ meets $\mathrm{LowerArc}(C)$.

(26) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 < \mathrm{len}\,\mathrm{Gauge}(C, n+1)$ and $1 < i_2$ and $i_2 < \mathrm{len}\,\mathrm{Gauge}(C, n+1)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C, n+1)$ and $\mathrm{Gauge}(C, n+1) \circ (i_1, k) \in \mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n+1)))$ and $\mathrm{Gauge}(C, n+1) \circ (i_2, j) \in \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n+1)))$. Then $\mathcal{L}(\mathrm{Gauge}(C, n+1) \circ (i_2, j), \mathrm{Gauge}(C, n+1) \circ (i_2, k)) \cup \mathcal{L}(\mathrm{Gauge}(C, n+1) \circ (i_2, k), \mathrm{Gauge}(C, n+1) \circ (i_1, k))$ meets $\mathrm{UpperArc}(C)$.

(27) For every simple closed curve C and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{W\text{-}bound}(C) < p_{\mathbf{1}}$ and $p_{\mathbf{1}} < \mathrm{E\text{-}bound}(C)$ holds $p \notin \mathrm{NorthArc}(C)$ or $p \notin \mathrm{SouthArc}(C)$.

(28) For every simple closed curve C and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_{\mathbf{1}} = \frac{\mathrm{W\text{-}bound}(C) + \mathrm{E\text{-}bound}(C)}{2}$ holds $p \notin \mathrm{NorthArc}(C)$ or $p \notin \mathrm{SouthArc}(C)$.

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